THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WILLIAM HENRY BUSSEY, Editor-in-Chief
HERBERT ELLSWORTH SLAUGHT
AUBREY JOHN KEMPNER

WITH THE COÖPERATION OF

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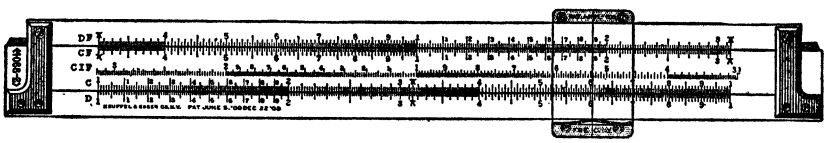
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THE SIXTH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The sixth regular meeting of the Southern California Section of the Mathematical Association of America was held at Occidental College, Los Angeles, on Saturday, Nov. 5, 1927. Professor W. P. Russell presided.

There were fifty present, including the following thirty seven members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, H. Bateman, Clifford Bell, E. T. Bell, W. N. Birchby, J. R. Campbell, M. Collier, P. H. Daus, H. H. Gaver, H. E. Glazier, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, M. M. Keith, Deca Lodwick, W. E. Mason, Ada McClellan, G. F. McEwen, B. Podolsky, A. Prater, M. Reinert, L. E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, D. V. Steed, P. W. Stoner, H. C. Van Buskirk, M. Ward, L. E. Wear, H. C. Willett, Clyde Wolfe, E. R. Worthington.

The meeting began with a luncheon at the college commons, and the Association was welcomed to Occidental College by Acting President, Robert Cleland. After the luncheon, the meeting adjourned to room 28, Fowler Hall for the following program. The numbers of the abstracts correspond to the numbers of the papers.

1. "The relation between certain functions analogous to hyperbolic functions and the Pellian cubic." by Professor P. H. DAUS, University of California at Los Angeles.

2. "The solution of linear differential equations in series by operational divisors," by Mr. MORGAN WARD, California Institute of Technology.

3. "Descriptive geometry of four dimensional space," by Professor L. D. AMES, University of Southern California.

4. "Some aspects of graduate instruction in mathematics" by Professor E. T. BELL, California Institute of Technology.

5. "Suggestions regarding the drawing of plane curves" by Professor E. R. HEDRICK, University of California at Los Angeles.

1. By making certain changes in an article by L. E. Ward in the June-July 1927 issue of this Monthly, Professor Daus considered certain functions analogous to hyperbolic functions, and discussed an isomorphism between them and the solutions of the Pellian cubic, $X^3+DY^3+D^2Z^3-3DXYZ=1$.

2. The connection between the ordinary method of solving a linear differential equation in series and the operational method was given and the practicability of the latter discussed.

3. Descriptive geometry furnishes a method by which space of three dimensions can be represented on a plane. The methods used are easily generalized. An obvious generalization of familiar principles of perspective can be used to represent n-dimensional aggregates on an aggregate of $n-1$ dimensions.

Professor Ames showed a three dimensional perspective model of a certain four dimensional aggregate. The application of these principles seems to involve no essential difficulties not met in three dimensions.

4. The speaker reported on changing conditions in graduate instruction in mathematics, owing to greatly increased enrollment, and outlined probable modifications in the curriculum leading to the doctor's degree.

5. Professor Hedrick discussed certain methods of tracing curves, which involve at most, the use of the fundamental drawing instruments. Besides the well known methods of addition, multiplication and division discussed in texts on Analytic Geometry, he discussed the mechanical construction of a function of a function exhibiting how such curves can be quickly traced. This paper was followed by a discussion by Professors Albert, Stoner, and Wolfe.

P. H. DAUS, *Secretary.*

MATHEMATICS FOR STUDENTS OF CHEMISTRY[1]

By FARRINGTON DANIELS, Department of Chemistry, University of Wisconsin.

1. Introduction. Chemistry has graduated from the class of descriptive sciences into the class of exact sciences and has taken its place by the side of physics and engineering as a branch of mathematics. All chemical phenomena and many biological phenomena are now being interpreted in terms of physical chemistry, and physical chemistry cannot be mastered without a foundation in mathematics which extends through calculus.

Research chemists are keen to realize their dependence on mathematics and many of them feel blocked in their progress by their weakness in mathematics. Another illustration of the need for mathematics is the fact that an increasing number of graduate students in chemistry are taking mathematics as a minor.

Unfortunately the undergraduate student and his advisers do not realize the necessity for mathematics until it is too late. The one overwhelming problem in the teaching of physical chemistry is the handicap of inadequate prepa-

[1] Presented at the meeting of the American Mathematical Association in Madison, Wis., Sept. 6, 1927. The material was presented with the help of special lantern slides. In making these slides, the writing was done with India ink and a smooth pen on an old kodak film which had been soaked and scraped to remove the gelatine film. The film was then mounted between two cover glasses, like those ordinarily used for lantern slides, and bound with gummed paper. The expense and time of a photographic process was thus eliminated. The cover glasses and the binding paper for ordinary lantern slides may be purchased for a few cents at any photographic store. In case many slides are to be made, thin transparent sheets of "cellophane" may be used in place of the kodak films. This material is now used in the confectionery trade and may be purchased very cheaply. It should be cut to the proper size, stretched over the edges of one of the glass plates while wet, and allowed to dry.

ration in mathematics. Calculus should be a prerequisite for every course in physical chemistry but in many cases it is not a prerequisite. Only the simplest differentiations and integrations are used but the ability to think in mathematical terms is absolutely essential.

A few specific applications of mathematics in physical chemistry are given in the following sections. They have been chosen either because they are particularly important or because experience has shown that they constitute stumbling blocks for the student.

2. Calculus. Simple differentiations and simple integrations are common in physical chemistry, and integration tables are freely used. The nature of physical chemical experiments with their complicated apparatus is such as to demand a close personal contact between student and instructor and when a student begins to fall behind the first question always is: "Have you had calculus?" There seem to be but three replies: "No"; "Yes, but it was a long time ago."; and "I was exposed but it didn't take." Now in most cases the preparation is ample for the demands of physical chemistry but the students seem to lack confidence. Can it be that they think that they are expected to remember all the half hundred rules of calculus? The principles of calculus rather than the details of calculus are needed in understanding physical chemistry and only the simplest parts of calculus are actually used.

3. The compound interest law:

$$k = \frac{2.303}{x_2 - x_1} \log_{10} \frac{y_1}{y_2} .$$

This equation is used extensively in the treatment of reaction rates, the absorption of light, the cooling of heated bodies and in many other phenomena. The logarithmic form is usually used rather than the exponential form $y = y_0 e^{-kx}$.

4. The exponential equation:

$$y = e^{\frac{-2500}{2T}} .$$

There is no surer way to make the average industrial chemist skip a page of a scientific article than to place on it a few exponential equations of this type. Practice in transforming and solving these equations is very necessary.

5. Partial differentiation. Chemical thermodynamics is largely founded on partial differentiation and the following factors may be involved: pressure, temperature, volume, entropy, heat, work, internal energy, heat content, free energy, thermodynamic potential. In thermodynamic systems involving any three of the ten variables there are 720 partial first derivatives and there are

11 billion thermodynamic relations involving any four derivatives. Bridgman[1] has shown how these may be reduced to 45 types and only a half dozen of these are important in chemistry.

6. Statistical mechanics is just entering the field of chemistry, and research chemists feel that with this tool will be found the solution of many of the unsolved problems of physical chemistry. In practically no case does the chemist's precision extend beyond a millionth of a gram molecule of material or 6×10^{17} molecules. There is always, then, a sufficiently large number of individuals to assure the application of the laws of probability with accuracy. An important book by Tolman,[2] just published, emphasizes clearly the value of statistical mechanics in chemical problems.

7. Errors in experimental measurements have been treated with the help of probability theorems and the average error, the probable error, and the root-mean-square error have been used. Only a few of the measurements of physical chemistry are sufficiently accurate to warrant the use of more than the average error, because temperature fluctuations and constant errors are usually more serious than the errors of measurement.

8. The differential equations:

$$(a)\qquad \frac{(d\log_e k)}{dT}=\frac{Q}{(2T^2)}\,; \qquad (b)\qquad dx/dt = k(a-x)(b-x).$$

Many of the differential equations of physical chemistry are so simple that the student does not realize that he is solving differential equations.

Equation (a) in which Q is assumed to be independent of T gives the influence of temperature on a great many chemical phenomena such as chemical equilibrium, rate of reaction, solubility, vapor pressure, and others.

Equation (b) involves the rate of reaction between two different substances, and there are cases of several reactions occurring simultaneously which are more difficult. Differential equations are sufficiently important in chemistry to warrant the publication of a special book.[3]

9. Algebra. The algebraic transformations illustrated by the following examples are much used.

(a) *If* $P_1 - P_2 = P_2C_v/(C_p - C_v)$ then $P_1/(P_1 - P_2) = C_p/C_v$.

$$(b)\qquad \left(\frac{2x}{1+x}\right)^2 \div \frac{1-x}{1+x} = \frac{4x^2}{1-x^2}.$$

[1] Bridgman, *A Condensed Summary of Thermodynamic Formulae*, Harvard University Press, Cambridge, Mass., 1925.

[2] Tolman, *Statistical Mechanics with Applications to Physics and Chemistry*, Chemical Catalog Co., New York, 1927.

[3] Hitchcock and Robinson, *Differential Equations in Applied Chemistry*, Wiley and Sons, New York, 1924.

If the algebraic process is pointed out, the student understands these steps perfectly; but few students take the initiative in proving these steps as they occur in the derivation of chemical formulae.

10. Quadratic equations are avoided by students whenever possible, except by engineers and those who have worked many problems. The following equation is typical:

$$K = \tfrac{1}{4}x^{-2}(a - x)(b - x) \; ; \quad x = ?$$

11. Large and small numbers. Molecules in a gram molecule $= 6.06 \times 10^{23}$; the charge on the electron $= 1.59 \times 10^{-19}$; Planck's constant $= 6.56 \times 10^{-27}$; the wave length of blue light $= 4 \times 10^{-5}$; the frequency of blue light $= 7.5 \times 10^{14}$.

These numbers were not necessary in the descriptive chemistry of the past but they are absolutely necessary now. Important advances have been made by visualizing the individual units of matter, electricity and radiation, the molecule, the electron and the quantum.

Twenty years ago it was taught that oxygen is sixteen times as heavy as hydrogen but that the absolute weights could not be known. Now it is known with precision that one atom of oxygen weighs $16/(6.06 \times 10^{23})$ gram. Again, visualizing the process of electrolysis, an ion receives an electron and is discharged as an atom, and the number of atoms (6.06×10^{23}) multiplied by the charge on one electron (1.59×10^{-19}) gives 96,500 coulombs, a quantity which was formerly considered an empirical constant in Faraday's law.

Students are afraid of these large and small numbers and it would be desirable if they could acquire familiarity with them in elementary courses by solving problems such as "How many millimeters to the moon?" or "How many molecules of water in Lake Michigan?" Students seem to enjoy problems of this kind.

12. Negative logarithms.

(*a*) $$10^{-5.7} = x, \quad x = ?$$

(*b*) $$.0592 \log 0.0093 = \frac{0.3984}{x}, \quad x = ?$$

Problems of this kind are very common in electro-chemistry and particularly in the treatment of acid and alkaline solutions. If x in equation (*a*) represents the concentration of hydrogen ions, 5.7 is the so-called pH of the solution. This term is universally used in the biological and botanical sciences, but not one student in five can solve these problems.

13. The slide rule. The slide rule is sufficiently accurate for most chemical calculations and it develops judgment in matters of accuracy. For example, when students titrate 30.0 cc of alkali against 40.0 cc of standard acid, the one

who uses a slide rule and is accustomed to evaluating the decimal point writes down a normality factor of 1.333. A routine calculator, using long division or logarithms, may write down 1.333333 and imply an accuracy far greater than is justified.

14. Displaced origins and unequal scales. Most students can use graph paper satisfactorily but there are always a few who insist on starting the scales at zero and wasting most of the paper. If the volume of gas is to be plotted against temperatures between 273 and 293, there are some who will start with zero rather than with 270 or 273. Many students have difficulty in finding the equation of a line unless the divisions along the x-axis are equal to those of the y-axis. It would be better if the elementary books didn't always start the scales at zero and didn't always have the same scales along the two axes.

15. Finding empirical equations.

$$(a)\quad y = a + bx + cx^2\,; \qquad (b)\quad \log y = a + bx.$$

Equations of these two types are used very often to express experimental data. The first one is purely empirical but the second one has a theoretical backing and it runs through a great many phenomena of physical chemistry. If students of analytical geometry could have more direct experience in curve fitting and in evaluating the constants of these equations from a set of data, it would help greatly in the sciences.

16. Methods for expressing data. The experimental determination of the vapor pressure of chloroform at different temperatures illustrates nicely the advantage of mathematical treatment. The student obtains the following data in the laboratory:

Temperature	(T)	293°	303	313	323	333
Vapor Pressure	(P)	160.5mm	247.5	369.3	535.0	755.5

He then plots vapor pressure against temperature and obtains an exponential curve. He next plots $\log P$ against $1/T$ and obtains a line which is nearly straight. Finally, he finds the equation of this straight line and writes

$$\log P = \frac{-1641.2}{T} + 7.8089.$$

The student gets real pleasure in work of this kind and he is convinced that the mathematical equation is much the best method for expressing the relation between variables.

17. Graphical calculations. A science which has recently become a mathematical science must limp along with graphical calculations for a while. For example, there are few graphical calculations left in astronomy, while in economics, a subject which involves more factors than chemistry, the graphical methods are still more prevalent than in chemistry.

There is at present a great need for more precise graphical methods, or for reasonably simple substitutes for the graphical calculations.

In many cases there are too many factors involved in the phenomenon to offer any immediate hope of finding a theoretical equation connecting the variables, but from a single graph many other properties may be calculated with an accuracy which depends only on the precision of the graphical method.

G. N. Lewis has emphasized the value of graphical calculations in chemistry and devised methods for increasing their accuracy. The methods are included in his book on thermodynamics.[1] The calculations consist chiefly in finding the tangents to a curve or in evaluating the area under a curve.

The velocity of a chemical reaction is determined by plotting the amount of material against the time and determining the tangent to the curve. The properties of a dissolved substance in solution are now defined in terms of the rate of change of the property with the change in concentration. For example, the partial volume of a salt in water is determined by plotting the volume of the solution against the concentration and drawing a tangent at any specified concentration. This obviously simple method for expressing the properties of a solution was devised by Lewis only a few years ago and it brought order out of chaos in the study of solutions. Previously, different investigators had worked over different finite concentrations and their results could not check. Ingenious methods have been devised for increasing the accuracy of these calculations. In one method for example the difference between the true differential value and an approximate finite value is plotted, and any error of graphing applies only to a small correction term rather than to the quantity itself.

When the specific heat of a substance is plotted against the temperature, the area under the curve down to absolute zero is a direct measure of the "entropy" of the substance, and the entropy change in a chemical reaction is a measure of the tendency to react. Chemical processes will soon be calculated from tables of entropy just as engineering problems are now solved with the help of tables. In accumulating the material for these entropy tables from experimental measurements the graphical methods must play an important part because the equations for the specific heat curves are unknown except in the region of the absolute zero.

Many of the calculations of physical chemistry may be made exact instead of approximate by substituting "activities" for concentration, and the activity of a dissolved substance may be calculated from the activity of the dissolving liquid by evaluating the area under a curve which is plotted from experimental data. Frequently these curves approach the axes asymptotically in such a

[1] Lewis and Randall, *Thermodynamics and the Free Energy of Chemical Substances*, McGraw-Hill Book Co., New York, 1923.

way as to render the calculation difficult, and special functions have been invented for the graphing, to avoid these inconvenient figures and permit an accurate determination of the area.

Any advances which permit an increased accuracy in graphical calculations will be welcomed by the chemist.

18. Conclusion. It has been the purpose of this article to point out some of the mathematical needs of modern chemistry. It would be appreciated if courses in mathematics could emphasize some of these things, along with the calculations of volumes, lengths of curves, and moments of inertia and other calculations which are included for the benefit of the engineer. The present courses should not be changed, however, for mathematics is much more valuable to the student of chemistry as a mental training than as a source of technical methods. In physical chemistry the chief aim is to emphasize the research point of view and to interest the student in the mechanism of natural phenomena and there is no better way to develop the necessary originality in a student than to have him solve hundreds and thousands of problems in pure mathematics.

A SIMPLIFICATION OF CERTAIN PROBLEMS IN ARITHMETICAL DIVISION

By MORGAN WARD, California Institute of Technology

1. Like all inverse operations, the process of dividing one integer by another requires certain tentative steps irresolvable into the direct operations of addition and multiplication. Any method of division has then a three-fold aim: to reduce as far as possible (1) the number of these tentative steps, (2) the difficulty of each step, and (3) the amount of addition, multiplication and mere copying necessary to combine the results of the separate steps into the correct quotient. These aims conflict to a certain extent and our ordinary "long" division is a sort of compromise between them. It may in fact be considerably improved as regards the third requirement.[1]

The object of this paper is to exhibit a method of division satisfying the requirements above and applicable to a class of problems where solution by ordinary division is excessively laborious, if not impossible. Our main result is given in section 4; its proof, which rests upon the most elementary properties of congruences and power residues, is developed in sections 2 and 3. The concluding section is devoted to numerical examples.

[1] See a discussion of "long" division by L. S. Dederick in this Monthly, vol. 33 (1926), pp. 143–144.

2. In what follows small italic and Greek letters stand for positive integers or zero.

Denote the quotient obtained on dividing a by $b \neq 0$ by

$$(1) \qquad [a/b] = q \quad \text{so that} \quad a = qb + r \qquad 0 \leqq r < b.$$

Then $q=0$ when $b>a$ and is otherwise a positive integer. It is to be understood that b is never zero in the symbol $[a/b]$. Let us develop some of the properties of the symbol.

First, if $b=b_1 \cdot b_2$, it is easily shown[1] that

$$(I) \qquad \left[\frac{a}{b_1 b_2}\right] = \left[\frac{\left[\frac{a}{b_1}\right]}{b_2}\right] = \left[\frac{\left[\frac{a}{b_2}\right]}{b_1}\right]$$

Suppose $a=a_1 \cdot a_2$. Let $[a_1/b]=q_1$ and $[a_2/b]=q_2$ so that $a_1=bq_1+r_1$ and $a_2=bq_2+r_2$ $(0 \leqq r_1, r_2 < b)$. Then $a_1a_2=(bq_1q_2+q_1r_2+q_2r_1)b+r_1r_2$. Hence

$$(II) \qquad \left[\frac{a_1a_2}{b}\right] = b\left[\frac{a_1}{b}\right]\left[\frac{a_2}{b}\right] + r_2\left[\frac{a_1}{b}\right] + r_1\left[\frac{a_2}{b}\right] + \left[\frac{r_1r_2}{b}\right].$$

The following special cases of II are to be noted:

First, if $r_2=0$,

$$(IIa) \qquad \left[\frac{a_1a_2}{b}\right] = b\left[\frac{a_1}{b}\right]\left[\frac{a_2}{b}\right] + r_1\left[\frac{a_2}{b}\right] = a_2\left[\frac{a_1}{b}\right] + r_1\left[\frac{a_2}{b}\right].$$

Secondly, if $q_2=0$, then $a_2<b$, $[a_2/b]=0$, $r_2=a_2$, and

$$(IIb) \qquad \left[\frac{a_1a_2}{b}\right] = a_2\left[\frac{a_1}{b}\right] + \left[\frac{a_2r_1}{b}\right].$$

Also if $a=a_1+a_2$,

$$(III) \qquad \left[\frac{a}{b}\right] = \left[\frac{a_1}{b}\right] + \left[\frac{a_2}{b}\right] + \left[\frac{(r_1+r_2)}{b}\right], \quad \text{where}$$
$$0 \leqq \left[\frac{(r_1+r_2)}{b}\right] \leqq 1.$$

3. Assume now that a is greater than b and prime to it,

$$(2) \qquad U_m = \left[\frac{a^m}{b}\right]. \quad \text{Then} \quad (2a) \quad U_0 = U_1 = 0.$$

We seek the general expression for U_m.

[1] Reid: *Elements of the Theory of Algebraic Numbers*, p. 27.

Let r be the least positive integer for which

$$a^{r+1} \equiv a \pmod{b}. \tag{3}$$

That is, r is the exponent to which a belongs, modulo b, so that $a, a^2, a^3, \cdots, a^r$ are all distinct. Let $a^s \equiv a_s \pmod{b}$ $(0 \leqq a_s < b)$ and

$$V_s = [aa_s/b], \quad \text{so that} \quad V_0 = 0, \quad V_s < b. \tag{4}$$

Then by (3), (4), $V_s = V_t$ when and only when $s \equiv t \pmod{r}$.

By (IIb), $U_m = [a \cdot a^{m-1}/b] = a[a^{m-1}/b] + [a \cdot a_{m-1}/b]$, $m \geqq 1$; or, by (2) and (4),

$$U_m = aU_{m-1} + V_{m-1}. \tag{5}$$

Thus $U_2 = aU_1 + V_1 = V_1$; $U_3 = aU_2 + V_2 = aV_1 + V_2$. Assume for some $s \geqq 2$

$$U_s = a^{s-2}V_1 + a^{s-3}V_2 + \cdots + aV_{s-2} + V_{s-1} = \sum_{K=1}^{s-1} a^{s-1-K}V_K.$$

Then

$$U_{s+1} = aU_s + V_s = \sum_{K=1}^{s-1} a^{s-K}V_K + V_s = \sum_{K=1}^{s} a^{s-K}V_K$$

or

$$U_{s+1} = \sum_{K=1}^{(s+1)-1} a^{(s+1)-1-K}V_K,$$

so that, by induction, for any $m \geqq 1$

$$U_m = \sum_{K=1}^{m-1} a^{m-1-K}V_K. \tag{6}$$

Now suppose

$$m - 1 = kr + \alpha \qquad (0 \leqq \alpha < r, \ k \geqq 0). \tag{7}$$

Divide K in (6) by r and let the quotient be τ and the remainder σ $(0 \leqq \tau \leqq k, 0 \leqq \sigma < r)$. Set $K = \tau_1 r + \sigma_1$ $(0 \leqq \sigma \leqq \alpha)$ and $K = \tau_2 r + \sigma_2$ $(\alpha < \sigma \leqq r-1)$. Then (6) may be written

$$\begin{aligned} U_m &= \sum_{\tau_1=0}^{k} \sum_{\sigma_1=0}^{\alpha} a^{(k-\tau_1)r+\alpha-\sigma_1}V_{\tau_1 r+\sigma_1} + \sum_{\tau_2=0}^{k-1} \sum_{\sigma_2=\alpha_1}^{r-1} a^{(k-1-\tau_2)r+\alpha+r-\sigma_2}V_{\tau_2 r+\sigma_2} \\ &= \sum_{\tau_1=0}^{k} a^{(k-\tau_1)r} \sum_{\sigma_1=0}^{\alpha} a^{\alpha-\sigma_1}V_{\sigma_1} + \sum_{\tau_2=0}^{k-1} a^{(k-1-\tau_2)r} \sum_{\sigma_2=\alpha_1}^{r-1} a^{\alpha+r-\sigma_2}V_{\sigma_2} \\ &= \sum_{\tau_1=0}^{k} a^{(k-\tau_1)r} \sum_{\sigma_1=0}^{\alpha} a^{\alpha-\sigma_1}V_{\sigma_1} + \sum_{\tau_2=0}^{k-1} a^{(k-1-\tau_2)r} \sum_{\sigma_1=1}^{r-1-\alpha} a^{\alpha+\sigma_1}V_{\alpha+\sigma_1} \end{aligned}$$

on replacing in the second expression $r-\sigma_2$ by σ_1 and changing the order of summation. But

$$\sum_{\tau_1=0}^{k} a^{(k-\tau_1)r} = \frac{a^{(k+1)r}-1}{a^r-1}; \qquad \sum_{\tau_2=0}^{k-1} a^{(k-1-\tau_2)r} = \frac{a^{kr}-1}{a^r-1}.$$

Hence we obtain the following theorem.

4. THEOREM: "Let b be any integer greater than a but not a power of a, and let m be any positive integer. Let k and α denote the quotient and remainder obtained on dividing $m-1$ by the least positive integer r for which $a^{r+1}-a$ is divisible by b. Then the quotient on dividing a^m by b is given by

$$\text{(IV)} \qquad \left[\frac{a^m}{b}\right] = \frac{a^{(k+1)r}-1}{a^r-1}(a^{\alpha-1}V_1 + a^{\alpha-2}V_2 + \cdots + aV_{\alpha-1} + V_\alpha) + \frac{a^{kr}-1}{a^r-1}a^\alpha(aV_{\alpha+1} + a^2V_{\alpha+2} + \cdots + a^{r-\alpha-1}V_{r-1}),$$

where $V_i=[a\cdot a_i/b]$ $(1\leqq i\leqq r-1)$ and a_i is the least positive residue of a^i modulo b."

The numbers V_i must be found by trial. They are always less than b and can be readily computed in conjunction with the a_i's; for from (4) we have $V_s=[a\cdot a_s/b]$, $a_{s+1}\equiv a^{s+1}\equiv a\cdot a^s\equiv a\cdot a_s \pmod b$, $0\leqq a_{s+1}<b$, or $a\cdot a_s = b\cdot V_s + a_{s+1}$, so that V_s is the quotient and a_{s+1} the remainder on dividing $a\cdot a_s$ by b. We can thus determine $V_1, V_2, \cdots, V_{r-1}$ step by step. If we have found by any means the residue system $a_1, a_2, \cdots, a_s, \cdots, a_{r-1}$, say by a table of indices modulo $b/(b, a)$, we have

$$(8) \qquad V_s = (a\cdot a_s - a_{s+1})/b$$

which for small b's gives V_s by inspection.

The expressions in brackets in IV are indeed numbers in the scale a.

If the V's are assumed known, IV involves only additions and multiplications, for the first terms in each line are merely the sums of geometric progressions. Hence we have here a *formula* for the quotient of a^m by b.

There are several special cases of IV. First $\alpha=0$ if $m-1$ is exactly divisible by r, and IV becomes

$$\left[\frac{a^m}{b}\right] = \frac{a^{kr}-1}{a^r-1}(aV_1 + a^2V_2 + \cdots + a^{r-1}V_{r-1}) = \frac{a^m-1}{a^r-1}(a^{r-1}V_{r-1} + a^{r-2}V_{r-2} + \cdots + aV_1)$$

so that we have the result:

(IVa) $$\left[\frac{a^m}{b}\right] \equiv 0 \pmod{a} \qquad \text{if} \quad m \equiv 0 \pmod{r}.$$

If $b = a^r - 1$, $V_1 = V_2 = V_3 = \cdots = V_{r-2} = 0$, $V_{r-1} = 1$, and

(IVb) $$\left[\frac{a^m}{b}\right] = \frac{a^{kr} - 1}{a^r - 1} a^{r-1}.$$

If $a = 10$, the expressions in round brackets in IV become ordinary numbers with digits $V_1 \cdots V_\alpha$; $V_{r-1} \cdots V_{\alpha+1}$, and the geometric progressions are numbers of the form

(IVc) $$100 \cdots 0100 \cdots 0100 \cdots 01 \cdots$$

where the 1's are separated by $r-1$ zeros.

The formulas I–IV enable us to solve any problem in division by separating a and b into sums and products in suitable ways. But IV is the only result essentially novel. Let us apply it to some numerical examples.

5. Example 1. To find the quotient when 10^{11} is divided by 13.

Here

$$a = 10, \quad b = 13, \quad m = 11, \quad a_1 = 10, \quad a \cdot a_1 = 100 = 7 \cdot 13 + 9.$$

$$\therefore v_1 = 7, \quad a_2 = 9, \quad a \cdot a_2 = 90 = 6 \cdot 13 + 12; \quad \therefore v_2 = 6, \quad a_3 = 12,$$

$$a \cdot a_3 = 120 = 9 \cdot 13 + 3;$$

$$\therefore v_3 = 9, \quad a_4 = 3, \quad a \cdot a_4 = 30 = 2 \cdot 13 + 4; \quad \therefore v_4 = 2, \quad a_5 = 4,$$

$$a \cdot a_5 = 40 = 3 \cdot 13 + 1;$$

$$\therefore v_5 = 3, \quad a_6 = 1, \quad r = 6, \quad m - 1 = 6 + 4. \quad \therefore k = 1, \quad \alpha = 4.$$

$$\left[\frac{10^{11}}{13}\right] = \left[\frac{10^{12} - 1}{10^6 - 1}\right] \cdot 7692 + \frac{10^6 - 1}{10^6 - 1} \cdot 10^4 \cdot 32$$

$$= (10^6 + 1) \cdot 7692 + 10^4 \cdot 32 = 7692307692.$$

The work here is exactly the same as in the ordinary process of short division:

$$13\overline{)100000000000}$$
$$7692307692.$$

Example 2. To find the quotient when 9^μ, where $\mu = (9^9)$, is divided by 19. Here $a = 9$, $b = 19$, $m = 9^9$, and 9, 5, 7, 6, 16, 11, 4, 17, 1 are the residues of $a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9$, so that $r = 9$.

$$m - 1 = 9^9 - 1 = (9^8 - 1)9 + 8, \quad \text{so that} \quad k = 9^8 - 1, \quad \alpha = 8.$$

Using (8), we readily find 4, 2, 3, 2, 7, 5, 1, 8 for V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8. Therefore if $\mu=(9^9)$ and $\nu=9^8$,

$$\left[\frac{9^\mu}{19}\right]=\left(\frac{9^\nu-1}{9^9-1}\right)(4\cdot 9^7+2\cdot 9^6+3\cdot 9^5+2\cdot 9^4+7\cdot 9^3+5\cdot 9^2+1\cdot 9+8)$$

Example 3. To find the quotient when $2^{257}-1$ is divided by 1023.

Here $a=2$, $b=a^{10}-1$, $r=10$, and IVb is applicable. Since $2^{257}\equiv 2^7 \pmod{1023}$,

$$\left[\frac{2^{257}-1}{1023}\right]=\left[\frac{2^{257}}{1023}\right]=\frac{2^{250}-1}{2^{10}-1}\cdot 2^9=2^9+2^{19}+2^{29}+\cdots+2^{249}.$$

FORMAL UNIFICATION OF GRADIENT, DIVERGENCE, AND CURL, BY MEANS OF AN INFINITESIMAL OPERATIONAL VOLUME

By VLADIMIR KARAPETOFF, Cornell University

In vector analysis, the results of the three differential operations known as taking the gradient, the divergence, and the curl of a function are radically different from each other, both from a mathematical and from a physical point of view. Nevertheless there is some formal connection among the operations themselves in that the same Hamiltonian operator "nabla" or "del" (∇) is used in all three.[1] This permits to denote the three operations as ∇, $\nabla\cdot$, and $\nabla\times$, respectively. In elementary text-books on vector analysis, the three operations and the Hamiltonian operator itself are introduced in Cartesian coördinates, thus perhaps leaving in the mind of the reader an unconscious impression that it is absolutely necessary to begin a problem by using the projections of both the operator and the operand.

On the other hand, the very purpose of vector analysis being to do away with resolving directed quantities into their components, as much as possible, a direct and unified interpretation of the foregoing differential operators in space is quite desirable. In some advanced German works[2] this unification has been obtained by defining the gradient, the divergence, and the curl as follows:

$$\text{(A)}\qquad \nabla P=\lim_{\Delta\doteq 0}(1/\Delta v)\int_S ds P\,;\qquad \text{(B)}\qquad \nabla\cdot F=\lim_{\Delta\doteq 0}(1/\Delta v)\int_S ds\cdot F\,;$$

$$\text{(C)}\qquad \nabla\times F=\lim_{\Delta\doteq 0}(1/\Delta v)\int_S ds\times F.$$

[1] In this article, by the Hamiltonian operator is meant the operator ∇ expressed in orthogonal coordinates, that is, $\nabla=i(\partial/\partial x)+j(\partial/\partial y)+k(\partial/\partial z)$

[2] C. Runge, *Vector Analysis* (English Translation), p. 95; W. von Ignatowsky, *Die Vektoranalysis*, vol. 1, p. 16; J. Spielrein, *Lehrbuch der Vektorrechnung*, p. 111.

In these expressions, P is a scalar point function, F is a vector point function, Δv is a small volume of any desired form, surrounding the point under consideration, S is the total surface of this volume, and $d\mathbf{s}$ is an element of this surface, thought of as a vector in the direction of the positive normal to the element. Since ∇P, $\nabla \cdot F$ and $\nabla \times F$ are point characteristics, the results of integration, after division by Δv, are independent of the form or the size of the chosen volume, provided that this volume and its boundary surface are in the limit reduced to zero, that is, to the point under consideration. The values of both the integrals and Δv being infinitesimally small, the process of finding ∇P, $\nabla \cdot F$, and $\nabla \times F$ is a generalized operation of obtaining an ordinary derivative of a function, and may be properly called *space differentiation*.

Equations (A), (B), (C) serve as general definitions of the operations under consideration. In specific applications, it is necessary to choose such a shape of the volume Δv as leads to the result in the most direct manner, preferably without actual integration and subsequent reduction of Δv to zero. It is thought that an infinitesimal volume, dv, of the shape proposed in Fig. 1 satisfies this con-

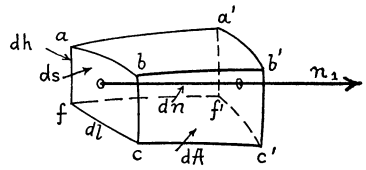


FIG. 1.

dition. It approaches a parallelopiped in shape, except that all its faces may be slightly curved, and the two opposite sides, $abcf$ and $a'b'c'f'$, slightly different and skewed with respect to each other. The vector $\boldsymbol{n}_1$ is a unit normal at the center of the face $abcf$. The area $abcf$ is denoted by ds and $cc'f'f$ by dA. The distance between the faces $abcf$ and $a'b'c'f'$ is dn, and the length of one of the vertical edges, such as fa, is dh.

In each particular problem, the shape of the operational volume, shown in Fig. 1, is to be so chosen and the volume itself so placed in the field, as to satisfy the following two conditions:

(1) Only the faces $abcf$ and $a'b'c'f'$ should contribute to the integrals in the expressions (A), (B), (C);

(2) The expressions themselves should reach maximum values.

The operator ∇ which enters in equations (A), (B), (C) may then be rewritten in the form

$$\nabla = \max d_n(ds\,\boldsymbol{n}_1)/dv. \tag{1}$$

To obtain a gradient, the scalar function P, upon which this operator is to act, is to be placed in the blank space after $\boldsymbol{n}_1$. To obtain a divergence or a curl, the

vector function $\boldsymbol{F}$, preceded by a dot or a cross, is to be placed in the same blank space. The symbol d_n stands for differential and signifies a differentiation along the normal $\boldsymbol{n}_1$. Expression (1) is thus a general symbol for space differentiation corresponding to equations (A), (B), (C), provided that dv satisfies the two conditions stated above. In place of writing $d\mathbf{s}$ as a vector, it is more convenient in applications to use a scalar ds, followed by unit vector $\boldsymbol{n}_1$. For the sake of brevity we shall refer to the operator ∇, in the form of equation (1), as the *Gibbsian operator.*

It will now be explained how the shape and the position of the volume dv are to be chosen in the particular cases of gradient, divergence, and curl; and it will also be shown that the results so obtained agree with those derived from the usual elementary definitions of these operators. The advantage of definition (1), as compared with (A), (B), (C), is that it is not necessary to go through an integration and a limiting process in each particular application. The prescribed shape of the infinitesimal volume reduces the problem to a simple differentiation.

(I) *Gradient.* Let the Gibbsian operator (1) act upon a scalar point function P and let the operational volume dv be a true parallelopiped, so placed that the face $abcf$ coincides with the equi-potential surface at the point under consideration. In this case $\boldsymbol{n}_1 = \text{Const.}$; $ds = \text{Const.}$, and both may be taken outside the sign d_n. Moreover, $dv = ds \cdot dn$. Thus we have:

$$\nabla P = d_n(ds\, \boldsymbol{n}_1 P)/dv = ds\, \boldsymbol{n}_1\, d_n P/(ds\, dn) \tag{2}$$

where dn is the distance between the faces $abcf$ and $a'b'c'f'$. We thus obtain the usual expression

$$\nabla P = \boldsymbol{n}_1\, dP/dn, \tag{3}$$

where the operation (1) is performed in the direction in which the rate of change of P is a maximum.

(II) *Divergence.* Let the Gibbsian operator be provided with a "dot" after $\boldsymbol{n}_1$, and let it act upon a continuous vector function $\boldsymbol{F}$. Let the operational volume, dv, be so shaped and placed that the edges aa', bb', cc', ff', coincide with some lines of force in the immediate vicinity of the point under consideration, and the faces $abcf$ and $a'b'c'f'$ are normal to the flux. Then

$$\nabla \cdot \boldsymbol{F} = d_n(ds\, \boldsymbol{n}_1 \cdot \boldsymbol{F})/dv. \tag{4}$$

Since $\boldsymbol{n}_1 \cdot \boldsymbol{F}$ is the magnitude of the flux density along the normal, $ds(\boldsymbol{n}_1 \cdot \boldsymbol{F})$ is the total flux entering through $abcf$. Calling this flux $d\phi$, we obtain,

$$\nabla \cdot \boldsymbol{F} = [d_n(d\phi)]/dv \tag{5}$$

which is the usual definition of divergence, viz., excess flux per unit volume.

(III) *Curl.* Let the Gibbsian operator (1) be provided with a "cross" after $\mathbf{n}_1$ and let it act upon a continuous vector function F. A small portion of the flux near the point under consideration is shown in Fig. 2 by the horizontal and vertical projections of lines of force, such as pq and rs. The osculating plane at the point w of pq is horizontal, so that the vertical projection of pq is a very nearly horizontal straight line. The element fc of pq is supposed to be parallel to the axis XY.

Of all the lines of force adjacent to pq, and distant from it by the same amount dn, that one is chosen (rs) for which the "twist" is the greatest, that is, the one whose osculating plane at x forms the greatest angle with the osculating plane of pq at w.

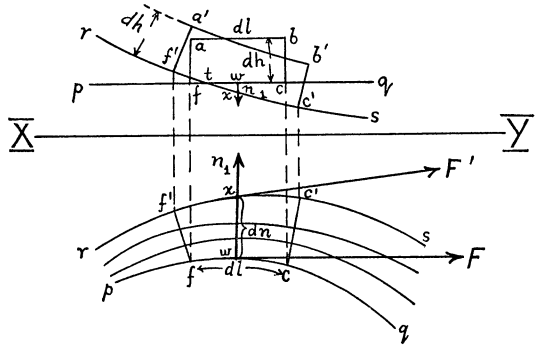


FIG. 2.

The operational volume is constructed as follows: The edge fc (Fig. 1) is placed on pq, the edge $f'c'$ on rs; the faces $bb'c'c$ and $aa'f'f$ are made normal to the lines of force. The average height, dh, of the face $abcf$ is taken to be equal to that of $a'b'c'f'$ (up to infinitesimals of the second order). The face $a'b'c'f'$ is skewed with respect to $abcf$ about the axis $\mathbf{n}_1$, in accordance with the above mentioned twist between the elements cf and $c'f'$ of the two lines of force. Both faces are normal to the respective osculating planes. We then have:

$$\nabla \times F = d_n(ds\, \mathbf{n}_1 \times F)/dv \tag{6}$$

The area ds may be replaced by the product $dh\, dl$, and the volume dv by the product $dh\, dA$. Factoring out and cancelling the constant quantity dh in the numerator and denominator, we get

$$\nabla \times F = d_n(\mathbf{n}_1 \times F\, dl)/dA \tag{7}$$

where

$$dA = \text{area } cff'c'.$$

$\boldsymbol{F}dl$ is the line integral of F along fc; and because it forms a vector product with n_1, this line integral is to be thought of as a vector normal to the osculating plane of pq. The symbol d_n means that a geometric difference is to be formed between this line integral and the corresponding value $n_1' \times \boldsymbol{F}'dl'$, along rs. The sum of the line integrals of F along cc' and ff' being an infinitesimal of a higher order, the numerator of equation (7) represents, in the limit, the total line integral of the vector function $\boldsymbol{F}$ along the contour $cc'f'f$. The ratio of this integral to the area, dA, of the contour, as in equation (7), is the curl of $\boldsymbol{F}$, according to the usual definition.

Let the flux be two-dimensional (co-planar); in other words, let there be no skewing of rs with respect to pq, both lines of force being curves in the same plane. Then both partial line integrals, $n_1 \times \boldsymbol{F}dl$ and $n_1' \times \boldsymbol{F}'dl'$, are normal to the plane of the curves, and consequently the curl is perpendicular to the planes of the flux lines. Skewing the flux, that is, making it three-dimensional, adds a component of the curl in the direction of the flux itself. Since rs is chosen in the direction of maximum skewing, expression (7) contains not only the whole component of the curl normal to the plane of pq, but the whole component along $\boldsymbol{F}$ as well; in other words, it gives the whole curl vector.

The foregoing investigation supplements the author's earlier article on the *Divergence and curl in a vector field, in terms of curvature and tortuosity*, Philosophical Magazine, vol. 31 (1916), p. 528.

EXPRESSIONS FOR THE GENERAL DETERMINANT IN TERMS OF ITS PRINCIPAL MINORS

By E. B. STOUFFER, University of Kansas

It was shown by MacMahon[1] in 1893 and later in a more simple manner by Muir[2] that only n^2-n+1 of the 2^n-1 principal minors (including the determinant itself) of the general determinant of the nth order are independent. Thus, for $n=1, 2, 3$ the principal minors are all independent but for $n>3$ there exist 2^n-n^2+n-2 relations among them.

In 1924 the author of the present paper gave a method[3] of expressing the general determinant of order $n>3$ in terms of n^2-n+1 of its principal minors of order less than n. Several expressions for the general determinant of order four in terms of its principal minors of order three or less had already been obtained[4]

[1] Philosophical Transactions of the Royal Society of London, vol. 185, pp. 111–160.

[2] Philosophical Magazine and Journal of Science, (5), vol. 38, pp. 537–541.

[3] Transactions of American Mathematical Society, vol. 26, pp. 356–368.

[4] See, e. g., MacMahon, loc. cit.; Nansen, Philosophical Magazine and Journal of Science, (5) vol. 44, pp. 362–67; Muir, Transactions of the Royal Society of Edinburgh, vol. 39, pp. 323–339.

but previous efforts to solve the problem in a general way had apparently been stopped by an erroneous theorem by MacMahon.[1] This theorem states that a general determinant of odd order cannot be expressed in terms of its principal minors of lower order. The error in MacMahon's proof arises from the assumption that certain relations among the principal minors are all independent. Such is not the case. It should be added that MacMahon's apparent proof that there are only two independent relations involving the principal minors of a general determinant of even order is also incorrect. The source of the error is the same.

It is the purpose of the present paper to give a method of expressing the general determinant of any order in terms of only fourteen of its principal minors of lower order. The expressions obtained are much simpler than those given by the previous method.

Let A represent a determinant of the nth order in which the element in row i and column j is denoted by a_{ij}. Furthermore, let $(A_{klmn}\cdots)$ denote the principal minor of A obtained by striking out all rows and columns except those numbered $k, l, m, n, \cdots$.

Muir[2] has shown that the values of the principal minors of a determinant A with general elements a_{ij} are not affected if certain $n-1$ elements are each put equal to unity. One such set consists of the elements $a_{1j}(j>1)$ and we shall accordingly assume in the remainder of this paper that those elements are equal to unity.

The following theorem[3] is important for our future considerations:

The principal minors

$$(A_i),\ (A_{ij}),\ (A_{1ij}),\quad i,j = 1,2,\cdots,n;\quad i<j, \tag{1}$$

constitute a total of n^2-n+1 *independent minors.*

We shall actually use this theorem only for $n=4$. For this case the direct proof by means of the functional determinant is quite simple.

For $n=4$ with $a_{12}=a_{13}=a_{14}=1$ we have the following results:

$$\begin{aligned}
a_{ii} &= (A_i) \quad (i = 1,2,3,4);\\
a_{j1} &= (A_1)(A_j) - (A_{1j}) \quad (j = 2,3,4);\\
a_{ij}a_{ji} &= (A_i)(A_j) - (A_{ij}) \quad (i,j > 1;\ i \neq j);\\
a_{j1}a_{ij} + a_{i1}a_{ji} &= (A_{1ij}) - (A_1)(A_{ij}) + 2(A_1)(A_i)(A_j)\\
&\qquad - (A_j)(A_{1i}) - (A_i)(A_{1j}),\quad (i,j > 1;\ i \neq j).
\end{aligned} \tag{2}$$

[1] MacMahon, loc. cit.

[2] Muir, Philosophical Magazine and Journal of Science, (5), vol. 38, pp. 537–541.

[3] Stouffer, loc. cit., p. 357.

Thus, for the case $n=4$, the expressions for all the elements of A in terms of the minors (1) follow immediately from these equations. The substitution of these expressions for the elements of A gives an expression for A itself and for its one remaining principal minor (A_{234}) in terms of thirteen of its principal minors of order three or less.[1]

The process outlined above works for determinants of any higher order. It has the advantage of involving principal minors of order no higher than three but the expression for A becomes very complex as the number n^2-n+1 of minors involved increases.

We shall now show that it is possible to go directly from the expression for the determinant of order four in terms of thirteen of its principal minors to an expression for the determinant of any order in terms of fourteen of its principal minors. For this purpose we make use of the so-called law of extensible minors which has been stated by Muir[2] as follows: *If any identical relation be established between a number of the minors of a determinant or between the determinant itself and a number of its minors, the determinants being denoted by means of their principal diagonals, then a new theorem is always obtainable by merely choosing a line of letters with new suffixes and annexing it to the end of the diagonal of every determinant, including those of order zero, occurring in the identity.*

It is obvious that the "extension" of a principal minor of one determinant is always a principal minor of the new determinant. Consequently, by "extending" each principal minor in the expression for the determinant of order four we can obtain an expression for a determinant of any desired order in terms of its principal minors. The necessity of keeping the expression homogeneous requires the introduction of the principal minor of maximum order consisting of new elements only. We thus have proved our main theorem:

Expressions for the general determinant of order $n>4$ in terms of fourteen of its principal minors follow immediately from the expressions for the determinant of order four in terms of thirteen of its minors.

The expressions thus obtained involve one principal minor of order $n-4$, four of order $n-3$, six of order $n-2$, and three of order $n-1$.

It is obvious that other expressions for the general determinant in terms of its principal minors may be obtained by replacing the principal minors of order ofur or greater involved in the above expressions by their equivalents in terms

[1] It is interesting to note that there are in general eight distinct determinants of order four whose principal minors (1) have the same values. This becomes evident when we observe that equations (2) contain three pairs of simultaneous equations, each pair consisting of a linear and a quadratic equation. Similarly, there are eight minors (A_{234}) determined by a given set of minors (1). The elements of A are uniquely determined in general when in addition to the values of (1) there is known a particular one of the eight associated values of (A_{234}).

[2] Muir, Transactions of the Royal Society of Edinburgh, vol. 30, pp. 1–4.

of their principal minors of lower order. A continuation of the process gives expressions for the original determinant in terms of its principal minors of order three or less. Of course, not more than 2^n-n^2+n-2 of the relations obtained in this way can be independent.

It is obvious from the well-known relations among the minors of a determinant and its adjoint that another relation involving only the determinant and its principal minors follows immediately from any one of the expressions obtained as indicated above if we replace each principal minor which appears by its cofactor, provided the expression is again made homogeneous by means of the determinant itself.

A THEOREM ON ALGEBRAIC QUADRATIC FORMS AND ITS APPLICATION IN THE GENERAL THEORY OF RELATIVITY

By P. Y. CHOU, California Institute of Technology

In the general theory of relativity the value of $\cos\theta$, where

$$\cos\theta = g_{\alpha\beta}\frac{dx_1^\alpha}{ds_1}\frac{dx_2^\beta}{ds_2}, \qquad ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \tag{1}$$

and θ is defined to be the angle between the tangents of the two curves s_1 and s_2 intersecting at a point (x^1, x^2, x^3, x^4) in the 4-dimensional space S_4, is ambiguous. The like holds for the extremals obtained from

$$\delta\int ds = 0. \tag{2}$$

These extremals may be either maxima or minima. The indeterminateness arises in each case from the fact that the fundamental quadratic form ds^2 is indefinite. It is usually proved in books on Riemannian Geometry[1] that $\cos\theta$ defined in (1) is less than unity provided the fundamental quadratic form ds^2 is positive definite. However, under the relativity hypothesis that any infinitely small space-time continuum can be always reduced to a Galilean space, we can show that $|\cos\theta| \geqq 1$ and that the geodesics of (2) are maximizing lines in S_4. In order to substantiate these two statements we have to establish the following theorem:

If for the real indefinite quadratic form $a_{\alpha\beta}x_\alpha x_\beta$ of rank n, signature $2-n$,

[1] L. P. Eisenhart, *Riemannian Geometry*, p. 37.

there are two sets of values of $x_i (i=1, \cdots, n)$, y_i *and* z_i *such that* $a_{\alpha\beta}y_\alpha y_\beta = A_1^2 > 0$, $a_{\alpha\beta}z_\alpha z_\beta = A_2^2 > 0$, *then the polar form,* $a_{\alpha\beta}y_\alpha z_\beta$, *satisfies the condition,*

$$|a_{\alpha\beta}y_\alpha z_\beta| \geqq |A_1 A_2|. \tag{3}$$

Since $a_{\alpha\beta}x_\alpha x_\beta$ has rank n and signature $2-n$, by a real non-singular linear transformation of x_i,

$$x_i' = c_{i\alpha}x_\alpha,$$

where the determinant

$$|c_{ik}| \neq 0, \tag{4}$$

we obtain[1]

$$a_{\alpha\beta}x_\alpha x_\beta = x_1'^2 - x_2'^2 - x_3'^2 - \cdots - x_n'^2.$$

Therefore

$$a_{\alpha\beta}y_\alpha y_\beta = y_1'^2 - y_2'^2 - \cdots - y_n'^2 = A_1^2;$$

$$a_{\alpha\beta}z_\alpha z_\beta = z_1'^2 - z_2'^2 - \cdots - z_n'^2 = A_2^2.$$

Unless y_i and z_i are all zero, which case we exclude, y_1' and z_1' can not vanish. Hence

$$1 = \frac{A_1^2}{y_1'^2} + \frac{y_2'^2}{y_1'^2} + \cdots + \frac{y_n'^2}{y_1'^2}; \qquad 1 = \frac{A_2^2}{z_1'^2} + \frac{z_2'^2}{z_1'^2} + \cdots + \frac{z_n'^2}{z_1'^2}. \tag{5}$$

We are dealing with real quantities, so the following relation is evident:

$$\left\{\frac{|A_1|}{|y_1'|} - \frac{|A_2|}{|z_1'|}\right\}^2 + \left\{\frac{|y_2'|}{|y_1'|} - \frac{|z_2'|}{|z_1'|}\right\}^2 + \cdots + \left\{\frac{|y_n'|}{|y_1'|} - \frac{|z_n'|}{|z_1'|}\right\}^2 \geqq 0. \tag{6}$$

By (5), (6) becomes

$$1 - \frac{1}{|y_1'z_1'|}[|A_1A_2| + |y_2'z_2'| + |y_3'z_3'| + \cdots + |y_n'z_n'|] \geqq 0,$$

and therefore

$$|y_1'z_1' - (y_2'z_2' + \cdots + y_n'z_n')| \geqq |y_1'z_1'| - |y_2'z_2'| - \cdots - |y_n'z_n'| \geqq |A_1A_2|. \tag{7}$$

By the inverse transformation from x_i' to x_i on the left hand side of (7), we obtain our final result,[2]

$$|y_1'z_1' - (y_2'z_2' + \cdots + y_n'z_n')| = |a_{\alpha\beta}y_\alpha z_\beta| \geqq |A_1A_2|. \tag{8}$$

[1] M. Bôcher, *Introduction to Higher Algebra*, p. 148.

[2] Ibid., p. 128.

In the definition (1) of $\cos\theta$, the rank of the quadratic form $g_{\alpha\beta}(dx^\alpha/ds)\,(dx^\beta/ds)$ is 4 and the signature is -2; and A_1 and A_2 are both unity. By (8), $|\cos\theta|\geqq 1$. Hence in order that θ should be real we may adopt the following definition of the angle θ between two intersecting curves in S_4,

$$\cosh\theta = \pm\, g_{\alpha\beta}(dx_1^\alpha/ds_1)(dx_2^\beta/ds_2).$$

It follows that for two intersecting time-like world lines, i.e., world lines for which $ds^2>0$, the condition of orthogonality can never be satisfied.

The Weierstrass' E-function for the integral

$$(9)\qquad I = \int_{t_1}^{t_2} f\left(x,\frac{dx}{dt}\right)dt, \qquad f\left(x,\frac{dx}{dt}\right) = \left\{g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right\}^{1/2} > 0,$$

where for convenience's sake we write x, dx/dt for x^i, dx^i/dt $(i=1,\cdots,4)$, respectively, in the argument of f, and where t is the independent parameter which can eventually be put equal to s, defined by $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$, is given by

$$(10)\qquad E(x,\dot{x},\dot{y}) = f(x,\dot{y}) - f(x,\dot{x}) - (\dot{y}^\delta - \dot{x}^\delta)\frac{\partial}{\partial\dot{x}^\delta}f(x,\dot{x})$$

$$= -\frac{1}{f(x,\dot{x})}(\dot{y}^\delta - \dot{x}^\delta)g_{\delta\beta}\dot{x}^\beta = -\frac{1}{f(x,\dot{x})}(g_{\delta\beta}\dot{x}^\beta\dot{y}^\delta - 1),$$

where $\dot{x}^i=dx^i/ds$ and the direction, $\dot{y}^i$, is sufficiently close to the direction $(\dot{x}^1,\cdots,\dot{x}^4)\neq(0,\cdots,0)$. A necessary condition for the extremal

$$(11)\qquad x^i = x^i(t),\quad (i=1,\cdots,4);\quad x_1^i = x_1^i(t_1),\quad x_2^i = x_2^i(t_2)$$

of (9) to be a maximizing line is[1]

$$(12)\qquad E(x,\dot{x},\dot{y}) \leqq 0.$$

By a slight modification the above condition also insures that (11) be a relatively strong or relatively weak maximum.[2] As the theorem stands in its present form in (3) it is not directly applicable to (10). Hence a modified proof of the theorem for the present purpose is necessary.

Replace $|y_i'|$ and $|z_i'|$ in (6) by y_i' and z_i' respectively. By (5) we obtain

$$(13)\qquad 1 - \frac{1}{y_1'z_1'}[\,|A_1A_2| + y_2'z_2' + y_3'z_3' + \cdots + y_n'z_n'\,] \geqq 0.$$

The transformation (4) is one-to-one and $x_1'=c_{1\alpha}x_\alpha$ is a continuous function of x_i $(i=1,\cdots,n)$. Moreover, $x_1'\neq 0$, unless $x_i=0$ $(i=1,\cdots,n)$. Hence for a set of values of x_i, z_i, which are in the closed neighborhood of the set

[1] G. A. Bliss, *Calculus of Variations*, p. 138. Replace the word "minimum" by "maximum."

[2] Ibid., p. 157.

$(y_1, \cdots, y_n) \neq (0, \cdots, 0)$ the signs of y_1' and z_1' can not be different because otherwise if we vary x_i from y_i to z_i $(i=1, \cdots, n)$, x_1' must vanish at least once, which is impossible. Consequently we may multiply (13) throughout by $y_1'z_1'$ and get

$$y_1'z_1' - (y_2'z_2' + y_3'z_3' + \cdots + y_n'z_n') \geqq |A_1A_2| .$$

By the inverse transformation on this, we obtain $a_{\alpha\beta}y_\alpha z_\beta \geqq |A_1A_2|$.

The above result is directly applicable to (10) in which both A_1 and A_2 are unity. Consequently Weierstrass' necessary condition (12) is established for the integral (9).

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. NOTE ON STEREOGRAPHIC PROJECTION

By THEODORE BENNETT, University of Illinois

Two fundamental theorems concerning stereographic projection state that (1) any circle on the stereographic sphere is projected into a circle on the stereographic plane, and conversely; and (2) the angle of intersection of two curves on the sphere is equal to the angle of intersection of their stereographic projections. The writer has examined a number of texts which treat this subject, and in all of them these two theorems are proved analytically. It is not difficult to prove them without algebraic assistance, using only the standard facts concerning planes and spheres which one learns in a first course in solid geometry; but such a proof does not seem particularly elegant. It is the purpose of this note to prove these theorems by the methods of synthetic projective geometry. The interest is therefore in the method used rather than the results established.

Since the theorems are of a metric character we shall first consider the relationship between projective and Euclidean geometry. Euclidean geometry in the plane is a specialization of projective geometry characterized by a special fixed line (the line at infinity) and two fixed points on that line. These fixed points, I and J, are called the circular points at infinity. Every circle in the plane passes through I and J, and every conic which passes through these points is a circle. The lines CI and CJ, which join I and J to a finite point C,

are called isotropic lines. Notwithstanding the fact that these lines are imaginary, they play a fundamental role in Euclidean geometry; and they are of particular importance when we are specializing a theorem of projective geometry to obtain a corresponding theorem of Euclidean geometry. These lines may be used to find the angle between two given lines. If the given lines are CA and CB, they form with CI and CJ a pencil which has a definite cross ratio $(C-ABIJ)$; then

$$\angle ACB = \tfrac{1}{2}i \log (C - ABIJ), \quad \text{where } i = \sqrt{(-1)}.$$

This formula, which is called the formula of Laguerre,[1] will be used later.

Euclidean geometry of three dimensions is a specialization of projective geometry characterized by a special fixed plane (the plane at infinity) and a conic in that plane which is called the absolute[2] or sphero-circle at infinity. Every sphere passes through the absolute, and every quadric which contains it is a sphere. Any finite plane meets the absolute in two points which are the circular points at infinity in that plane.

Finally, let us recall a few facts concerning quadric surfaces. Any non-singular quadric has two sets of rectilinear generators; we call each set a regulus. No two lines of the same regulus intersect, but any line of one regulus meets every line of the other. Through any point of the quadric there is one line of each regulus. The hyperboloid of one sheet and the hyperbolic paraboloid have real generators; other types of non-singular quadrics have imaginary generators. Any plane section of a quadric is a conic; if the plane is tangent to the quadric the conic degenerates into the two generators which pass through the point of tangency. In particular, the generators of a sphere are isotropic lines, and any tangent plane cuts the sphere in the two isotropic lines (in the tangent plane) which pass through the point of tangency. Two quadrics intersect in a twisted curve of order four, which sometimes degenerates into two or more curves of lower order. For example, if two quadrics have two straight lines in common the residual intersection is a conic, which must be a plane section.

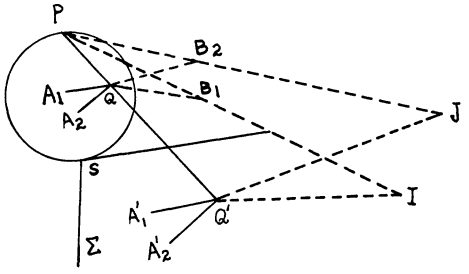
Let us now consider the stereographic sphere, with P and S two diametrically opposite points. From P we project Q, any point on the sphere, into Q', on the plane Σ, which touches the sphere at S. If Π is the plane tangent to the sphere at P, we know that Π and Σ have a common line at infinity, and therefore they have the same circular points, I and J, in which their common line at infinity meets the absolute.

[1] Pascal, *Repertorium*, vol. 2, p. 28.

[2] Snyder and Sisam, *Analytic Geometry of Space*, p. 52.

Any circle on the sphere is cut out by a plane, p, which we assume does not pass through P. The lines PI and PJ lie entirely on the sphere, since they are the generators through P. Therefore the points in which they pierce the plane p are two points of the circle in which p meets the sphere. Moreover, when these points are projected upon Σ (from P) they fall at I and J. Therefore the projection of the circle is a conic (on Σ) passing through the circular points I and J, whence it is a circle.

Conversely, if P is joined to every point of an arbitrary circle in Σ, we have a quadric cone, K, which has PI and PJ for two of its generators, since the circle must pass through I and J. But PI and PJ lie on the sphere, and this line pair forms part of the intersection of K with the sphere. The remaining part is a conic, which must be a plane section of the sphere. Since all plane sections of a sphere are circles, we see that any circle on Σ is the projection of a circle on the sphere.



If the plane p passes through P, the projection of the circle cut out by p is composite. The points of the circle other than P are projected into the points of the straight line in which p and Σ intersect. The point P is projected into the whole line at infinity in Σ. Since the line at infinity contains the circular points, this projection forms a degenerate circle. This completes the proof of the first theorem stated at the beginning of this note.

It may be well to state here that a single finite line and a circle are not equivalent from the standpoint of projective geometry, although they are equivalent from the standpoint of inversive geometry (the geometry associated with the function theory in the complex plane). The reason is that in one case the infinite region of the plane is defined to be a line, and in the other case a single point.

We now prove the second fundamental theorem. Let the curves on the sphere meet at Q, with tangents QA_1 and QA_2. Since a projection sends a curve and one of its tangent lines into a curve and a corresponding tangent line, the curves which meet at Q are sent into two curves which meet at Q', with tangents $Q'A_1'$ and $Q'A_2'$, where $Q'A_1'$ is the projection (from P) of QA_1, and similarly for $Q'A_2'$. The plane QA_1A_2 touches the sphere at Q, and therefore cuts out

the two generators through Q, namely the isotropic lines QB_1 and QB_2. The isotropic lines PI and PJ are the generators through P, and of these two lines one meets QB_1 and the other meets QB_2. Therefore the line pencils $(Q-A_1A_2B_1B_2)$ and $(Q'-A_1'A_2'IJ)$ are projective, whence their cross ratios are equal. But in the plane QA_1A_2 the lines QB_1 and QB_2 are the two isotropic lines through Q, and similarly, in Σ, $Q'I$ and $Q'J$ are the two isotropic lines through Q'. Therefore, from the formula of Laguerre, the angles A_1QA_2 and $A_1'Q'A_2'$ are proportional to the logarithms of the cross ratios of the corresponding line pencils. Hence the equality of the angles follows from the equality of the cross ratios.

II. Some Geometric Illustrations for the Elementary Course in Differential Equations

By J. D. Tamarkin, Brown University

1. The equation linear in y and y':

$$\frac{dy}{dx} + p(x)y + q(x) = 0. \tag{1}$$

The family of integral curves of (1) possesses a curious property as follows:

(*A*) *Draw the line $x=\xi$ and at the points of intersection of this line with the integral curves of (1) draw tangents to all integral curves. All these tangents either pass through the same point (which depends on ξ), or they are parallel.*

The proof of (*A*) is based upon another property of the integral curves of (1), which is well known:

(*B*) *If we draw any two particular integral curves (C_1), (C_2) of (1), then the general integral curve of (1) is the locus of points which divide in a constant ratio the segments of vertical lines ($x=\xi$; ξ is a parameter) intercepted by (C_1) and (C_2).*

This property follows immediately from the fact that if y_1 and y_2 are any two particular solutions of (1), then the general solution y of (1) is given by

$$y = y_1 + C(y_2 - y_1),$$

where C is an arbitrary constant.

Now to prove (*A*), draw any three integral curves of (1)

$$(C_1)\ y = y_1(x)\ ;\quad (C_2)\ y = y_2(x)\ ;\quad (C_3)\ y = y_3(x).$$

Let P_i, P_i' $(i=1, 2, 3)$ be the points of intersection of (C_i) with the vertical lines $x=\xi$ and $x=\xi'$ respectively. It follows from (*B*) that

$$\overline{P_1P_2}/\overline{P_2P_3} = \overline{P_1'P_2'}/\overline{P_2'P_3'}\ ;$$

hence the chords $P_i P_i' (i=1, 2, 3)$ either are concurrent or else they are parallel. If we make $\xi' \to \xi$, the chord $P_i P_i'$ approaches its limiting position, that is the tangent to (C_i) at P_i. Hence the tangents to (C_i) $(i=1, 2, 3)$ at the points P_i are either concurrent or else parallel, which proves (A).

It is readily seen that the tangents at the points P_i are parallel when and only when the slopes of the integral curves of (1) for $x=\xi$ do not depend on the ordinate y of the curve, that is when the coefficient $p(\xi)$ of y in (1) is zero for this particular value $x=\xi$.

Denote by $T_\xi(x, y)$ the point of concurrency of the tangents to the integral curves of (1) at their points of intersection with the vertical $x=\xi$. As ξ varies the point T_ξ describes a certain locus which may be of great use for the graphical integration of (1). To obtain the equation of the locus of T_ξ, write the equation of the tangent to the integral curve of (1) at the point (ξ, y):

$$Y - y = -[p(\xi)y + q(\xi)](X - \xi).$$

Substitute any two particular values of y, as for instance, $y=0$ and $y=1$, and solve the resulting equations with respect to X and Y. This gives

$$X = \xi + [1/p(\xi)]; \quad Y = -\frac{q(\xi)}{p(\xi)}.$$

2. The equation linear in x and y:

$$y = x\phi(y') + \psi(y'). \tag{2}$$

If we are given any differential equation of the first order

$$F(x, y, y') = 0, \tag{3}$$

the notion of the "curves of equal slope" of (3) is very important in the discussion of the geometric properties of the integral curves. The curve of equal slope (equal y_0') is the locus of the points at which the tangents to the integral curves have the same slope y_0'.

The family of these curves of equal slope is obtained immediately from (3) if we consider y' as a parameter which has a fixed value for any particular curve of the family.

In the case of equation (2), linear in x and y, the curves of equal slope are straight lines. The tangents to all the integral curves of (2) drawn at their points of intersection with any line

$$y = x\phi(y_0') + \psi(y_0')$$

are parallel (with the slope y_0').

Now, the integral curves of (2) have the property analogous to (*B*), the only difference being that the "segments of the vertical lines" should be replaced by the "segments of the lines of equal slope."

(*C*) *If we draw any two particular integral curves* (C_1), (C_2) *of* (2), *then the general integral curve of* (2) *is the locus of points which divide in a constant ratio the segments of the lines of equal slope intercepted by* (C_1) *and* (C_2).

To prove (*C*) we have to remember that equation (2) "is integrated by differentiation." Take differentials and observe that $dy = y'dx$:

$$y'dx = dx\phi(y') + x\phi'(y')dy' + \psi(y')dy'. \tag{4}$$

This is a linear equation in x and dx/dy', which can be solved by quadratures, giving x as a function of y' and of an arbitrary constant, C. On substituting in (2) we get y in terms of y' and C. If we consider y' as a parameter we obtain, for each fixed value of C, a particular integral curve of (2). Take any two particular integral curves (C_1) and (C_2) which correspond to any two particular values C_1 and C_2 of C. Let

$$x = x_i(y'), \quad y = y_i(y') = x_i\phi(y') + \psi(y'),$$

be the resulting equations of (C_i) $(i=1, 2)$. Equation (4), being linear in x and dx/dy', has the general solution $x = x_1 + C(x_2 - x_1)$. Accordingly the equations of the general integral curve of (2) become

$$x = x_1 + C(x_2 - x_1) \; ; \quad y = y_1 + C(y_2 - y_1)$$

which shows that the segment P_1P_2 joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is divided by the point $P(x, y)$ in a constant ratio independent of y'. Thus (*C*) is proved.[1]

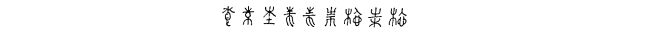
III. The Origin of the Term "Root" in Chinese Mathematics

By C. C. Ma, Columbia University

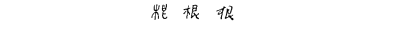
In the *Shih Wu Chi Yuan*, a historical work written by Kao Cheng in the year 1448, the invention of writing, in the proper sense of the term, is attributed to Tsang Chieh who is said to have lived in the twenty-eighth century, B.C. Tradition states that he devised his characters by imitating the footprints of birds, which is quite as reasonable a theory as several that have been ad-

[1] For many years I used to give the properties (*A*) and (*C*) in the elementary course in differential equations. As far as I am aware, these properties are not found in any of the standard textbooks on differential equations, except in the recent book by E. L. Ince, *Ordinary Differential Equations* (Longmans, Green Co., London, 1927), which contains an incomplete formulation of property (*A*); the case where the tangents are parallel is omitted (p. 15, problem 9).

vanced concerning the origin of Western alphabets.[1] Whether or not the Chinese theory is a mere fable, the fact remains that the better established history of the development of such characters as the one representing a mathematical root is filled with interest. The older ideograms have been studied with much care by Fu Ying Kei in his work entitled *Liu Shu Fen Lei*, published in 1765. In this he gives the early forms of the word which we translate as "root" as follows:



In modern Chinese works the word appears in several forms, among them the following:



It is pronounced *kun* and means the lower part of a plant, that part which fixes itself in the earth and draws nutriment from the soil. Thus we speak of the roots of grass, or the root of a tree, and hence metaphorically of the bottom of any object; of the root of a mountain; of the root, origin, cause of anything; of the root or source of virtue; of an ancestor (the root of a family tree); of the fundamental note of a chord; or of the root of bitterness, error, sin, or evil. In our mathematical works it always refers to square root or cube root,—that is, to the basic number from which we obtain special products (squares or cubes).

Although the form of the ideogram has changed from time to time, the fundamental meaning of the word *kun* has remained the same. It is because of this fact that the writer was surprised at reading, in Dr. Gandz's interesting article *On the origin of the term root*,[2] the statement, "The Chinese, indeed, do use the word *kun* to mean root, grass, and shrub, and the Hindus also use the word *mula* for the root of a plant, but *this is very likely due to the Arabic influence*, which is so often seen in China and which may have spread into India by way of China."

It is true that in the records of our Tang Dynasty (618–907) there begins to appear the evidence of Arabic influence, but before that period we had fully thirty centuries of culture. In this extended period we developed our own type of mathematics and our own terminology. The word *kun* was familiar to thousands of our scholars and appears in many of our early books, including the Book of Rites, the Book of Poetry, the Book of History and others equally well-known in the East. Not only does it appear, but it has always the same meanings, and hence it is historically impossible that we should have derived it from the relatively late and transitory civilization of the Arabs.[3]

[1] On a new theory of the origin of all alphabets, see I. W. D. Hackh, *The history of the alphabet*, Scientific Monthly, August, 1927.

[2] In this Monthly, vol. 33 (1926), pp. 261–265.

[3] The author wishes to express his thanks to Professor David Eugene Smith for assistance in preparing this note for publication.

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EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

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SPEISER, A. *Theorie der Gruppen von endlicher Ordnung.* Zweite Auflage. Berlin, Julius Springer, 1927. ix+251 pages. 15 marks.

VEBLEN, OSWALD. *Invariants of Quadratic Differential Forms.* (Cambridge Tracts, Number 24) Cambridge, University Press, 1927. viii+102 pages.

WELLS, W. and HART, W. W. *Modern Plane and Solid Geometry.* New York, D. C. Heath and Company, 1927. x+480 pages. $1.64.

REVIEWS

Le Calcul des Probabilités, son Évolution Mathématique et Philosophique. By L. G. DU PASQUIER. Paris, J. Hermann, 1926. xxi+304 pp.

The nine chapter headings with the principal topics discussed in each are: *Historical Introduction.* The three currents in the evolution of probability. The invention of the calculus of probability. The mathematical current. The sociological and stocastical currents. The philosophical current. *Elements of the Calculus of Probability.* The classical definition of mathematical probability. Fundamental theorems. Combinatory analysis. Stirling's formula. *The Series of Repeated Trials.* The empirical law of chance or the law of great numbers. Various concepts of discrepancy. Repeated trials and Newton's binomial theorem. The first part of Bernoulli's theorem. *Bernoulli's Theorem and the Theory of Discrepancies.* The mathematical probability that the fluctuation of a series of repeated trials takes a fixed value. The mathematical probability that the discrepancy of a series of repeated trials lies between fixed limits. Standard deviation. Equiprobable limits. Formulae of chance. Precision. Dispersion. Mean value and mathematical expectation. *Various Interpretations of the Calculus of Probability.* The psychological, practical, logical, empirical,

and inductive interpretations. The principle of zones of comportment. *Mathematical Probability Based on the Theory of Assemblages. The Logical Definition of Probability.* Mathematical probability and the theory of assemblages. Paradoxes of the calculus of probability. Mathematical probability and ignorance. Mathematical probability and the principle of zones of comportment. Mathematical probability and statistical regularity. *Applications to Physics.* The introduction of the calculus of probability in physical chemistry. Review of the kinetic theory of gases. Distribution of molecular velocities. The law of Maxwell-Boltzmann. Reversible phenomena and the principle of Carnot. Statistical determinism and fluctuations. Other applications to physics. *Mathematical Probability Based on the Theory of Aggregates. The Stocastical Definition of Probability.* The empirical aggregate. The mathematical aggregate. A new definition of mathematical probability. Four simple operations for deducing new mathematical aggregates. *Mathematical Probability and Experience.* Two fundamental postulates and experience. The postulate of fortuity. Logical systems and experience. The appendix contains two tables of values for the probability function.

This detailed analysis shows us several new topics in the subject. The author has a viewpoint which is more mathematical than that of Keynes but more philosophical than those of Arne Fisher, Coolidge, Borel, Poincaré, or Bachelier.

Stocastics is defined to be the science which applies the theorems of the calculus of probability to numbers collected by statistics in such a way as to show the existences of real causes whose action combines with that of fortuitous causes (p. 23). A more formal definition is that given in chapter VIII, which also illustrates the use of other terms peculiar to the author. "Definition 5.—An assemblage of objects or things, which are included in a general notion and which are arranged in the order of a measurable or numerically expressible character, forms an aggregate or *syllepte.* It is characterized by its *law of distribution, frequency function,* or *diadose.* The study of these aggregates from the point of view of the calculus of probabilities constitutes stocastics."

F. A. FORAKER

College Algebra. By W. B. FORD. Revised edition. New York, The Macmillan Co., 1926. vii+288 pages.

This book follows the lead of most college algebras which have appeared in the last few years, in that it has included more review material and has excluded some of the more advanced topics, such as partial fractions and limits and series. Also most of the exercises of an involved character have been replaced by simpler ones. Some teachers will probably have a feeling that such a procedure is not entirely justified.

The first forty-two pages are essentially a review. This is followed by the usual topics included in such a text. The following are a few features of the development of these topics. The chapter on "Variation" is treated more fully than usual and a large number of problems from physics is inserted. The derivative is defined and used to find maximum and minimum values of a function. With this definition at hand one wonders why Newton's method of approximation was not included in the chapter on the theory of equations. On the other hand, most readers will be pleased to find Sylvester's method of elimination and the definition of eliminant included in the chapter on determinants.

The book is well written. The explanations are clear, and all principles are illustrated by examples carefully worked out. One ambiguity should be noted. On pages six and forty-eight the word "cancel" is used. On page six it means divide and on page forty-eight it means subtract. This double meaning of the word is confusing to the student. Hence it might be better to replace the word "cancel" by the word which suggests the process involved.

Very few typographical errors occur, and the general make-up of the pages is good.

J. H. WEAVER

Charts and Graphs. By K. G. KARSTEN, New York, Prentice-Hall, Inc., 1925. xl+734 pages. Price $6.00.

Graphical methods have been found extremely useful for at least three general purposes: (1) the correlation and simplification of vast amounts of historical, economic, experimental or any other kind of data, (2) the use of these data to predict development beyond present knowledge, (3) the preparation in graphical form of any kind of information so that it may be more easily understood by the general public or any special group of people.

The increasing use and universal appeal of graphical methods create a need for a book describing the evolution, the uses and abuses of all kinds of graphs. This need has been met by the author, who sets as his object the telling of the whole story of charts and graphs in the simplest words at his command. The result is a book of 734 pages with 498 illustrations, well indexed and written in an interesting, non-technical style. The small amount of necessary mathematics has been thoroughly explained so that it can be understood without special training. The general headings are:

I. Non-mathematical Charts (4 chapters); II. Amount-of-change Analysis (26 chapters); III. Rate-of-change Analysis (8 chapters); IV. Special Analyses (such as curves of error, probability curves and curve fitting—6 chapters); V. Calculating Charts (such as curves for formulae, slide rules and alignment charts or nomographs—5 chapters); VI. Two- and Three-dimension Data

(such as models, surfaces and relief maps—7 chapters); VII. Conclusion (statistical materials and function of charts—2 chapters).

In reading this book the reviewer was favorably impressed with the wealth of illustration covering every type of chart, graph, and diagram and also with the complete description of the construction and use that accompanies each illustration.

The author has provided a liberally illustrated textbook for those whose business it is to collect, study, and disperse statistical information. At the same time he has prepared a reference book for persons, such as business executives, engineers, and experimentalists, who can use to their own advantage a greater variety of charts and graphs than they do at present.

HUBERT H. RACE

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE NAPERIAN CLUB OF DEPAUW UNIVERSITY, Greencastle, Indiana.

The officers for the year 1926–1927 were: President, William Nusbaum, '27; vice-president, Ruth Quebbeman, '27; secretary, Josephine Rode, '27; treasurer, Dever Colson.

The following programs were given at the regular monthly meetings:

November 11, 1926. "The life and work of John Napier" by Boyd Crawley, '27.

January 13, 1927. "Theorem of the nine point circle" by Charles La Hue, '28.

February 8. "Number Systems" by Professor Greenleaf. "Geometric fallacies" by Ruth Quebbeman, '27.

March 10. "Acoustics in buildings" by Dever Colson, '27.

April 21. "The teaching of mathematics for the last three quarters of the present century" by Helen Huhn, '28.

On April 28 and 29 the Indiana section meeting of the Mathematical Association of America was held at Depauw University. The Naperian Club was very much interested in helping obtain Professor Kunz of the University of Illinois as one of the speakers.

(Report by Josephine Rode)

THE NEWTONIAN SOCIETY OF THE STATE COLLEGE OF WASHINGTON, Pullman, Washington.

President, Miss Florence Johnson; secretary, Miss Arlene Perry; reporter, Mrs. Opal Priest.

November 3, 1926. Prof. C. A. Isaacs—"Numbers".

November 17. Mr. Schulkenoff—"Null spaces."

December 15. Dorothy Webster—"Transfinite numbers." Arlene Perry—"Zeno's paradoxes."

December 29. Mrs. Mills—"Euclidean and Riemannian geometry."

February 10, 1927. Mr. Calogaris—"A recent trip to Greece."

February 24. Miss Swannack—"Elliptic geometry." Mr. Schulkenoff—"Mathematical worlds."

March 3. Miss Verna Betz—"History of determinants." Prof. Isaacs—"Some determinants connected with the triangle and the tetrahedron."

March 10. Miss Davis—"Evolution of the solar system." Mr. Calogaris—"Laplacian law of densities."

March 17. Miss Pell—"Solving a cubic equation." Prof. Irwin—"Approximation curves."

March 29. Miss Freeman—"Approximation curves." Prof. Colpitts—"The students' loan association."
April 14. Prof. Hutchinson—"The application of calculus to physics."

(Report by Professor C. A. Isaacs)

THE MATHEMATICS CLUB OF WELLESLEY COLLEGE, Wellesley, Massachusetts.

The Mathematics Club of Wellesley College made some interesting studies in the year 1926–1927:

October 22, 1926. Discussion and review by members of books available on the Mathematics Browsing Shelf in the College Library.

November 19. Discussion of topics for which the year 1926 might be considered an anniversary: The work of Eratosthenes, Edmund Gunter, Albert Girard, Wilbar Snell. One member spoke on mathematics as it was first introduced into colleges.

January 7, 1927. Meeting in the treasure room of the College Library. Prof. Lennie P. Copeland pointed out certain unusual details in the old mathematical volumes kept there.

February 25. "Problems of antiquity and their solution:" (1) "Geometric representation of the roots" by Elizabeth Mitchell; (2) "Trisection of an angle" by Frances Baume; (3) "Duplication of a cube" by Elizabeth Peek; (4) "Spirals" by Doris Miller.

March 11. "The Geometry of imaginaries" by Dr. William C. Graustein of Harvard University.

May 10. The department of mathematics entertained the club.

May 28. Club dinner and election of officers for year 1927–1928: President, Leona Bailey, '28; vice president, Elizabeth Peek, '28; treasurer, Mary Parsons, '28; secretary, Doris Raine, '29; Junior executive, Esther Kirkbride, '29; faculty advisor, Prof. Marion E. Stark.

(Report by Frances A. Hartman)

THE GORDON N. ARMSTRONG MATHEMATICS CLUB OF OHIO WESLEYAN UNIVERSITY, Delaware, Ohio.

The program of the club for the year 1926–1927 was as follows:

October 13, 1926. "Irrational numbers by Prof. Rowland.

November 10. "Maclaurin's theorem" by Mr. Foote.

December 8. Social meeting.

January 19, 1927. "The Circle" by Mr. McCullough.

On April 16, 1927 the Beta Chapter of Ohio of Pi Mu Epsilon was installed at Ohio Wesleyan University. Meetings are held bi-monthly and one meeting each month is held in conjunction with the Gordon N. Armstrong Mathematics Club.

The Pi Mu Epsilon program for the remainder of the year was as follows:

April 27, 1927. "Integration by imaginary substitution" by Miss Leonard.

May 11. "Tricks in mathematics" by Prof. Crane.

May 25. "The relation of mathematics to psychology" by Mr. Valentine.

(Report by Edward T. Prosser)

PI MU EPSILON, University of Kentucky, Lexington, Kentucky.

A chapter of Pi Mu Epsilon was established at the University of Kentucky this year with the following officers: Director, Dean P. P. Boyd; vice-director, Professor J. Morton Davis; secretary, Professor F. E. LeStourgeon; treasurer, Mr. E. J. Canaday; librarian, Professor E. L. Rees. The White Mathematics Club was continued with Dean P. P. Boyd as president, and Professor F. E. LeStourgeon as secretary. Meetings of the two organizations alternated during the year. The combined program was as follows:

December 2, 1926. "Trisection of an angle" by Professor E. L. Rees.

January 20, 1927. "Double tangents of quartics" by Dean P. P. Boyd.

February 10. "Line, surface, and space integrals" by D. E. South, Instructor.

February 24. "Groups of substitutions" by Professor F. E. LeStourgeon.

March 10. "Elliptic functions" by Mr. D. O. Streyffeler, Instructor. "Application of mathematics to architecture" by Mr. E. J. Canaday, Instructor.
March 24. "Sir Isaac Newton" by Mr. W. W. Chambers, Instructor. "Certain mathematical equations in the theory of gases" by Mr. Grant Willey, '27.
April 28. "Coupled oscillations" by Mr. R. B. Scott, Graduate. "The path of an α-particle through an atom of a heavy metal" by Mr. Madison Cawein, Graduate.
May 30. "Pedal curves" by Mr. M. C. Brown, Instructor.

(Report by Professor F. E. LeStourgeon)

The Mathematics Club of Cooper Union, New York City.

The officers for the year 1926–1927 were: Edwin T. Schwarz, '27, president; Henry Wilhelm, '27, vice-president; Fred Mertens, '29, secretary-treasurer. Meetings were held at intervals of two or three weeks throughout the year, with the following program:
November 9, 1926. "Applications of probability to automatic telephony" by William H. Spahn of the Bell Telephone Laboratories.
December 1. "Tricks on the slide-rule" by Edwin T. Schwarz, '27.
December 22. "Integral right triangles" by A. H. Beiler of the Interboro Rapid Transit Company.
January 12, 1927. "Unicursal curves" by Henry Wilhelm, '27.
January 26. "Summation of algebraic series" by Samuel Lubkin, '27.
February 16. "π by chance" by Henry Eckhardt, '27.
March 9. "Mental calculations, with an exposition of methods" by Abraham Rosenbaum, '28.
March 26. Joint meeting of the Cooper Union Student branches of the A. S. M. E., A. I. E. E., A. S. C. E., the Mathematics Club, and the Chemistry Club. "How to become a successful engineer" by Farley Osgood, past president of the A. I. E. E.
April 6. "Paradoxes of infinity" by Walter Prenowitz, Instructor.
April 20. "A Photographic representation of a crinkly curve" by Edwin T. Schwarz, '27. At this meeting the following officers were elected for the coming year: Peter Douglas, '28, president; James McConaghy, '30, vice-president; Edward Trapani, '29, secretary-treasurer.

(Report by Professor H. W. Reddick)

The Undergraduate Mathematics Club of the University of Iowa, Iowa City, Iowa.

The officers of the Undergraduate Mathematics Club of Iowa University for the year 1926–27 were: Faculty advisor, Professor Roscoe Woods; president, Mr. C. R. Wilson, graduate student; secretary-treasurer, Miss Eva E. Latta, '28.

The following programs were given at the club meetings during the year:
November 4, 1926. "Paper folding" by Professor R. P. Baker.
November 18. "The nine-point circle" by Miss Ruth Balluff, M. S.
December 2. "Number systems" by Mr. A. H. Blue, M. S.
January 13, 1927. "Logarithmic graphs" by Mr. C. S. Carlson, Graduate.
February 3. "Algebra of points" by Mr. C. A. Messick, M. S.
February 17. "Hyperbolic functions" by Mr. W. M. Davis, Graduate.
March 3. "The slide rule" by Mr. Harry Rice, Graduate.
March 31. "Some unusual numbers" by Miss Phebe Williams, '28.
May 5. "Measurement of distances in astronomy" by Mr. Walter Long, '29.
May 19, 1927. "Transcendentalism of e" by Dr. N. B. Conkwright.

At the last meeting, the following officers were elected for the year 1927–28: Faculty advisor, Dr. N. B. Conkwright; president, Mr. H. A. Meyer, graduate; secretary-treasurer, Miss Dorothy McCoy, graduate.

The club meets on Thursday afternoons at four o'clock; tea is served each time before the beginning of the program.

(Report by Eva E. Latta)

THE MATHEMATICS CLUB OF MIAMI UNIVERSITY, Miami, Ohio.

November, 1926. "The attraction of spherical shells" by Mr. MacDougal. "Prime numbers" by Mrs. Reed.

December. "Squaring the circle" by Mr. DeLong. "DeMoivre's theorem" by Miss Hughes.

January, 1927. "The trisection of an angle" by Miss Runyon. "Curves used in surveying" by Mr. Halbedel. "The theory of the planimeter" by Mr. Armstrong.

February. "The Einstein theory" by Mr. Rumbaugh.

March. "The Pythagorean theorem" by Miss Young. "A graphical solution of the cubic equation" by Miss Williams. "Arithmetic progressions and interpolations" by Mr. Bulow.

April. "Central forces" by Miss Fisher. "The general number" by Miss Treichler. "Geometric interpretation of an imaginary angle" by Mr. Serviss.

May. "Logarithms of negative numbers" by Miss Zwick. "Nomography" by Mr. Tischer.

All of the talks were given by students of the various classes under the direction of the mathematics professors. Its main purpose was to bring to the students an opportunity of acquiring a knowledge of topics rarely discussed in the classroom. (Report by Harold F. Sielaff)

THE HARVARD MATHEMATICAL CLUB, Harvard University, Cambridge, Mass.

The officers for 1926–27 were: Mr. Robin Robinson, president; Mr. Morris Marden, secretary. Officers for 1927–28 are: Mr. Clarence I. Lubin, president; Mr. Griffith B. Price, secretary.

Meetings for 1926–27

October 20, 1926. "Riemann's theory of algebraic functions and functions related to them" by Professor W. F. Osgood.

November 3. "Some properties of skewsquares" by Mr. S. B. Sommerville.

November 17. "Some topics in the calculus of variations" by Mr. T. L. Smith.

December 1. "European mathematics of today" by Professor G. D. Birkhoff.

December 15. "Legendre's normal forms and some applications" by Mr. T. R. Long.

January 5, 1927. "Motor calculus" by Mr. C. I. Lubin.

January 19. "Multiple integrals with singular integrands dependent on a parameter" by Mr. A. B. Brown.

February 16. "Some mathematical ideas in electrical engineering" by Dr. Joseph Slepian, Research Consulting Engineer, Westinghouse Electric and Manufacturing Company.

March 2. "An application of theta functions in the theory of numbers" by Mr. C. N. Liu.

March 16. "A recent treatment of maxima and minima" by Mr. S. S. Cairns.

March 30. "The determination of analytic functions" by Professor Philip Franklin, Massachusetts Institute of Technology.

April 13. "The parallelism of Levi-Civita" by Mr. H. B. Hammatt.

April 27. "Some circles related to the triangle" by Mr. G. B. Price.

May 11. "Some properties of entire functions" by Mr. F. C. Jonah.

May 25. "Parallelism in Finsler space" by Dr. Harry Levy.

(Report by Robin Robinson)

SEVER MATHEMATICAL CLUB OF HARVARD UNIVERSITY, Cambridge, Mass.

The officers for 1926–27 were: Mr. Stuart B. Sommerville, president; Mr. Edward H. Cutler, secretary.

The officers for 1927–28 are: Mr. John J. Hinrichsen, president; Mr. Joseph K. Peterson, secretary.

Meetings for 1926–27

October 27, 1926. "The stability of floating bodies" by Mr. Robin Robinson.

November 10. "The structure of the world" by Mr. D. D. Kosambi.

December 8. "Diophantine analysis" by Mr. S. S. Cairns.

January 12, 1927. "Some solutions of the cubic and the biquadratic" by Mr. J. J. Hinrichsen.

February 9. "Sir Isaac Newton's pasturage problem" by Mr. R. M. Barclay.
February 23. "Some mathematical scandals" by Mr. Charles Wexler.
March 9. "Point sets" by Mr. W. S. Seidel.
March 28. "The four-color map problem" by Miss M. E. Steinmetz.
April 11. "The lateral area of an oblique cone with circular base" by Mr. C. W. T. Bunnell.
May 4. "Recurring series" by Mr. E. C. Berkeley.
May 18. "On the roots of polynomials" by Mr. Morris Marden.

(Report by S. B. Sommerville)

The Junior Mathematics Club of the University of Chicago, Chicago, Illinois

The officers for the year 1926–1927 were: R. G. Archibald, president; F. R. Bamforth, chairman of committee on program; May M. Beenken, chairman of committee on arrangements; A. O. Hickson, secretary and treasurer.

The officers for the year 1927–1928 are: B. W. Jones, president; May M. Beenken, chairman of committee on program; Ruth Mason, secretary and treasurer.

The program for the year 1926–1927 was as follows:
October 20, 1926. "Some interesting topics in algebra" by Miss L. W. Griffiths.
November 3. "Pythagorean theorems and Pythagorean triads" by Dr. J. S. Georges.
November 17. "The four color problem" by Dr. Orrin Frinck.
December 8. "Projective equivalents of trigonometric functions" by Miss M. M. Beenken.
January 5, 1927. "Introduction to the structure of algebras" by Mr. A. A. Albert.
January 19. "On some questions regarding non-euclidean geometries" by Miss Louise Lange, Ph. D.
February 2. "Popular probability problems" by Mr. C. E. Rhodes.
February 16. "Mathematical card tricks" by Mr. B. W. Jones.
March 2. "Magic squares" by Mr. R. A. Hefner.
March 30. "Some notes concerning the theory of relativity" by Miss Louise Lange, Ph. D.
April 9. Social evening at the home of Professor H. E. Slaught.
April 13. "The algebraic method of obtaining the derivative" by Mr. H. H. Downing.
April 27. "On the congruences of lines generated by a projective relationship between two planes" by Miss L. Hoffman, Ph. D.
May 11. "A theorem of Bertrand" by Mr. Lincoln LaPaz.
May 25. "Hyperbolic functions" by Mr. C. J. Rees.

(Report by Professor H. E. Slaught)

The Mathematics Club of Hunter College, New York City.

The officers for 1926–27 were Miss Lena Kalan, president; Miss Antoinette Finocchi, secretary; Miss Ruth Chatwin, treasurer; Dr. Roger Johnson, advisor. The following meetings were held during the year:
October 5, 1926. Business meeting.
October 17. Hike and picnic.
October 19. Party for freshmen.
October 26. "Sturm's theorem" by Professor Tomlinson Fort.
November 4. "The numbers of Fibonacci" by Dr. Roger Johnson.
November 14. "Projective ornament" by Mr. Claude Bragdon.
December 3. "Infinite series" by Professor W. B. Fite of Columbia University.
February 15, 1927. Business meeting.
March 8. Party.
March 15. "The cycloid" by Jean Hutchinson.
March 29. "Magic squares" by Josephine Stephenson.
April 5. "Comets" by Lewis D. Hill, Professor of Physics.
April 19. "The conchoid" by Lillian Glass.

(Report by Miss Finocchi)

The Mathematics Club of McKendree College, Lebanon, Illinois.

The Mathematics Club of McKendree College, Lebanon, Illinois, was organized in October, 1926, with twelve charter members. Students who finish the first semester of freshman mathematics with a good record are eligible to associate membership. Students who enroll for sophomore calculus are eligible to active membership. Meetings are held monthly. The programs for the school year 1926–27 were as follows:

November 4, 1926. Social meeting.

December 8. Topic: "Invariants of the general equation of the second degree in two variables." Papers by Mr. V. R. W. Hardy, Mr. Lorin Mitchell, Mr. Emery Martin, and Professor C. J. Stowell.

January 7, 1927. Topic: "The teaching of high school mathematics." Papers by Miss Emma Bergmann, Miss Verdie Correll, Mr. Eugene Smith and Professor C. E. Vick.

March 3. "Analytic geometry before Descartes" by James Stuart. "The contribution of Descartes to analytic geometry" by Mr. Clifton Gould. "The early development of calculus" by Mr. Charles Jack.

April 14. "Elements of the theory of probability" by Mr. Lee Baker. "The method of least squares" by Mr. J. Wendell Dunn.

May 10. Social meeting.

(Report by Professor Stowell)

The Mathematics Club of Hood College, Frederick, Maryland.

The officers for 1926–27 were: President, Elizabeth Slifer; vice President, Anna Lee Schaidt; Secretary, Grave Lough; Reporter, Dorothy Kerper.

October 12. 1926. Initiation of new members.

November 9. "A mathematical melange" by Dr. Apple.

December 14. "Blaise Pascal" by Elva Kurtz. "Anecdotes about famous mathematicians" by Ruth Anderson, Dorothy Dougherty, Katherine Kehm, Dorothy Kerper, Ethel Landgraff, Ethel Sauble.

January 11, 1927. "Trend of mathematics in secondary schools" by Prof. Remsberg.

February 15. "Development of the calendar:" "Early calendars" by Anna Lee Schaidt, "Julian and Augustan" by Irene Speicher, "New calendar" by Ethel Dallmeyer.

March 18. "Binary scales of notation" by Susan Hirsch. "Cardan's rings" by Katherine Woodfill.

April 12. "Map making" by Dorothy Bobb. "Newton" by Gladys Allison, Grace Lough. "Euclid" by Nellie Burrows. Election of officers.

April 28. Picnic.

May 10. "Mathematics and music" by Miss Wiggin.

At each meeting magazine reports from mathematical magazines found in the library were given by different members of the club.

(Report by Miss Lough)

The Mathematical Society of Rutgers University, New Brunswick, New Jersey.

The Mathematical Society at Rutgers University held regular monthly meetings during the year 1926–1927 at which the following papers were presented by members of the faculty and by the student members:

Prof. W. V. N. Garretson—"A new representation of the complex number."

Prof. E. P. Starke—"Applications of the complex variable."

Prof. S. E. Brasefield—"Vectors."

Prof. Richard Morris—"Positive integral solutions of an indeterminate equation."

Prof. C. M. Huber—"Non-euclidean geometry."

Mr. A. E. Meder, Jr.—"Mathematical analysis of statistics."

Mr. Carl Gronquist—"Analysis of a structure by the method of least work."

Mr. H. M. Walter—"Actuarial mathematics."

Messrs. Veatch and Wittes—"Partial fractions."
Mr. Klaessig—"Stereographic projections."
Messrs. Eason and Worden—"Euler's theorem on homogeneous functions."
Mr. Clayton—"The Jacobian a covariant."

(Report by Professor Morris)

The Mathematical Society of the New Jersey College for Women, New Brunswick, New Jersey.

During the year 1926–1927, the Mathematical Society of the New Jersey College for Women was featured by a number of social events in addition to the regular monthly meetings.

The Society sent one of the student members as a delegate to the meetings of the Association in Philadelphia.

The social events consisted of a tea for the new members, a christmas party, a dinner in honor of Professors Titsworth and Morris, a subscription dance, a farewell tea to the seniors.

The Society designated and adopted a pin for members.

During the month of March, a symposium on the teaching of mathematics was held once each week.

The following papers and talks were given at the regular meetings:
Prof. A. A. Titsworth—"Reminiscences of the work on the geodetic survey of New Jersey."
Prof. C. M. Huber—"An account of non-euclidean geometries."
Mr. A. E. Meder, Jr.—"Graphical methods for computation."
Prof. Richard Morris—"Positive integral solutions of an indeterminate equation."
Miss Ruth Thompson—"Mathematics in investments."
Prof. E. P. Starke—"Tri-linear co-ordinates."
Miss Elsie Lundt gave a summary of her trip to the meetings of the Association at Philadelphia.
Miss Blackford—"The Slide Rule."
Miss Van Etten—"Fallacies."
Miss Spinetti—"Mathematics and philosophy."
Miss Bilderback—"A sketch of Isaac Newton."
Miss Bump—"Fluctions."
Miss Robinson—"Notes on 'Principia'."

A roll call was held at which every member responded with some mathematical problem or interesting mathematical fact.

(Report by Professor Morris)

Mu Theta Epsilon, University of California, Berkeley, Calif.

The officers for the year 1926–1927 were: Henrietta Sommer, president; E. B. Roessler, vice-president; Lucye E. Morris, secretary; Dean Smith, treasurer.

The program for the year consisted of the following:

August 25, 1926. "Perfect numbers" by D. H. Lehmer.

September 22. "What is mathematics?" by Professor Bernstein.

October 20. "The determination of time" by C. H. Smiley. "Seismology" by Henrietta Sommer.

December 1. "Generalization of measurement of distances and angles" by Beryl Britton. "The derivation of the secant column formula" by E. B. Roessler.

January 19, 1927. "Algebraic numbers" by D. C. Duncan.

February 5. Annual initiation and banquet.

February 16. "Geometrical inversion" by Helena Kusick. "The Naperian base" by D. H. Lehmer.

March 10. "Simplification of an equation in astronomical work" by H. Miles. "Primality" by Emma Trotskaia.

April 14. "The rate of reaction" by Jerome Martin.

(Report by Miss Irma Wicht)

THE IRRATIONAL CLUB OF THE UNIVERSITY OF WYOMING, Laramie, Wyoming.

The following is the program of the Irrational Club at the University of Wyoming for the year 1926–27:

November 9, 1926. Organization meeting at the home of Professor and Mrs. O. H. Rechard. Election of officers and social meeting. Officers for the year: Positive square root, Stella Lavergne; negative square root, Harry Cole; keeper of the log and bones, Reiva Niles.
November 23. "Magic squares" by Fredia Conner.
December 7. "Determinates" by Phillip Pepoon.
December 20. "What we see in the heavens" by Greta Neubauer.
January 20, 1927. "Einstein and his work" by O. H. Rechard.
February 3. "An evening with the telescope" by C. F. Barr.
February 17. Social meeting with program of mathematical stunts.
March 3. "Non-euclidean geometry" by Marion Linnville.
March 17. Historical topics by various members of the club.
April 12. "Squaring the circle"—Discussion by several members.
May 5. Club Picnic—Final meeting.

(Report by Reiva Niles, Keeper of the Log and Bones)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3302. Proposed by R. E. Gaines, University of Richmond.

Determine the envelope of the chord of a central conic which subtends a right angle at a given focus.

3303. Proposed by Otto Dunkel, Washington University.

Show that the sequence $C_0, C_1, C_2, C_3, \cdots$, defines the arithmetic nth root of N, where

$$C_{i+1} = C_i \frac{(n+1)N + (n-1)C_i^n}{(n-1)N + (n+1)C_i^n},$$

n is a positive integer, and N and C_0 are any positive numbers. See this MONTHLY [1927, 131]. Simpler sequences for this purpose are indicated on pages 367–368.

3304. Proposed by J. V. Uspensky.

The Eulerian numbers $E_1, E_2, E_3, \cdots$, are defined by the development

$$\sec x = 1 + E_1\frac{x^2}{1\cdot 2} + E_2\frac{x^4}{1\cdot 2\cdot 3\cdot 4} + E_3\frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots,$$

whence it follows that $E_1=1$, $E_2=5$, $E_3=61, \cdots$. Let p be a prime number $=4m+1$. Denoting by f an *odd* number between 0 and p which satisfies the congruence $f^2+1\equiv 0 \pmod p$, show that

$$E_1 + E_3 + \cdots + E_{2m-1} \equiv \frac{1-(-1)^\mu}{2} \pmod p, \quad \text{where } \mu = \frac{f-1}{2},$$

$$1 + E_2 + E_4 + \cdots + E_{2m} \equiv \frac{1-(-1)^\mu}{2} \pmod p, \quad \text{where } \mu = \frac{f-1}{2}.$$

For example, for $p=17$, we have $m=4$ and $f=1$, and

$$E_1 = 1 \equiv 1; \qquad E_3 = 61 \equiv -7; \qquad E_5 = 50521 \equiv -3; \qquad E_7 = 199360981 \equiv 9$$

$$E_1 + E_3 + E_5 + E_7 \equiv 0 \pmod{17}.$$

UNSOLVED PROBLEMS

Solutions are desired for the following unsolved problems proposed from January 1, 1913 to December 31, 1917. The number of each problem is printed in bold face type, with the year and page numbers following. Some of these problems were reprinted as indicated by the second reference. The numbers in parentheses are the incorrectly printed numbers. A list of unsolved problems proposed after December 31, 1917 will be printed later.

Algebra. **461,** 1916, 209 and 1919, 414.

Geometry. **446,** 1914, 191; **470,** 1915, 228 and 1919, 414; **472,** 1915, 267; **477,** 1915, 337; **478,** 1915, 338; **494,** 1916, 210; **510,** 1917, 231; **523,** 1917, 426.

Note. 478 is the same problem as 523.

Calculus. **348,** 1913, 312 and 1919, 268; **353,** 1914, 55 and 1919, 312; **415,** 1916, 301; **429,** 1917, 231; **432,** 1917, 287; **434,** 1917, 328 and 1918, 119; **436,** 1917, 388.

Mechanics. **279** (274), 1913, 222 and 1919, 214; **287,** 1914, 55 and 1919, 312; **291,** 1914, 122; **299,** 1914, 267; **308,** 1915, 268; **309,** 1915, 162; **313,** 1915, 202; **315,** 1915, 309; **343,** 1917, 124; **344,** 1917, 177; **351,** 1917, 328.

Number Theory. **191** (187), 1913, 196 and 1919, 214; **196,** (192), 1913, 223 and 1919, 214; **202,** 1913, 313 and 1919, 312; **205,** 1914, 55; **232,** 1915, 202; **234,** 1915, 268; **238,** 1916, 19; **263,** 1917, 171; **266,** 1917, 288; **270,** 1917, 389; **272,** 1917, 427; **274,** 1917, 467; **275,** 1917, 467.

SOLUTIONS

3228 [1926, 525]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The perpendiculars dropped from the vertices of a triangle upon the lines joining the mid-points of the opposite sides to the orthocenter of the triangle meet the respective sides of the triangle in three collinear points.

The line joining these three points is perpendicular to the Euler line of the triangle.

I. Solution by T. C. Esty, Amherst College

Take the origin, O, at the orthocenter of the triangle ABC and let α, β, γ be the position vectors of the vertices A, B, C respectively. Then $\alpha \cdot (\beta - \gamma) = \beta \cdot (\gamma - \alpha) = \gamma \cdot (\alpha - \beta) = 0$, whence

$$\alpha \cdot \beta = \beta \cdot \gamma = \gamma \cdot \alpha. \tag{1}$$

The position vectors of the mid-points A', B', C' of the sides BC, CA, AB are, respectively $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\gamma+\alpha)$, $\frac{1}{2}(\alpha+\beta)$. The equation of BC is $\rho = t\beta + (1-t)\gamma$. If AL is perpendicular to OA', its equation is $(\rho-\alpha) \cdot (\beta+\gamma) = 0$. Hence the point of intersection, L, of BC and AL is given by the last two equations, with

$$t = (\beta+\gamma) \cdot (\alpha-\gamma)/(\beta^2-\gamma^2) = (\gamma \cdot \alpha - \gamma^2)/(\beta^2-\gamma^2) = (\alpha \cdot \beta - \gamma^2)/(\beta^2-\gamma^2).$$

Hence the position vector of L is $\lambda=[(\alpha\cdot\beta-\gamma^2)\beta+(\beta^2-\alpha\cdot\beta)\gamma]/(\beta^2-\gamma^2)$. By a cyclic permutation of the letters α, β, γ and the use of equation (1), we find the position vectors of the corresponding points M, N on the sides CA, AB to be

$$\mu=\frac{[(\alpha\cdot\beta-\alpha^2)\gamma+(\gamma^2-\alpha\cdot\beta)\alpha]}{(\gamma^2-\alpha^2)};\qquad \nu=\frac{[(\alpha\cdot\beta-\beta^2)\alpha+(\alpha^2-\alpha\cdot\beta)\beta]}{(\alpha^2-\beta^2)}.$$

A necessary and sufficient condition for the collinearity of three points whose position vectors are λ, μ, ν is the existence of scalars x, y, z not all zero such that

$$x\lambda+y\mu+z\nu=0 \quad\text{and}\quad x+y+z=0.$$

Since
$$(\alpha\cdot\beta-\alpha^2)(\beta^2-\gamma^2)\lambda+(\alpha\cdot\beta-\beta^2)(\gamma^2-\alpha^2)\mu+(\alpha\cdot\beta-\gamma^2)(\alpha^2-\beta^2)\nu=0$$

and
$$(\alpha\cdot\beta-\alpha^2)(\beta^2-\gamma^2)+(\alpha\cdot\beta-\beta^2)(\gamma^2-\alpha^2)+(\alpha\cdot\beta-\gamma^2)(\alpha^2-\beta^2)=0,$$

it follows that L, M, and N are collinear.

To prove that the line LMN is perpendicular to the Euler line, we note that a vector parallel to the former is given by

$$\mu-\lambda=(\gamma^2-\alpha\cdot\beta)(\beta^2-\gamma^2)^{-1}(\gamma^2-\alpha^2)^{-1}[(\beta^2-\gamma^2)\alpha+(\gamma^2-\alpha^2)\beta+(\alpha^2-\beta^2)\gamma].$$

Since the position vector of the centroid is $\frac{1}{3}(\alpha+\beta+\gamma)$, we can prove our proposition by showing that $(\alpha+\beta+\gamma)\cdot(\mu-\lambda)=0$, which follows from the fact that

$$(\alpha+\beta+\gamma)\cdot[(\beta^2-\gamma^2)\alpha+(\gamma^2-\alpha^2)\beta+(\alpha^2-\beta^2)\gamma]=0.$$

II. Solution by Otto Dunkel, Washington University

Let H be the intersection of the altitudes AA_h, BB_h, CC_h of the triangle ABC; and let A_hB_h cut AB in C_1; B_hC_h cut BC in A_1; and C_hA_h cut CA in B_1. It will be shown that these are the three points of the problem and that they are collinear. The points H, A_h, C, B_h determine a complete quadrilateral and from it we see that A, C_h, B, C_1 are harmonic. Similarly, A, B_h, C, B_1 are harmonic. The harmonic pencil $A_1(A, C_h, B, C_1)$ cuts AC in A, B_h, C and a fourth point which must be B_1. Hence A_1, B_1, C_1 lie in a straight line. Through H draw HK parallel to AB and HJ meeting CC_1 perpendicularly at J. Then $H(B_h, K, A_h, J)$ has its rays perpendicular, respectively, to those of $C(A, C_h, B, C_1)$; it is therefore harmonic and it cuts AB in B, ∞, A, C'. Therefore C' is the middle point of AB. A similar proof applies for A' and B'.

We now prove that $A_1B_1C_1$ is the radical axis of the nine-point circle (N) and the circumcircle (S) with the centers N and S, respectively. It will then follow that NS, the Euler line, is perpendicular to $A_1B_1C_1$.

Let E be the middle point of HC; produce CS and JHC' until they meet in C_2. Now SC' is equal and parallel to CE, and it follows that $SC_2=CS$. Hence J and C_2 lie on (S). A circle (E) with center E passes through C, J, H, B_h, A_h. Thus (E) and (S) have the common chord CJ; also (E) and (N) have the common chord B_hA_h. Therefore the radical axis of (S) and (N) passes through C_1; similarly, it passes through A_1 and B_1, and the proof is complete.

Also solved by H. E. Arnold, C. F. Barr, J. W. Clawson, Max Coral, H. A. DoBell, and B. C. Patterson.

3230 [1926, 525]. Proposed by C. N. Schmall, New York City.

Determine the point in an ellipse at which the two focal distances include the greatest angle.

Solution by T. L. Smith, Harvard University

Let P be a point on the first quadrant of the ellipse with the foci F' and F. Let the bisector of the angle P of the triangle $F'FP$ cut $F'F$ in Q. Set $\angle QPF=\theta$, $\angle FQP=\psi$, $F'P=x$, $FP=y$, $FF'=2c$. Then $x+y=2a$ and hence $QF=cy/a$. The law of sines applied to the triangle FQP gives $\sin\theta=c\sin\psi/a$, where θ is acute. Hence $\psi=90°$ gives the maximum $\theta=\sin^{-1}(c/a)$.

Also solved by R. P. AGNEW, J. M. BARBOUR, A. G. CLARK, H. A. DOBELL, J. B. MEYER, A. PELLETIER, J. B. REYNOLDS, and F. W. WINTERS.

3232 [1927, 45]. Proposed by J. Rosenbaum, Milford, Conn.

Prove that in a tetrahedron, if three of the face angles at one vertex are equal, each less than or equal to a right angle, then any of the angles of the face opposite that vertex is less than the supplement of one of the three equal angles.

SOLUTION BY THE PROPOSER

Denote the tetrahedron by $ABCD$, and let D be the common vertex of the three equal angles, each angle being equal to α. Also denote the sides of the triangle ABC by a, b, and c, in the usual manner; and the edges DA, DB, and DC by x, y, and z respectively.

If α is a right angle then, $a^2=y^2+z^2$, $b^2=z^2+x^2$, $c^2=x^2+y^2$; and hence $a^2<b^2+c^2$.

Therefore, angle A, that is any angle, of triangle ABC is acute, and hence less than the supplement of α.

For the case when α is acute, we denote $\cos\alpha$ by k, and prove that for any angle such as A, of triangle ABC, the opposite side, a, satisfies the inequality $a^2<b^2+c^2+2kbc$, thus:

From triangles DCA, DAB, $x-z<b$, $y-x<c$.

Since these inequalities hold also when their left hand members are replaced by their absolute values, we have a right to multiply these inequalities obtaining

$$-yz < x^2 - zx - xy + bc.$$

Multiply by the positive quantity $2k$,

$$-2kyz < 2kx^2 - 2kzx - 2kxy + 2kbc.$$

Replacing the positive term $2kx^2$ of the right hand member by the greater quantity $2x^2$, and adding y^2+z^2 to each side of the resulting inequality, we obtain

$$y^2 + z^2 - 2kyz < z^2 + x^2 - 2kzx + x^2 + y^2 - 2kxy + 2kbc.$$

From the triangles DBC, DCA, and DAB, this reduces to the inequality which we desired to prove; and it completes the proof.

Also solved by J. B. REYNOLDS.

3234 [1927, 45]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The median through a given vertex of a triangle meets the Apollonian circle passing through this vertex and the circumcircle of the triangle in two points which, with the other two vertices of the triangle, determines a parallelogram.

SOLUTION BY C. H. CHEPMELL, Hove, Sussex, England

Let ABC be a given triangle, with base $BC(=2\alpha)$, bisected in D; ABC, its circumcircle; FAE its Apollonian circle, cutting BC in E and F. Then AF and AE are, respectively, the external and internal bisectors of the angle A, cutting CB produced in F and E. Let AD cut the Apollonian circle in P and the circumcircle in H. Since F, B, E, C form an harmonic set of points, $\alpha^2=DE\cdot DF$. Also $DE\cdot DF=DP\cdot DA$ and $\alpha^2=HD\cdot DA$, from the elementary properties of circles. Hence $DP=HD$, and BC, PH bisect one another at D. Hence these last two lines are diagonals of $PCHB$, which is therefore a parallelogram.

NOTE: If the angle BAC is acute, P falls inside the triangle ABC; but if it is a right angle, P coincides with A; and, if it is obtuse, P falls outside ABC.

The cases, where ABC is isosceles or equilateral, have an interest of their own; but their adequate discussion might take us too far afield.

Also solved by RUTH A. BALLIEFF, MAX CORAL, J. W. CLAWSON, A. G. CLARK, H. A. DOBELL, T. C. ESTY and H. K. JUSTICE.

3235 [1927, 45]. Proposed by Paul Capron, U. S. Naval Academy.

If the curtate cycloid, $x=a\phi-b\sin\phi$, $y=a-b\cos\phi$, $(b>a)$, has a node at which $x=0$ when $\phi=\alpha$, and if h_1, h_2, and $-h_3$ are its maximum ordinate, the ordinate of a node, and its minimum ordinate, show that the area of a loop is $\alpha(h_1h_3-ah_2)$.

If the loops are just large enough to be tangent externally, show that the area of a loop is about $39.2a^2=8.52ab=1.85b^2$.

SOLUTION BY R. P. AGNEW, Cornell University

If A is the area of a loop, then an expression for the area of the half-loop which lies to the left of the y-axis is

$$\frac{A}{2} = -\int_{\phi=0}^{\phi=\alpha} x\,dy = -\int_{\phi=0}^{\phi=\alpha} (ab\phi\sin\phi - b^2\sin^2\phi)\,d\phi,$$

where α is the positive value of the parameter ϕ which corresponds to the node on the y-axis. Then

$$A = 2ab(\alpha\cos\alpha - \sin\alpha) + b^2(\alpha - \sin\alpha\cos\alpha).$$

Since $b\sin\alpha=a\alpha$, this reduces to $A=\alpha(ab\cos\alpha-2a^2+b^2)$. Since $b\cos\alpha=a-h_2$, $a+b=h_1$, and $b-a=h_3$, it follows immediately that $A=\alpha(h_1h_3-ah_2)$.

Imposing the condition that adjacent loops be tangent externally, it is found that $a/b=\cos\psi$, where ψ is the positive first-quadrant solution of the equation $\tan\psi=\psi+\pi$. Then $\psi=1.35182$, and $a/b=.21724$. Solving the equation $\sin\alpha=.21724\alpha$ for α, we find $\alpha=2.5536$, and $\cos\alpha=-.8320$. Substituting the values of α, $\cos\alpha$, a/b, and b/a in the general expression for A, we find $A=39.2a^2=8.52ab=1.85b^2$. All the numerical results are approximations.

Also solved by E. M. BERRY, A. G. CLARK, J. B. REYNOLDS, and the PROPOSER.

3237 [1927, 98]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

A variable chord of an ellipse subtends a right angle at the center of the curve. Find the envelope of the chord.

SOLUTION BY R. E. GAINES, University of Richmond

Let κ be the length of the perpendicular from the center to the chord, and ρ_1, ρ_2, the lengths of the semi-diameters to the ends of the chord. From the right triangle, it follows that $\kappa^{-2}=\rho_1^{-2}+\rho_2^{-2}$; and, from the properties of the ellipse, the expression on the right is equal to $a^{-2}+b^{-2}$. Hence κ is a constant and the envelope is a circle with its center at the center of the ellipse and a radius $\kappa=ab/(a^2+b^2)^{1/2}$.

Also solved by R. P. AGNEW, THEODORE BENNETT, H. C. BRADLEY, A. G. CLARK, MICHAEL GOLDBERG, J. E. WILLIAMS and R. M. WINGER.

3238 [1927, 98]. Proposed by Malcolm Foster, Williams College.

Determine the necessary and sufficient condition that the loci of the centers of first and second curvature of a given curve have orthogonal tangents at corresponding points.

SOLUTION BY THEODORE BENNETT, University of Illinois

Let the coordinates of any point (x, y, z) on the original curve be expressed in terms of the arc length s. Then the loci of the centers of first and second curvature may be written

$$X_1 = x + \rho l, \qquad Y_1 = y + \rho m, \qquad Z_1 = z + \rho n,$$

$$X_2 = x + \tau\lambda, \qquad Y_2 = y + \tau\mu, \qquad Z_2 = z + \tau\nu,$$

respectively, using the notation given in Eisenhart's *Differential Geometry*, pp. 9–17. By differentiating these equations, and simplifying by means of the Frenet-Serret formulas, we find

$$X_1' = l\rho' - \lambda\rho/\tau, \qquad X_2' = \alpha + l + \lambda\tau',$$

and similar expressions for Y and Z. For the tangents to be orthogonal the necessary and sufficient condition is

$$X_1'X_2' + Y_1'Y_2' + Z_1'Z_2' \equiv \rho' - \rho\tau'/\tau = 0.$$

By integrating this last equation we find at once that $\rho = C\tau$, whence the radii of first and second curvature must have a constant ratio, *i.e.* the original curve must have been a cylindrical helix. See Eisenhart's Differential Geometry, pp. 20, 21.

Also solved by the PROPOSER.

3240 [1927, 98]. Proposed by Lü Ling, Tsing Hua University, Peking.

On a given base, construct a triangle such that its vertex may be on a given straight line, and the difference between its base angles may be equal to a given angle.

SOLUTION BY THEODORE BENNETT, University of Illinois

Let AB be the given base and let the angle at B exceed the one at A by θ. It is known from geometry that the third vertex P describes an equilateral hyperbola through A and B with center at O the mid-point of AB, with parallel tangents at A and B making an angle $\pi-\theta$ with AB, and with asymptotes making angles $(\pi-\theta)/2$ and $\pi-(\theta/2)$ with AB. A standard construction of projective geometry enables us to find the points in which this hyperbola is cut by the given line on which the third vertex must lie. The construction is modified only in minor details by the fact that some of the points are coincident at infinity.

NOTE BY OTTO DUNKEL

It is not necessary to consider the locus of P. The projective pencils AP and BP cut the given line l in two projective ranges of points, and we have to find on l the self-corresponding points of the two ranges. Construct on l the mates B_1, B_2, B_3 to three of its points A_1, A_2, A_3 chosen in any way. Draw a circle in the plane, and join any fixed point M of the circle with these six points. Let the six rays thus determined cut the circle in $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$, $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$. Draw the straight line m joining $(\overline{A}_1\overline{B}_2, \overline{A}_2\overline{B}_1)$ with $(\overline{A}_1\overline{B}_3, \overline{A}_3\overline{B}_1)$. If the line m cuts the circle in I, then the intersection of MI and l gives the desired third vertex of the triangle. There are two solutions of the problem if m cuts the circle in two distinct points; one solution if m is tangent to the circle; and no solution if it does not cut the circle.

3241 [1927, 98]. Proposed by F. A. Berger, Washington University.

With a center on the circumference of a given circle an arc of a second circle is drawn which bisects the area of the first circle. Find the ratio of the radii of the two circles.

SOLUTION BY J. E. WILLIAMS, Virginia Polytechnic Institute

Draw a circle with its center at O and a radius equal to r. Let A be one end of a diameter. With A as a center draw a circle with radius R, cutting the first circle at the points B and K, and let H be the mid-point of the arc of the circle A which lies within the circle O. Let the angle AOB be represented by ϕ and the angle OAB by θ.

One-half of the area between the circles is made up of the sector $HAB = \frac{1}{2}R^2\theta$ and the difference between the sector $AOB = \frac{1}{2}r^2\phi$ and the triangle $AOB = \frac{1}{2}Rr\sin\theta$. Since the area of the first circle is to be bisected, we have

$$R^2\theta + r^2\phi - Rr\sin\theta = \tfrac{1}{2}\pi r^2.$$

We also have $\phi = \pi - 2\theta$ and $R/r = 2\cos\theta$. With the aid of these two results the above equation reduces to

$$\sin 2\theta - 2\theta\cos 2\theta - (\pi/2) = 0.$$

It is readily seen from a figure that there is only one root between 0 and $\pi/2$ and that it must be less than $\pi/3$. By approximation processes it is found that

$$\theta = 54^\circ\, 35'\, 38'', \qquad R/r = 1.15874.$$

Also solved by THURMAN ANDREW, THEODORE BENNETT, MICHAEL GOLDBERG, A. PELLETIER, and S. D. TURNER.

3242 [1927, 98]. Proposed by R. S. Underwood, Auburn, Alabama.

A man finds that a pile of cocoanuts is exactly divisible by n after giving an extra nut to a monkey. He takes away $1/n$th of the remaining nuts and leaves the rest. A second man repeats the process with the rest giving one nut to the monkey and taking away exactly $1/n$th of the rest. This is continued and the nth man leaves at the end a pile of nuts which is exactly divisible by n. How many nuts were there at the beginning and how many at the end?

SOLUTION BY ROBERT E. MORITZ, University of Washington

Let x equal the number of nuts at the beginning, y, the number of nuts at the end. Then by the conditions of the problem,

(1) $$x-1=kn,\ y=hn,$$ where k, h, and n, are unknown integers.

Denote $(n-1)/n$ by p. Then the number left by the first man is knp, while the number left by the rth man is

(2) $$knp^r - (p^{r-1} + p^{r-2} + \cdots + p^2 + p) = \frac{(k+1)(n-1)^r - (n-1)n^{r-1}}{n^{r-1}}.$$

Setting this last expression equal to hn for $r=n$, we have

(3) $$(k+1)/[(h+1)n - 1] = n^{n-1}/(n-1)^n,$$

that is, $k+1$ and $(h+1)n-1$ must be equi-multiples of n^{n-1} and $(n-1)^n$ respectively, say

(4) $$k + 1 = mn^{n-1}, \quad (h+1)n - 1 = m(n-1)^n.$$

The second of the equations (4) may be written in the form

(5) $$h = (n-1)n^{-1}[m(n-1)^{n-1} - 1]$$

from which it appears that $m(n-1)^{n-1}-1$ must be a multiple of n. It is evident also that if one value of m say m', has been found such that $m'(n-1)^{n-1}-1$ is a multiple of n, the most general value of m which will make h integral is $m=m'+nc$, where c is any integer. If m' is the smallest positive integral value of m, all possible integral values of h are found by giving to c the successive values 1, 2, 3, $\cdots$. The first of equations (4) determines the corresponding values of k, and (1) the values of x and y. The general solution corresponding to any given value of n is thus found to be

(6) $$x = (m' + nc)n^n - (n-1), \qquad y = (m' + nc)(n-1)^n - (n-1),$$

where m' is the smallest positive integer that will make $m'(n-1)^{n-1}-1$ a multiple of n.

Now, on expanding the expression $(n-1)^{n-1}$ in powers of n, we see that the least positive value of m which will satisfy the congruence

$$m(n-1)^{n-1} - 1 \equiv 0, \mod n,$$

is $m'=1$ when n is odd, and $m'=n-1$ when n is even. We have therefore the general solution of our problem in the form

(7)
$$x = (1 + nc)n^n - (n-1), \qquad y = (1+nc)(n-1)^n - (n-1), \quad n \text{ odd},$$
$$x = (n - 1 + nc)n^n - (n-1), \qquad y = (n-1+nc)(n-1)^n - (n-1), \quad n \text{ even},$$

and these two forms may be combined into the single form

(8) $$x = Kn^n - (n-1), \quad y = K(n-1)^n - (n-1), \quad 2K = n(1+2c) + (-1)^{n+1}(2-n),$$

in which n and c are any two positive integers.

The following table gives the numerical values of both the principal solutions ($k=k'$), and the general solutions of x and y for all values of n from $n=2$ to $n=10$.

n	m'	x_0	y_0	x_c	y_c
2	1	3	0	$3+8c$	$0+2c$
3	1	25	6	$25+81c$	$6+24c$
4	3	765	240	x_0+4^5c	$y_0+4.3^4c$
5	1	3,121	1,020	x_0+5^6c	$y_0+5.4^5c$
6	5	233,275	78,120	x_0+6^7c	$y_0+6.5^6c$
7	1	823,537	279,930	x_0+7^8c	$y_0+7.6^7c$
8	7	117,440,505	40,353,600	x_0+8^9c	$y_0+8.7^8c$
9	1	387,420,481	134,217,720	$x_0+9^{10}c$	$y_0+9.8^9c$
10	9	89,999,999,991	31,381,059,600	$x_0+10^{11}c$	$y_0+10.9^{10}c$

Also solved by R. P. AGNEW, H. C. BRADLEY, E. M. BERRY, P. S. DWYER, L. H. FELTMAN, R. E. GAINES, J. S. GEORGES, MICHAEL GOLDBERG, H. O. HANSON, P. A. KNEDLER, JOSHUA MATZ, RICHARD MORRIS, J. R. MUSSELMAN, A. PELLETIER, B. C. PATTERSON, J. E. ROWE, G. E. RAYNOR, L. S. SHIVELY, F. L. WILMER, and the PROPOSER.

3247[1927, 156]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

When can two conics be projected into two orthogonal circles by a real projection?

SOLUTION BY THE PROPOSER.

The operation being projective, and therefore reversible, we may try to answer the proposed question by considering the contrary problem: *If two orthogonal circles are transformed by a real projection, in what relation will the resulting conics be to each other*? Or, in other words, do two orthogonal circles have a characteristic property which is projective?

It is known that the radical axis of two orthogonal circles has for its pole with respect to either of the circles the center of the other cricle. This property holds when both circles are real, or when one circle is real and the other imaginary. Let us now establish the converse of this proposition.

If the radical axis r of two circles (A), (B) is the polar of the center A of one of the circles with respect to the other circle (B), then: (a) the radical axis r is also the polar of the other center B with respect to the first circle (A); (b) the two circles are orthogonal.

Let a, b be the radii of the two circles, and R the point of intersection of the radical axis with the line of centers AB. The point R has the same power with respect to both circles; hence we have, both in magnitude and in sign, (1) $RA^2-a^2=RB^2-b^2$. On the other hand R, A are, by assumption, inverse points with respect to the circle (B); hence we have, again both in magnitude and in sign, (2) $BA\,.\,BR=b^2$. The relation (1) may be written, due regard being paid to the signs of the segments involved,

$$a^2 - b^2 = RA^2 - RB^2 = (RA + RB)(RA + BR) = (RA + RB)BA.$$

By adding this result to (2) we get $(RA+RB+BR)BA=a^2$, or (3) $AR.AB=a^2$. Thus the points B, R are inverse points with respect to the circle (A), which proves part (a) of the proposition.

Adding (2) and (3) we have

$$a^2+b^2=AR\,.\,AB+BA\,.\,BR=AR\,.\,AB+AB\,.\,RB=AB(AR+RB)=AB^2,$$

which proves part (b) of the proposition.

The radical axis r is a common chord of the two circles and joins two real common points of the two curves, if both circles are real, or a pair of conjugate imaginary common points of the two curves, if one of the circles is imaginary but has a real equation. The two circles have also in common the circular points at infinity of the plane, and the line at infinity is a real common chord of the two circles, and

furthermore the centers of the two circles are the poles of this line with respect to these circles. Thus the property of two orthogonal circles quoted above may be formulated in projective terms. We are therefore able to answer the proposed question as follows: Two conics (P), (Q), one of which, at least, is real, may be transformed into two orthogonal circles by a real projection, if they have two real common chords, s, t such that the pole of one of these chords, say s, with respect to one of the conics, say (P), coincides with the pole of the other chord t with respect to the other conic (Q).

If both conics are real one of the chords must join a pair of real common points and the other a pair of conjugate imaginary common points of the two conics.

REMARK: Two conics related in this manner will have all the descriptive properties of two orthogonal circles. Many of these properties become particularly interesting in their new application. For lack of space we shall consider only one example. It is known that the lines on which two conics determine two pairs of harmonic points envelop, in general, a conic. In the case of two conics which may be transformed into two orthogonal circles this envelope consists of two points, namely, the two points in which coincide the four poles of their common chords s, t.[1]

EXTENSION TO SPACE: The proposition proved above about two orthogonal circles applies without modification to two orthogonal spheres. The point R will be the trace on the line of centers of the radical plane of the two spheres. We may therefore extend our result to two orthogonal spheres. Thus we have the proposition: *Two quadric surfaces (P), (Q), at least one of which is real, may be transformed into two orthogonal spheres by a real projection if they have two real intersections in planes σ, τ such that the pole of one of these planes, say σ, with respect to one of the quadrics, say (P), coincides with the pole of the other plane τ with respect to the other surface (Q).*

If both surfaces are real, one of the common planes must cut the surfaces along a real conic and the other along an imaginary conic.

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NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Nashville meeting was a great success. The attendance was large and representative. The South did itself proud both in numbers and in active support. But the North and the East and the West were all in evidence also, both on the programs and at the joint dinner. Speaking of dinner, were there ever such meals served at such moderate prices as Ward-Belmont provided! We all vote for another meeting in the South.

The first volume of the Ahmes Mathematical Papyrus is now printed and bound, but the second volume is still delayed on account of tedious work connected with the perfecting of the colored plates. We are assured, however, that these difficulties are now practically surmounted, and that the completion of the second volume may confidently be hoped for sometime in March. The two volumes will be shipped together as soon as they are ready.

The following schedule of lectures has been arranged for Professor CONSTANTIN CARATHÉODORY of the University of Munich who will be the first

[1] Nathan Altshiller-Court, *College Geometry* (Johnson Pub. Co., Richmond, Va., 1925), p. 148 ex. 15.

Visiting Lecturer of the American Mathematical Society. The University of Pennsylvania, January 10–13; Ohio State University, January 16–17; The University of Iowa, January 19–20; The University of Chicago, January 23–24; The University of Michigan, January 25–27; Adelbert College, January 30–31; Cornell University, February 1–2. After these lectures, Professor Carathéodory will be in residence at Harvard University until about June 1.

The Board of Managers of the Franklin Institute has voted to award the Elliott Cresson Gold Medal to Professor VLADIMIR KARAPETOFF of Cornell University in consideration of the inventive ability, skill in design, and detailed theoretical knowledge of kinematics and electrical engineering displayed in the development of computing devices.

At the Bartol Research Foundation, Professor EINAR HILLE of Princeton University lectured on December 9 and 15, 1927 on "Boundary problems in differential equations with special reference to their application to Schroedinger's wave equation"; and Professor ARNOLD DRESDEN of Swarthmore College gave a lecture on January 10, 1928, on "Special devices used in the solution of problems by the matrix mechanics."

At the Nashville Meeting of the American Association for the Advancement of Science, Professor R. C. ARCHIBALD of Brown University was elected Vice-president and Chairman of Section A, and Professor C. N. MOORE of the University of Cincinnati was elected Secretary. At the same meeting Professor D. R. CURTISS of Northwestern University was elected a member of the Executive Committee.

Dr. HAROLD HOTELLING, research associate in the Food Research Institute until the fall of 1927, has been appointed associate professor of mathematics at Stanford University.

Dr. G. T. WHYBURN has been appointed assistant professor of mathematics at the University of Texas.

Professor J. L. RILEY, Ouachita College, a charter member of the Mathematical Association, died on July 12, 1927, from the effects of an operation.

Dr. G. A. OSBORNE, emeritus professor of mathematics at the Massachusetts Institute of Technology, died on November 20, 1927, at the age of eighty-eight years. He held membership in the Mathematical Association from December, 1916, to his death.

Professor C. D. GARLOUGH of Wheaton College, a member of the Mathematical Association since 1920, died on November 25, 1927.

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No. 3. *Mathematical Statistics,* by PROFESSOR H. L RIETZ. (First Impression, March, 1927.)

Nos. 4, 5 in preparation.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, April 27-28.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 9-10.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 3.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, May.

OHIO, Columbus, Ohio, April 6.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Durham, N. C., March.

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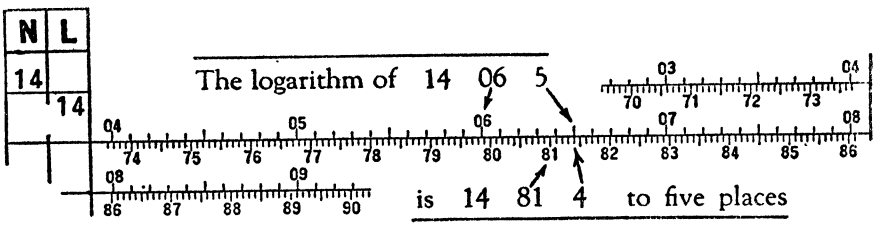
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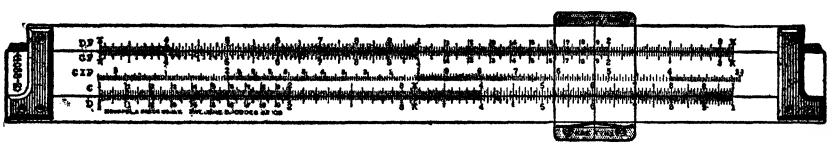
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THE NOVEMBER MEETING OF THE MISSOURI SECTION

The eleventh annual meeting of the Missouri Section of the Mathematical Association of America was held at Washington University, St. Louis, Missouri, on Friday and Saturday, November 25–26, 1927, in connection with the meeting of the Southwestern Section of the American Mathematical Society. A reception was held at the University Club on Friday evening under the auspices of the Washington University Chapter of the Pi Mu Epsilon Mathematical Fraternity. At this reception several musical numbers were given, and Professor E. B. Stouffer related some of his experiences in Italy, particularly at the University of Bologna, while engaged in research work as the holder of a Guggenheim Fellowship. A session held at nine o'clock Saturday morning was presided over by Miss Kathryn Wyant, chairman of the section, and by Professor W. H. Roever, vice-chairman, who occupied the chair while Miss Wyant presented a paper. This session was followed by a session of the Southwestern Section of the Society at ten-thirty. The Saturday afternoon session, which was a joint session with the Southwestern Section, was presided over by Professor Roever. Those attending the meetings were the guests of Washington University at luncheon on Saturday.

The attendance was about thirty, including the following twenty-two members of the Association: Florence Black, W. C. Brenke, S. C. Davisson, Otto Dunkel, F. J. Gerst, Cornelius Gouwens, L. D. Haertter, Byron Ingold, Louis Ingold, C. G. Jaeger, Harry Levy, Jesse Osborn, W. O. Pennell, P. R. Rider, G. Edna Robinson, W. H. Roever, Eugene Stephens, E. B. Stouffer, G. B. Sweazey, J. S. Turner, E. Kathryn Wyant, Jessica M. Young.

The following officers were elected for 1928: Chairman, W. A. LUBY, Kansas City Junior College; Vice-chairman, G. B. SWEAZEY, Westminster College; Secretary-Treasurer, P. R. RIDER, Washington University. The 1928 meeting will be held in Kansas City at the time of the meeting of the Missouri State Teachers Association in November.

The following two papers were presented at the meeting of the Missouri Section:

1. "The foundation of the theory of ideals" by Miss KATHRYN WYANT, University of Missouri.

2. "The trigonometry of hyperspace" by Miss NOLA LEE ANDERSON, University of Missouri. (By invitation.)

At the joint session Professor E. B. STOUFFER of the University of Kansas presented a paper on "Some canonical forms and associated canonical expansions in projective differential geometry" (by invitation of the program committee).

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles. The abstract of Professor STOUFFER's paper may be found in the report of the meeting of the Southwestern Section in the BULLETIN of the American Mathematical Society.

1. In attempting to prove Fermat's last theorem, Kummer introduced the notion of ideal numbers. These were replaced by Dedekind by ideals which are a little more concrete. In this paper Miss Wyant discussed the fundamentals of the theory of ideals based on a more general definition. She defined ideals, their sum, product, and direct sum, and stated the most important theorems concerning them, including several on divisors and multiples. She showed how to establish the unique factorization of an ideal into relatively prime primary ideals, defined equivalent ideals and classified ideals of a general ring in such a way that all ideals equivalent to one ideal belong to the class, and then showed that these classes form an Abelian group. This group is finite in the special case of algebraic number fields but it is not yet known in general whether the group is finite or infinite.

2. Consider n hypersurfaces in an n-dimensional space. These form by their mutual intersection various subspaces.

It was the object of Miss Anderson's paper to obtain formulas connecting the trigonometric functions of the angles formed by such a configuration. These formulas were obtained directly from known relations connecting certain differential invariants.

P. R. RIDER, *Secretary*

THE SECOND MEETING OF THE PHILADELPHIA SECTION

The second regular meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania on Saturday, November 26, 1927, Professor H. H. Mitchell presiding.

There were sixty present including the following thirty members of the Association: P. A. Caris, G. G. Chambers, J. W. Clawson, E. S. Crawley, J. E. Davis, A. Dresden, D. A. Flanders, T. Fort, H. V. Gummere, G. A. Harter, F. H. Jackson, J. R. Kline, M. S. Knebelman, P. A. Knedler, E. Latshaw, W. F. Long, H. M. Lufkin, J. A. Miller, H. H. Mitchell, R. Morris, F. W. Owens, H. B. Owens, C. J. Rees, G. Rosengarten, F. H. Safford, W. Sensenig, L. L. Smail, W. M. Smith, A. H. Wilson, R. R. Wood.

The following officers were elected for the next year: Professor F. W. OWENS, chairman; Professor J. R. KLINE, Secretary-Treasurer, Professors A. DRESDEN and J. M. THOMAS, Program Committee.

The following papers were presented. Short abstracts are given below.

1. "Descartes' Geometry" by Professor E. S. CRAWLEY, University of Pennsylvania.

2. "The Malfatti problem" by Professor F. W. OWENS, Pennsylvania State College.

3. "On matrix equations" by Professor ARNOLD DRESDEN, Swarthmore College.

4. "Space filling polyhedra" by Professor A. H. WILSON, Haverford College.

5. "Difference equations" by Professor TOMLINSON FORT, Lehigh University.

6. "Positive integral solutions of an indeterminate equation" by Professor R. MORRIS, Rutgers College.

1. Professor Crawley's talk on "Descartes' Geometry" was a brief sketch of the contents of this book, limited by the time available to a description of Descartes' method of attack of the problem in which he first introduces the use of co-ordinates, and a short reference to his application of his method to the problem of constructing a normal to a curve at a given point.

2. In this paper, Professor Owens gave a brief historical sketch of the Malfatti problem, i.e. the problem of constructing in a triangle three circles each one tangent to two sides of the triangle and to the other two circles. He concluded with an original solution somewhat different from any usually given.

3. This paper consisted of a brief report on a method recently developed by Mr. W. E. Roth of the University of Wisconsin for the determination of solutions of the equation $P(X)=A$, which are expressible as polynomials in A, where $P(\lambda)$ is a polynomial in λ with scalar coefficients and without constant term, A is a matrix of order n and X is to be determined as a matrix of order n. The method is very elementary in character and is applicable in many instances where the matrix A is singular. It has the further advantage over methods developed by earlier writers in not requiring the use of elementary divisors. Mention was also made of another method developed by Professor M. H. Ingraham.

4. Professor Wilson discussed first the analogous problem for two dimensions: to cover the plane by repetitions of a polygon. He enumerated the well known classes of polygons which have this property and gave an account of the work of MacMahon, showing charts of some of the repeating polygons developed by him.

For space of three dimensions, in addition to the prisms, the classical space filling polyhedra, the rhombic dodecahedron, the bees'-cell, and the tetrakaidecahedron were described. Models of these were shown and a brief sketch of their significance in nature was given, with especial reference to the work of the Harvard Medical School in determining the form of certain cells. He concludes with an account of the work of Sommerville, who solved the problem of finding all plane-filling triangles and all space-filling tetrahedra for non-

Euclidean geometry as well as Euclidean geometry. Models of the four tetrahedra for Euclidean space were shown.

5. Professor Fort outlined the history and literature of difference equations and stated the principal problems involved. In particular, he dwelt upon the advances in difference equations which have been made in America.

6. Professor Morris discussed a problem in Diophantine analysis which arose as a generalization of a problem recently proposed for solution in this journal.

J. R. KLINE, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-second regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Georgetown University, Washington, D. C., on Saturday, December 3, 1927. Two sessions were held, one in the morning and one in the afternoon, Professor J. R. Musselman, Chairman of the Section, presiding at both sessions.

Sixty-eight persons attended the meeting, including the following forty-six members of the Association: O. S. Adams, L. M. Blumenthal, C. C. Bramble, J. A. Bullard, J. F. Burley, Helen Calkins, Paul Capron, G. R. Clements, A. B. Coble, A. Cohen, Tobias Dantzig, L. S. Dederick, Alexander Dillingham, J. A. Duerksen, Harry English, P. J. Federico, G. L. Fentress, Michael Goldberg, Harry Gwinner, W. M. Hamilton, P. E. Hemke, L. S. Hulburt, H. P. Kaufman, L. M. Kells, W. D. Lambert, A. E. Landry, C. L. Leiper, T. W. Moore, W. K. Morrill, F. D. Murnaghan, J. R. Musselman, J. W. Peters, E. C. Phillips, Otto Ramler, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, W. F. Shenton, T. H. Taliaferro, John Tyler, Paul Wernicke, J. E. Willis, Elizabeth Wilson, E. W. Woolard.

At the beginning of the morning session, Rev. Charles W. Lyons, S. J., President of the University, made a short address of welcome to the members and guests of the Section. During the intermission between the morning and afternoon sessions those attending the meeting were entertained at luncheon by the University. Just before the formal opening of the afternoon session Professor Cohen, of Johns Hopkins University, as Maryland Representative of the American Mathematical Society, made a short informal address reminding those present that the Society and the Association were making an experiment in holding their annual meetings so far south as Nashville, and suggested that all those who possibly could should avail themselves of the opportunity thus afforded to attend these joint meetings with that of the A. A. A. S. A good attendance, incidentally, would convince the Society and the Association that their experiment was sympathetically received. At the close of the afternoon

session the Executive Committee held a brief meeting at which it was decided to hold the 1928 meetings as follows: the Summer Meeting on Saturday, May 5, at Annapolis, Maryland, and the Winter Meeting on Saturday, December 1, at Baltimore, Maryland.

The following eight papers were presented:

1. "On the equilibrium of an elastic blade" by Dr. TOBIAS DANTZIG, University of Maryland.

2. "A configuration of ten points analogous to the Desargues configuration" by Professor A. B. COBLE, Johns Hopkins University.

3. "An extension of the Gauss problem of eight queens" by Dr. L. M. BLUMENTHAL, Johns Hopkins University.

4. "Observers' patterns" by HOWARD S. RAPPLEYE, U. S. Coast and Geodetic Survey. (By invitation.)

5. "Rational paths with supersingular equations" by Rev. E. C. PHILLIPS, Georgetown University.

6. "Some remarks on a recent textbook of College Geometry" by Dr. J. N. RICE, Catholic University of America.

7. "On the application of differentials and definite integrals in physics" by Mr. E. W. WOOLARD, U. S. Weather Bureau.

8. "Some remarks on the danger of hasty generalization" by Professor R. E. ROOT, Postgraduate School, U. S. Naval Academy.

The authors' abstracts of their respective papers follow in the same order as enumerated above.

1. In the classical treatment of the beam problem certain assumptions are made which make it possible to reduce the problem to a linear differential equation of the second order, which gives a solution in finite form. These approximations become untenable in the case here considered. The author shows that the problem depends on the solution of the differential equation $(d^2\phi)/(ds^2) = s\cos\phi$ and seeks a power expansion which satisfies this equation. He considers the most general holomorphic expression of the form $y'' = x^r f(y)$ and shows that it admits a power expansion in terms of x^{m+2}. Applying this theorem he finds a series solution for the elastic line. With excellent approximation this can be approached by a quartic parabola with vertex at the free end. The locus of the free end as the length of the strip changes is sought. A practical application is the determination of the elastic modulus of a material and method for the measure of stiffness. The method used in this paper is applicable to the case of other modes of loading and support.

2. If four nodes of a rational plane sextic are isolated, the four pairs of nodal parameters and the six pairs of parameters of points in which the six lines joining the four nodes meet the sextic again furnish us a set of ten specially

related binary quadratics. If these quadratics are plotted as points with reference to a norm-conic a ternary configuration of ten points is obtained which contains ten triangles each perspective to a triangle whose sides are defined by the ten points. This configuration is analogous to that of Desargues but contains nine, rather than three, absolute constants.

3. The problem of ascertaining the number of ways in which eight queens may be placed on a chess board so that no queen can capture any other was proposed to Gauss by Nauck. In this paper is considered an extension of the problem to the case of n queens on a board of n^2 squares. It is shown that whenever n is not a multiple of two or three many solutions may be written down at once. These solutions are generated by a key solution. The problem is seen to be equivalent to one in combinatory analysis.

4. In the work of first-order leveling as carried on by the U. S. Coast and Geodetic Survey, the rods are read by estimating tenths of centimeter graduations. Each rod is read at three points. The height of the instrument and consequently the height at which the rods are read being purely accidental, there should be about an even distribution among the ten digits in the resulting estimated millimeters. This paper is a preliminary statement of results obtained by tabulating over 30,000 separate estimations of millimeters.

The results were grouped for different observers and for different observing conditions. The diagrams or "patterns," by means of which the results were shown, displayed some startling differences between the work of different observers and even for the same observers under different observing conditions.

The possibility of devising a test to determine the fitness of a particular observer for this class of work, before taking the field with a level party, was discussed. Certain other practical results, such as a limit on the length of sight which may possibly arise from this investigation, were noted.

5. The main purpose of this paper was to call attention to some advantages of the parametric equations of a plane curve over the usual Cartesian equation, especially when dealing with curves considered from the kinetic point of view as geometric loci traced by a moving point. As a simple example the parabolic trajectory of a bullet was cited. When the angle of elevation is 90°, the Cartesian equation degenerates or breaks up into linear factors ($x^2=0$). Such an equation gives very inadequate and in a certain sense misleading information as to the actual trajectory, whereas the parametric equations even in this special case still give us full and exact information. It was shown that the degeneration of the Cartesian equation results from imposing on the trajectory the condition of having a cusp, which is requiring the curve to be supersingular. Any path which, like the trajectory of the bullet, is an unambiguous, unicursal locus of finite order, whose Cartesian coordinates can be represented by rational

functions of a single real variable parameter, was called a rational path; it was shown that the parametric equations of such a path of order n can be made to satisfy the necessary and sufficient conditions for the existence of 2n–2 (or fewer) cusps, whereas a proper curve of order n can have at most $(3n-6)/2$ cusps; hence such loci may be called supersingular rational paths.

6. This paper consisted of remarks on Altshiller-Court's *College Geometry* based on class-room experience with the text.

7. Attention was drawn to certain loose statements about differentials frequently encountered in textbooks, as illustrated by the customary derivation of the hypsometric formula; and a rigorous derivation of this formula, making use of Duhamel's theorem, was outlined. It was particularly emphasized that if $y=f(x)$, dy is not, in general, the increment in y produced by the increment dx in x; that neither dy nor dx is "infinitely small"; and that $dy \neq \lim \Delta y$, $dx \neq \lim \Delta x$.

8. In the remarks by Professor Root, attention was called to the fact that, whilst mathematics is a deductive science, yet the methods and devices of inductive reasoning are very properly and effectively used to aid invention and discovery in mathematical fields. In expository and text book writing, analogy and generalization are frequently employed, the essential deductive proofs often being replaced by such rhetorical devices as "it is evident that," or "similarly it may be shown," etc. The dangers of these devices was shown by an example drawn from a text book in mechanics, the error arising from an incorrect use of analogy between the ellipse of inertia in a plane and the ellipsoid of inertia in space.

E. C. PHILLIPS, *Secretary*

THE ANALYSIS SITUS OF THE PLANE WHEN THE DIRECTED LINE IS TAKEN AS ELEMENT[1]

By JESSE DOUGLAS[2]

In previous papers on the geometry based on the line as element,[3] the writer has made use of a certain representation of the oriented lines of the plane on the points of a cylinder, which he has since found had already been used by Blaschke.[4] In the present paper, this and related representations are employed to study the analysis situs of the plane regarded as a manifold of oriented lines.

[1] This paper is a development of one presented to the American Mathematical Society, Oct. 28, 1922. Extension to space is made in §§5, 6.

[2] National Research Fellow in Mathematics.

[3] Abstracts are in the Bulletin of the American Mathematical Society, vol. 26 (1920), p. 437; vol. 27 (1921), p. 398.

[4] *Über einige unendliche Gruppen von orientierten Berührungstransformationen in der Ebene*, Mathematische Annalen, vol. 69 (1910), pp. 204–217; and *Über die Laguerresche Geometrie orientierter Geraden in der Ebene*, Archiv der Mathematik und Physik, vol. 18 (1911), pp. 132–140.

We shall use the term *spear* (due to E. Study) as alternative to oriented line. Let the spears σ of the plane be referred to (Hessian) coordinates (ω, p) as follows. Regard the plane as xy-plane of a Cartesian coordinate system in space, and let τ denote the spear passing through the origin O, perpendicular to σ, and oriented relative to σ as the positive x to the positive y axis. If M is the intersection of τ with σ, we define p as the algebraic length of the directed segment OM. Thus p will be positive or negative according as σ indicates positive or negative rotation about O. ω shall denote the angle made by τ with a fixed spear α through O; ω is determined up to the addition of a multiple of 2π, and its trigonometric functions are therefore uniquely determined. Then we take as representing cylinder the one of equation $x^2+y^2=1$, and make correspond to σ the point $(x, y, z)=(\cos\omega, \sin\omega, p)$.

A *beam* of spears will be the name given to the totality of spears parallel, including sense, to a given one. It is evident that in the representation just defined the spears of a beam correspond to the points of an element of the cylinder.

We shall find that the analysis situs of the plane regarded as a manifold of spears depends essentially on the conventions made relative to elements at infinity.

1. Cylinder. Suppose we take only the lines at finite distance in the plane (metric plane) and make each into two spears by giving it its two possible

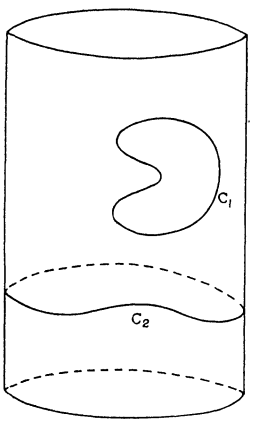


FIG. 1

orientations. The resulting spear manifold M_1 is by the above representation in one-one continuous correspondence with the points at finite distance of the cylinder. *The analysis situs of the metric plane of spears is identical with that of the metric cylinder regarded as a point manifold.*

The outstanding implication of this is that there are precisely two distinct types of simple, continuous, closed spear series in the plane non-equivalent as to one-one continuous transformation. These correspond respectively to the two cylindrical curves C_1 and C_2 in Fig. 1. Since it is characteristic of C_1 that upon it the total variation of ω ($\int_{c_1} d\omega$) is zero, while upon C_2 this total variation is 2π, we may distinguish the two types of closed spear series by saying that in the one the total rotation of the moving spear is zero, in the other, 2π. Any two simple, continuous, closed spear series both of the first or both of the second type are equivalent in the analysis situs sense, but never a series of the first type to one of the second. In illustration of the first type we may use Fig. 2, which, though open as a point locus, is closed as a spear locus. Typical for the second type is a pencil or oriented circle.

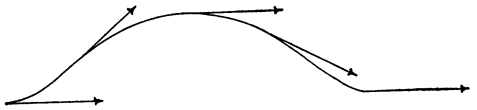


FIG. 2

As a second, curious, inference from our representation, we cite the following. Let a point move continuously so as to surround the cylinder twice and return to its original position. It is evident that it cannot do this without crossing its own path (Fig. 3). Translated to the plane, this signifies that if a spear move continuously through a total rotation of 4π in returning to its

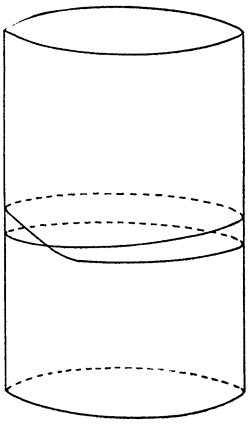


FIG. 3

original position, there must be some position which is occupied more than once by the moving spear. This is not nearly so clear to the intuition in the plane manifold of spears as it is in its equivalent form on the cylinder.

2. Projective cone. Imagine that we subject the space containing the cylinder to a projective transformation so that the cylinder becomes a cone.

The cylinder being regarded from the metric point of view, and the cone considered as including its conic at infinity through which it closes upon itself, the correspondence between the two surfaces is one to one and continuous, except that to the vertex of the cone there corresponds no point of the cylinder, and therefore no spear of the plane. But if a point on the cone is near to the vertex, it corresponds evidently to a spear at great distance from the origin O and oriented positively or negatively according as the point belongs to the one or the other nappe of the cone.

This impels us to adjoin to the spear manifold M_1 described in §1 an element at infinity in the form of the line at infinity of projective geometry, to which however, unlike the other lines, we attach no orientation. We define the neighborhood of a spear at finite distance in the usual way, and regard the neighborhood of the element at infinity as consisting of the spears of either positive or negative orientation as to O which are at great distance from O. The manifold M_2 so defined is homeomorphic with the projective cone, the line at infinity corresponding to the vertex of the cone.

A representation by means of a surface located altogether in the finite part of space is illustrated by Fig. 4.

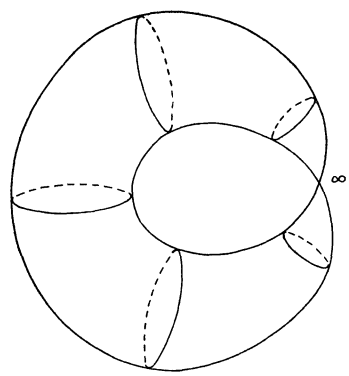


FIG. 4

3. Sphere. Consider the sphere $x^2+y^2+z^2=1$ in relation to the cylinder $x^2+y^2=1$. We can represent the one on the other by means of rectilinear rays through the origin. If we delete temporarily the north and south poles of the sphere, the correspondence is one-one and continuous. To points near the north pole correspond spears of great distance from O and positive orientation as to O; to points near the south pole spears of great distance from O and negative orientation as to O.

Thus, in order to obtain a correspondence which shall be without exception one-one and continuous, we must adjoin to the manifold M_1 of §1 two spears at infinity, one of positive, one of negative orientation as to the finite part of the

plane. This is done by using the projective plane and giving to the line at infinity as well as to every other line its two possible orientations. The analysis situs of the so constructed spear manifold is simply that of the sphere (genus zero).

4. Torus. The fourth and most interesting analysis situs is arrived at as follows. Start with a straight line and a circle tangent to it at a point T; then remove from the circle the point O diametrically opposite to T. Denoting the line regarded as metric line (no point at infinity) by λ, and the deleted circle by γ, we then set up a one-one continuous point correspondence between λ and γ by mating the points of λ and γ respectively which lie on the same straight line through O (Fig. 5).

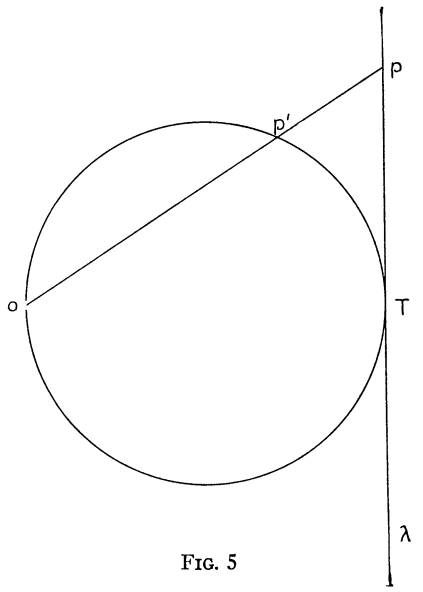


FIG. 5

If now the entire figure be revolved about an axis in its plane parallel to λ, and on the side of λ opposite to the circle, λ generates a cylinder, and γ a torus from which the outer equator has been removed; and we have furthermore a homeomorphism between this deleted torus and the cylinder, and therefore a homeomorphism between the deleted torus and the spear manifold M_1.

It is natural to restore to the torus its missing outer equator and to preserve the homeomorphic character of the correspondence between spears of the plane and points of the torus by inventing appropriate improper spears.

To this end, we recall that each beam of spears corresponds to an element of the cylinder, which corresponds in turn to a meridian of the torus, and that points near the missing equator correspond to spears at great distance from the origin. This indicates that we postulate in each beam a spear at infinity, and

regard the infinite spears of different beams as distinct. A neighborhood of such a spear at infinity will consist of those spears at great distance from the origin which differ little in direction from this spear at infinity.

The resulting manifold M_4 of spears is homeomorphic with the torus (genus one).

Since every continuous closed (not necessarily simple) curve on the torus is, if not reducible[1] to a point, equivalent[1] to a certain finite or discretely infinite number parallels together with a similar number of meridians, it follows that every continuous closed spear series, not of the type of Fig. 2, is equivalent to a discrete number of pencils together with a discrete number of beams.

As a preliminary to the extension to space to be made in §6, we remark that instead of the torus we may use the two-dimensional manifold of ordered point-pairs (P, Q)[2] of a circle as the homeomorphic equivalent of the spear manifold M_4.

Extension to Space.

5. Oriented planes or "pages." The preceding considerations may be extended to space in two ways, according as we regard space as a three-dimensional manifold of oriented planes or a four-dimensional manifold of spears.

Out of each plane we may construct two oriented planes, or, as we shall call them, *pages*, by distinguishing in a preferential way one of the sides of the plane from the other. The page will be said to *face* in the direction of the preferred side of the plane.

For page coordinates we adopt the following. Let a spear be drawn through the origin O of Cartesian coordinates perpendicular to the plane of the given page and oriented so that the page faces in the sense of the arrow-head. Let l, m, n, with $l^2+m^2+n^2=1$, denote the direction cosines of this spear, and p the plus or minus value of the directed segment described by a point moving along the spear from O to its piercing point with the plane. A homeomorphism between the pages of metric space and the points of the metric hypercylinder $x^2+y^2+z^2=1$ in four-dimensional space (x, y, z, w) is established by causing to correspond to the page (l, m, n, p) the point $(x, y, z, w)=(l, m, n, p)$.

As in two dimensions, we can replace the hypercylinder by a hypercone, a hypersphere, or a hypertorus, by the appropriate conventions as to elements at infinity.

6. Spears. Here, as before, the analysis situs depends entirely on the postulates adopted relative to elements at infinity. Of the many different types that

[1] Since we do not require the curve to be simple, we here understand "reducible" and "equivalent" in the sense of one-many continuous transformation.

[2] Q is allowed to be identical with P.

are possible,[1] we shall confine ourself to the one which results from the assumption that in each beam of spears there is a unique infinite spear, distinct from that of any other beam. It is to be remarked that this is quite different from the usual assumption of projective geometry. There a beam is simply a sheaf whose vertex is in the plane at infinity, so that, whether we deal with oriented or unoriented lines, there are evidently ∞^1 different elements at infinity in each beam. For example, in our convention there is exactly one spear at infinity directed vertically upward, and ∞^1 horizontal spears at infinity; whereas in projective geometry these two numbers are reversed.

It is readily shown that just as for the corresponding convention in the plane (§4) the analysis situs is that of the ordered point-pairs of a circle, here the analysis situs is that of the ordered point-pairs of a sphere.

To see this, let us consider all the spears of any given beam β. Having chosen a fixed sphere, center O, let us denote by P the point in which the spear belonging to β and passing through O pierces the sphere, in emerging from it. P will be the first point of the ordered point-pair corresponding to any spear σ of β.

The second point Q is determined as follows. Let π be the plane through O perpendicular to the direction of the beam β. We regard π as an inversion plane, and make correspond to each proper spear σ of β the point in which σ pierces π, and to the improper spear of β the unique infinite point of π. When we follow this representation of β on π by a stereographic projection of π on our fixed sphere, using P as pole, there is made to correspond, in a one-one continuous way, to each spear σ of β a point Q of the sphere. This Q shall be taken as second point of our ordered point-pair.

The correspondence $\sigma \rightarrow (P, Q)$ is evidently homeomorphic. The pairs (P, P) whose two points coalesce correspond to the improper spears of space.

A representation of our four-dimensional spear manifold by means of a four-spread in euclidean five-space can be had by starting with an (ordinary) two-sphere together with a non-intersecting two-flat, both located in the same three-flat of the five-space; then rotating the two-sphere about the two-flat freely in the five space (∞^2 positions). The manifold of points formed by the resulting four-spread, torus-like in a two-dimensional way,[2] is easily set into homeomorphism with the ordered point-pairs of a sphere, and therefore with the spears of space.

[1] Besides the one considered in this section there are the following: no elements at infinity; one such element; the lines at infinity of projective geometry, unoriented; the same, each given its two possible orientations.

[2] Not, of course, to be confused with the four-manifold (analogous in one higher dimension to the hypertorus of §5) which is obtained by rotating a three-sphere about a non-intersecting three-flat, both located in the same four-flat of our five-space. The resulting four-manifold may be said to be torus-like in a one-dimensional way (∞^1 positions of the moving three-sphere).

NAPIER'S LOGARITHMS AS HE DEVELOPED THEM

By W. D. CAIRNS, Oberlin College

In 1614 Napier published his *Mirifici Logarithmorum Canonis Descriptio* which incorporates his tables and in 1619 his *Mirifici Logarithmorum Canonis Constructio* which explains the method by which he formed his tables. Various editions and translations from the Latin originals are to be found, such as the combined edition of 1620, Edward Wright's 1616 translation of the *Descriptio*, the 1807 re-printing of the 1614 *Descriptio* by Maseres in the sixth volume of his *Scriptores Logarithmici*, and the 1889 translation of the *Constructio* by W. R. Macdonald. Reference may also be made to the facsimile reproduction of the title page and the first eleven pages of Wright's 1616 translation of the *Descriptio* in the Napier Tercentenary Memorial Volume (Longmans, Green and Co., 1915). Yet numerous colleges and universities have not been able to obtain material on Napier's logarithms suitable for their purposes; and it is the purpose of this paper to give the essentials of his treatment. The details of his computations throw an interesting light on the status of mathematical practice at the beginning of the seventeenth century. We leave it to the historians, however, to determine how much of this is due to Napier in particular.

Noting that he purposes to avoid all multiplications, divisions, and the more difficult extractions of roots and that his method depends on a correspondence between an arithmetic and a geometric progression, he proceeds to compute the latter as simply as possible, viz., by subtracting successively a simple fractional part. Thus, using 10^7 (in modern notation) as "radius," he takes the ratio in his "First Table" as .9999999, and one ten-millionth is subtracted successively one hundred times; the last number in this table, to the seven decimal places which he preserved, is 9999900.0004950. Replacing this last number by 9999900, the "Second Table" begins with 10^7, uses the ratio .99999 by subtracting one hundred-thousandth fifty times in succession; the last number in the table is 9995001.222927, a result slightly incorrect, for the decimal part should be .224804, as noted by Macdonald. Again, using 9995000 as "sufficiently near to" this last number, the "Third Table" begins with 10^7, uses the ratio .9995 by subtracting one two-thousandth twenty times in succession, the last number being 9900473.57808; this set of 21 numbers forms the first of 69 columns in this third table. The "easiest" number nearest to the last number, viz., 9900000, is taken as the first number in the second column, and in accordance with this, the first numbers in the 69 columns are formed, beginning with 10^7, by subtracting one hundredth 68 times in succession, that at the head of the 69th column being recorded by Napier as 5048858.8900. Next he fills in the other numbers in the 69 columns, row after row, by reducing those

in the first column by one-hundredth successively; for example, the last (twenty-first) row is

9900473.5780, 9801468.8423, 9703454.1539, · · · , 4998609.4034.

This last number, he notes, is practically half the radius, and he has thus computed a whole series of geometrical means between the radius and the half radius with the ratio .9995, and can insert 49 means between each pair of consecutive numbers by the numbers of the second table used as multipliers, and can further insert 99 means between the resulting numbers by means of the numbers of the first table. His choice of the base 10^7 is now justified by the fact that the numbers thus obtained differ consecutively by no more than unity and that the decimal parts may be neglected.

At this point in Napier's explanations he introduces his theory of logarithms, centering in the definition quoted here from the *Constructio*; the numbers of the sections are given for convenience in reference:

"26. The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine." The reader must recognize that to Napier the sine was a line, or the number measuring the line, as in the present-day line representation of functions of angles. Various theorems follow which are needed for the development of his tables, several of these being given here in substance.

27. Zero is the logarithm of the radius.

29. An upper and lower limit are derived from the definition in section 26. " · · · the given sine being subtracted from radius the less limit remains, and radius being multiplied into the less limit and the product divided by the given sine, the greater limit is produced · · · ."

33. By section 29 the first "proportional" of the first table, 9999999, has its logarithm between the limits 1.0000001 and 1.0000000, that of the second proportional, 9999998.0000001, between 2.0000002 and 2.0000000, etc. Hence these logarithms may be taken as 1.00000005, 2.00000010, etc.

36. "The logarithms of similarly proportioned sines are equi-different." This is proved by the definition of logarithms, also by a comparison of the upper and lower limits.

41. The method for finding the logarithms of sines or "natural numbers" not found in the first table will be shown clearly enough by his examples. Let the given sine be 9999975.5000000; the nearest sine in the table is the twenty-sixth number, 9999975.0000300, and is smaller than the given sine. By section 33 the limits of the logarithm of this last are 25.0000025 and 25.0000000. By the same method used in proving the rule in section 29, he proved that the difference of the logarithms of these two sines has the limits .49997122 and .49997124, whence to seven decimal places the required limits of the logarithm

are 24.5000313 and 24.5000288, and he uses 24.5000300. Similarly the limits for 9999900.0004950, the last sine in the first table, which are 100.0000100 and 100.0000000, yield 100.0005050 and 100.0004950 as the limits for the logarithm of 9999900, the second number in the second table, and from this the logarithms for all the other numbers in the second table are found by the rule for proportionals.

43. For the method of finding the logarithms of sines not found in the second table, an example will suffice. Let the given sine be 9995000, the second number of the first column in the third table. The nearest sine in the second table is the last, 9995001.222927, the limits for its logarithm being 5000.0252500 and 5000.0247500. He finds "a fourth proportional, which shall be to radius as the less of the given and table sines is to the greater." This he does "by multiplying the difference of the sines into radius, dividing this product by the greater sine, and subtracting the quotient from radius." By section 36 the logarithm of this fourth proportional differs from that of the radius by as much as the logarithms of the given and table sines differ from each other. He then finds the limits of the logarithm of the fourth proportional by aid of the first table, and adds them to or subtracts them from the limits of the logarithm of the table sine, to obtain the limits of the logarithm of the given sine. In this example, the fourth proportional is 9999998.7764614, and the limits of its logarithm, found by 41, are 1.2235387 and 1.2235386. Adding these to the former limits, he has 5001.2487888 and 5001.2482886 as the limits of the required logarithm, which is therefore taken (midway) between these, 5001.2485387.

44. From this the logarithms of all the other numbers in the first column of the third table are found by the rule for proportionals.

45, 46. It only remains to find by the same method the logarithm of 9900000, the first number in the second column, viz., 100503.3210291, after which the logarithms of all the numbers of the third table are found either by adding this last logarithm successively along any row, beginning with the first column, or by adding 5001.2485387 successively down each column. The logarithm of the last entry, 4998609.4034, is thus found to be 6934250.8007528.

47. The "Radical Table," next formed by Napier, merely enters all the numbers of the third table and their logarithms in 69 double columns, the "natural numbers" or sines to four decimal places, the logarithms to one; and from this he compiled his logarithm tables. For example, in getting the logarithm of 7489557, the nearest table sine being 7490786.6119, he finds the difference of these two to be 1229.6119. Multiply this by the radius and divide by "the easiest number," which may be either of the foregoing or the simpler 7490000, the result being 1640.1; this added to 2889111.7, the logarithm of the table sine, gives 2890752. Similarly for the logarithms of all sines within the limits of the radical table.

51, 52. By this same rule the logarithm of 5000000 is 6931469.22, of 8000000 is 2231434.68, and by a further use of the same rule the logarithm of 1000000 is 23025842.34.

54. These particular logarithms and the rule for proportionals give him the logarithms of all sines outside the table limits. For example, the logarithm of 378064, which is 1/20 of 7561280, a number in the limits of the table, is 32752756, the sum of 2795444.9 and 29957311.56, the logarithms of 7561280 and 20.

59. Napier's table is constructed in quite the same form as used at present, except that the second (sixth) column gives sines for the number of degrees indicated at the top (bottom) and of minutes in the first (seventh) column, the third (fifth) column gives the corresponding logarithm and the fourth column gives the "differentiae" between the logarithms in the third and fifth columns, these being therefore essentially logarithmic tangents or cotangents. Reproductions of some of these pages may be seen in Macdonald's translation and in Cajori's History of Mathematics.

Suffice it to say, in closing this note, that the second book of the *Descriptio*, "On the remarkable advantage (praeclaro usu) in trigonometry of the wonderful canon of logarithms," shows the solutions of a right triangle by logarithms in the manner familiar to us, the solutions of oblique triangles by the law of sines and of tangents, and of the case where three sides are given, the last by the expedient of drawing the altitude from the intersection of sides a and c and using an auxiliary isosceles triangle of which this is the altitude and side c is one of the equal sides; then follows the portion devoted to spherical triangles, wherein Napier uses his "circulares," known to us as Napier's circular parts, to solve quadrantal spherical triangles, and completes the theory of spherical triangles essentially in the way adopted at the present day.

ON THE ORIGIN OF THE TERM "ROOT." SECOND ARTICLE

By SOLOMON GANDZ, The Rabbi Isaac Elchanan Theological Seminary, New York City

In a recent number of this Monthly[1] the writer suggested that the term "root" goes back to the Arabic *jadhr*, a word originally meaning a concrete number designating a geometric magnitude, as contrasted with an abstract number. He further referred to some passages of the algebra of al-Khowârismî which clearly defined *jadhr* as the side of a square multiplied into a square unit, and proposed that *jadhr* should not be translated as "root" but rather as "basis" or "foundation."

[1] Vol. 33, pp. 261–265.

It is the purpose of this paper to inquire further into the same problem, to examine fully the algebra of al-Khowârismî, and to see how he understands this term *jadhr* in those numerous passages of his book in which the above definition cannot be applied. In connection with this special question the writer will attempt, in a short and necessarily incomplete sketch, to comment on the origin of the Arabic terminology of algebra in general.

An intensive study of al-Khowârismî's algebra shows that the Arabic mathematician does not use the word *jadhr* as a clear and unequivocal term, but that he combines with it different ideas and meanings. This fact was observed long ago by Rosen[1] in regard to the term *mâl* (square), and by the writer in regard to the term *Misâḥah*.[2] In the beginning of his book[3] we find the following definitions: "And I found that the numbers which are required in the operations of the algebra are of three classes, namely, *jadhr's*,[4] *mâl's*,[5] and the abstract number without reference to the *jadhr* or to the *mâl*. Of these the *jadhr* is any *shay'* which can be multiplied into itself, be it a unit, any number above a unit, or any fraction below it. The *mâl* is anything which is obtained as the result of the multiplication of the *jadhr* into itself. The abstract number is any number expressed in words[6] without reference to *jadhr* or *mâl*." The definition of the *jadhr* forming the basis of the other two definitions is rather obscure. It has two essential characteristics. There are the *shay'* and the *maḍrûb fi nafsihi*, "which can be multiplied into itself." Al-Khowârizmî himself does not tell us what the *shay'* is. But while he defines here the term *jadhr* by the term *shay'*, we find him in two other passages[7] defining the term *shay'* by the term *jadhr*, a procedure which, of course, does not make the definition any clearer. The same lack of clearness is seen in the second characteristic *maḍrûb fî nafsihi*, liable or capable of being "multiplied into itself."[8]

[1] *Algebra of Mohammed ben Musa*, English text, p. 50, note. See also Ruska, *Zur ältesten arabischen Algebra*, pp. 47 et seq.

[2] This Monthly, vol. 34, p. 80.

[3] Arabic text, p. 3; English text, p. 5 et seq. The writer's translation, however, differs considerably from that given by Rosen.

[4] In the text, *judhûr*, the Arabic plural of *jadhr*.

[5] In the text, *amwâl*, the Arabic plural of *mâl*.

[6] Pure numerals.

[7] Arabic text, p. 15, lines 4–5; p. 16, line 15; English text, p. 21, 23. "I shall now teach you how to multiply the things (*ashyâ'*), that is to say, the roots (*judhûr*)"; "Ten less thing, the signification of thing (*shay'*) being root (*jadhr*)."

[8] There is also a philological obscurity in the formulation of it. *Maḍrûb* is the Arabic passive participle, which admits two meanings: *Multiplicatus* "multiplied," and *multiplicandus*, "to be multiplied" or "capable of being multiplied." Now al-Khowârizmî uses the same word *maḍrûb* in the definition of the *jadhr* and of the *mâl*. The *mâl* is the result of the *jadhr almaḍrûb fî nafsihi*, "of the root multiplied into itself." We, of course, have to give the author the benefit of the best interpretation, and to say that, in the definition of the *jadhr*, *maḍrûb* means *multiplicandus*, "to be multiplied"; while in the

The quality of "liable or capable of being multiplied into itself" does not belong merely to the *jadhr* or *shay'*. No one will deny that an abstract number or a square is also capable of being multiplied in itself.

But we get more and better information when we turn to the later Arabic mathematicians. Omar al-Khayyâmî (*c.* 1100) introduces his algebra[1] with a clear and philosophical exposition of its fundamental principles and notions. He says: Algebra deals with two main classes of numbers, the absolute or abstract numbers and the concrete numbers. The concrete numbers are the geometric magnitudes and unknown quantities. It is customary to call the unknown quantity *shay'*, its square *mâl*, the third power "cube," etc. The first power is also called by al-Khayyâmî, *jadhr* and *ḍil'*. The same opinion is also represented by al-Karkhî (*c.* 1020) who refers to the algebraic powers as the *ajnâs al-majhûlât* or *marâtib al-majhûlât*[2] the classes or degrees of the unknown quantities. This theory is still more clearly stated in the arithmetic of al-Qalaṣâdî (*d.* 1477–1486) who says[3]: "Algebra is based upon three classes, namely, the [absolute] numbers, the things (*shay's*) and the squares (*mâl's*). Some people add also the cubes. The [absolute] number has no basis (*ass*), and the basis of the *shay's* is one, and the basis of the *mâl's* is two, and the basis of the cubes is three. And among these classes nothing is known save the number. *Shay'* and *jadhr* are the same thing, i.e., the name for the unknown quantity. Al-Qalaṣâdî was the commentator of Ibn al-Bannâ (*c.* 1222) and we find his theory in the latter's work[4]: The basis of the *shay'* is one, the basis of the *mâl* two, etc.[5] The same theory of the basis (*ass*) is also given in the *Muqaddamat* (Prolegomena) of Ibn Khaldûn (born 1332).[6]

This gives us the clue to the right understanding of al-Khowârizmî's definition of the *jadhr*. It is a *shay'*, a concrete number having a basis (x), and it can be multiplied into itself and produce a number of a higher power (x^2).[7] These are the two characteristics which distinguish the concrete numbers from the abstract numbers.

definition of the *mâl* it is *multiplicatus*, "multiplied." As we shall see later, however, a great deal of confusion was created by the use of this inaccurate language.

[1] Ed. Woepcke, Paris, 1851, Arabic text, p. 4 et seq.; French text, p. 5 seq.

[2] Woepcke, *Extrait du Fakhri*, p. 48.

[3] *Kashf al-asrâr'an'ilm ḥurûf al ghubâr*, translated by Woepcke, Rome, 1859, in an Extrait des Atti dell' Accademia Pontificia de' Nuovi Lincei, vol. 12, 1859. The Arabic manuscript was recently edited in lithograph copies, *sine anno et loco*. The passage occurs in the French text, p. 51; Arabic text, part 4, 3rd chapter on algebra; it is p. 36 of the unpaged text.

[4] *Talkhis*, translated into French by Aristide Marre, Extrait des Atti dell' Accademia Pontificia de' Nuovi Lincei, vol. 17, Rome, 1865, p. 30.

[5] He has also the rule that the quotient has as its exponent the difference between the exponents, and the product has as its exponent the sum of the exponents.

[6] Ed. Beirut, 1886, p. 422; ed. Paris, 1858, vol. 1, pp. 215-216; French translation by de Slane, p. 248.

[7] The pure number, say 4, if multiplied into itself, will produce only the number 16 of the same power.

Ruska[1] states that one who translates *mâl* as "square" has to take *jadhr* as the first power, which will lead to a very great difficulty in the terminology, because "eine 'erste' Potenz ist für Altertum and Mittelalter ein Unding." But the writer frankly owns that he does take *mâl* as the square, *jadhr* as the first power, and that this alleged "Unding" (absurdity) is, in his opinion, clearly stated in the works of the mathematicians al-Khowârizmî, al-Khayyâmî, al-Karkhî, Ibn al-Bannâ, Ibn Khaldûn, and al-Qalaṣâdî. Furthermore, the writer thinks that the very name *jadhr* had originally the meaning "basis," "foundation" to designate the first power of the concrete number, like the word *ass* (or *uss*) used by Ibn al-Bannâ, Ibn Khaldûn, and al-Qalaṣâdî which admits no other interpretation than "basis or foundation" and was used to indicate the exponent of the algebraic powers.[2] This term *ass* is much older than the Arabic authorities quoted above. It is found already in the Hebrew encyclopedia of knowledge written by Savasorda[3] (*c.* 1120) who cites it as a well-known Arabic term for the powers of ten in the decimal system, and translates it by the Hebrew *ush* belonging to the same root.[4] Ibn Ezra (*c.* 1150) calls it in Hebrew *yesôd* and *môsad*.[5] It goes back to the Greek πυθμένες, *pythmenes*, bases.[6]

There were two main characteristics in the science of algebra. The one consisted in the introduction of the method of equations to find the unknown quantity. This characteristic was given expression in the Arabic name *al-jabr wa-l-muqâbalah* which is the origin of our "algebra" and means "to set up an equation."[7] The second characteristic consisted in the introduction of operations with concrete numbers, or geometric magnitudes and algebraic powers, and this was emphasized by the Greek name "logistic" (λογιστική) which means a science dealing with sensible concrete things, not only with absolute numbers.[8] Hence the Arabic term *shay'* for the basic quantity, corresponding to Greek logos (λόγος) "thing," *res, cosa, coss*, Egyptian *hau*.[9] This conception alone leads us to understand why al-Khowârizmî introduced the *'adad al-mufrad*, the abstract number, and al-Khayyâmî, the *'adad al-muṭlaq*, the

[1] Loc. cit., p. 63.

[2] See Woepcke, translation of al-Qalasâdî, p. 10, note; p. 51, note 2; translation of a chapter of the *Prolegomena* of Ibn Khaldûn, p. 6, note 6. That *jadhr* is the first power was stated by Rosen, loc. cit., p. 6, note, and Woepcke, translation of al-Qalasâdî, p. 49, note 1.

[3] On this unpublished encyclopedia see Steinschneider, *Gesammelte Schriften*, pp. 388–404. Our quotation occurs *ibid.*, p. 394, note 10.

[4] The same form seems to occur also in the *Miqdash Me'aṭ* of Moses da Rieti, p. 14b. The passage reads: *Me'uqqab, meguddar we ishiyyi*, meaning a cube, square, and basic (number or magnitude). Read *ushiyyi* instead of *ishiyyi*. The *Miqdash Me'aṭ* is a Hebrew imitation of Dante's *Divina Commedia*, written about 1416.

[5] *Sefer ha-Eḥad*, p. 14; *Sefer ha-Mispar*, pp. 5, 6, 79; Steinschneider, loc. cit., p. 483.

[6] See Woepcke, translation of al-Qalasâdî, p. 51, note 2; Heath, Greek *Mathematics*, vol. 1, p. 55.

[7] See this Monthly, vol. 33, pp. 437 et seq.

[8] See Heath, *Greek Mathematics*, vol. 1, pp. 14 et seq.

[9] See Smith, *History of Mathematics*, vol. 2, p. 392 et seq.

absolute number, as contrasted with the *jadhr* and *mâl*. If *jadhr* and *mâl* were also only abstract roots and squares, as Ruska[1] and many others think, then there would be no reason to speak of the abstract number as being opposite and having no reference to *jadhr* or *mâl*.[2]

But the responsibility for the origin of this error must be ascribed, to a certain degree, to al-Khowârizmî, himself, and the inaccuracy of his terminology. The same term *jadhr* is very often used to mean the root of a number and as an equivalent of our modern symbol $\sqrt{}$. So, for instance, throughout the chapter "On addition and subtraction"[3] the term *jadhr* stands only for the root. The sentence: "Know that the root (*jadhr*) of two hundred minus ten, added to twenty minus the root of two hundred, is just ten," cannot be rendered otherwise than, as with Rosen,[4] by $20-\sqrt{200}+(\sqrt{200}-10)$. In the same meaning the term *jadhr* occurs in numerous other passages throughout the entire book.[5] The means of determining in what sense the term *jadhr* is used lies in the very nature of the two different ideas. The term *jadhr* meaning the root ($\sqrt{\ }$) has only some sense when applied in the *status constructus* (construct state); *i.e.*, in close connection and interdependence with another word, indicating the number whose root it is; as, for instance, "The root of 200, the root of the *mâl*, its root" etc.; but it can never stand in the *status absolutus* (absolute state), *i.e.*, alone by itself. This is the only way in which al-Khowârizmî distinguishes between the two fundamental meanings of the term *jadhr*. Standing alone it is the same as the *shay'* and denotes the unknown quantity of the first power (x).[6] Standing in the *status constructus*[7] it means the root of the following number.

So for instance is al-Khowârizmî's text of the equation,[8] dealt with by Ruska,[9] both puzzling and confusing. By applying the above mentioned test, however, it becomes clear. In the sentence *jadhr-mâl fa-jadhr-mâl illâ jadhr*

[1] Loc. cit., p. 64. Instead of squares he suggests "Zahlgrösse" for *mâl*.

[2] As a matter of fact, however, Ruska is not the only one to represent this opinion. It may suffice here to mention, as one instance, the medieval Hebrew mathematician Ibn Ezra (*c.* 1150). In the introduction to the chapter on arithmetical roots (*Sefer ha-Mispar*, ed. Silberberg, Hebrew text, p. 61; German text, p. 65; note 137 on p. 112) Ibn Ezra says: "All the numbers can be divided into three classes: the one (class) are roots, the second are squares, and the third are neither roots nor squares." Evidently this classification was taken from the algebra of al-Khowârizmî, but erroneously applied to pure arithmetic, where it is obviously meaningless and absurd.

[3] Arabic text, pp. 19–21; English text, pp. 27–31.

[4] Ibid., p. 27.

[5] P. 4, lines 2, 5, 6, 11, 14; p. 5, lines 10, 14, 15; p. 6, lines 8, 10, 18, 19; p. 7, lines 9–10, etc.

[6] Arabic text, p. 3, lines 10, 11, 13, 14, 16, 17; p. 4, lines 1, 4, 6, 8, 18, 19; p. 5, lines 1–3, 5, 7, 9, 11, 12, 15, 17, 19; p. 15, lines 4–5; p. 16, lines 15–16, etc.

[7] See note 5.

[8] Arabic text, p. 47 line 12 et seq.; English text, p. 66, line 17 et seq.

[9] Loc. cit., p. 65.

the first two words *jadhr* are in the *status constructus*[1] and denote the root $\surd$, while the third one stands in the *status absolutus* and means the first power (x). In transforming these words into algebraic symbols we shall have to set up the equation $\surd x^2+\surd(x^2-x)=2$ in the form $x+\surd(x^2-x)=2$ as Rosen did and not $\surd x+\surd(x-\surd x)=2$ as Ruska suggested.

It is true that, according to Ruska's theory, there is "eine vollständige Verwirrung des Textes,"[2] a complete confusion in the text of the two chapters "On Multiplication" and "On Addition and Subtraction";[3] for the chapter "On Multiplication" contains also the operations of addition and subtraction, while in the chapter "On Addition and Subtraction" the operations of multiplication and division are also given. There then follows a new chapter "On Division"[4] where we find also operations of multiplication. It seems, however, that the arrangement of the text is clear and justified. The confusion took place only in the headings of the chapters which were added by some ignorant scribe. The difference between the two main chapters does not consist in the difference between multiplication on the one side, and addition and subtraction on the other side, but it consists in the difference between the two kinds of *jadhr*, the one being a *shay'* (concrete number) and the other one being a root.[5]

It is thus clearly established that al-Khowârizmî connects two different ideas with the term *jadhr*, namely, the algebraic first power x and the abstract root of a number. We shall now attempt to explain why al-Khowârizmî introduced a new definition of the term *jadhr*, namely, the conception of the side multiplied into the square unit $(x \cdot 1^2)$, discussed in the writer's former article.

This new conception of the *jadhr* is to be found only in the chapter on geometry,[6] and while giving the geometric illustrations for the equations $x^2+21=10x$, and[7] $3x+4=x^2$. It is evident that, while trying geometrically to illustrate these equations, we have to represent x^2 by a square erected upon the side x, the number 21 as 21 square units, and $10x$ as ten rectangles having x for the length and 1 for the breadth $[10(x \cdot 1^2)]$. Otherwise, if 21 would represent only pure numbers, for example, and $10x$ only length-measures, there could be no equation. The same thing, of course, holds good for all quadratic equations in general. All the three members of such an equation must in reality be

[1] The hyphen between *jadhr* and *mâl* was added by the writer.

[2] Loc. cit., p. 25, note.

[3] Pp. 15–21; English text, pp. 21–31.

[4] Pp. 20–21; English text, pp. 29–31.

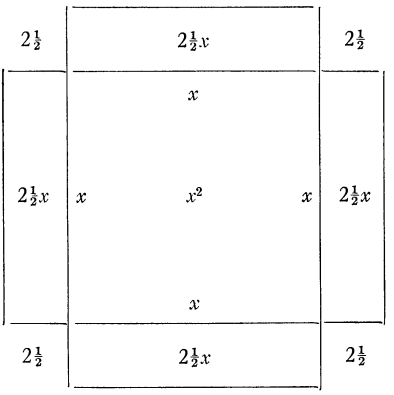
[5] That these headings were not in the original appears from the early Latin translations. Gherard of Cremona (Libri, *Histoire des sciences mathématiques*, vol. 1) has the headings *Capitulum multiplicationis* (p. 265), and *Capitulum aggregationis et diminutionis* (p. 269), but no chapter on division. Robert of Chester (ed. Karpinski) has entirely different titles.

[6] P. 51, lines 4–5; English text, p. 71.

[7] P. 11, line 11; p. 13, line 9 et seq.; English text, pp. 17–19.

conceived as square magnitudes in order to have a real equality, and this necessity is brought to our consciousness by trying to give geometric demonstrations for the equations.

Now it is certainly very strange that al-Khowârizmî does not give the same definition in the two geometric illustrations of the first quadratic equation $x^2+10x=39$.[1] It would seem that this can be explained only in a psychological way, namely, that al-Khowârizmî did not revise his work thoroughly and systematically enough. In the illustration of this first equation he still says that the *jadhr* is the side of the square, and $10x$ are ten sides of the square x^2. We have therefore, says al-Khowârizmî, to divide $10x$ into four parts, each part being $2\frac{1}{2}x$, and to add each part to one side of the original square x. The result is that we have four new rectangles added to each side, consisting of the length x and the breadth $2\frac{1}{2}$. Here the idea must have come into his mind



that if we assume $10x$ to be four rectangles having the length of x and the breadth of $2\frac{1}{2}$, or, as in the second demonstration, to be two rectangles of the length x and the breadth 5, then it can also be ten rectangles with the length x and the breadth 1; or, what is the same thing, the length x multiplied into the square unit. In this way he came to the perception that the *jadhr* cannot be a side, but rather a side multiplied into the square unit, and so he formed a new definition which he used in the two following demonstrations and in the chapter on geometry. But he did not care to rectify the definition given in the first demonstration,[2] that the side of a square is its *jadhr*.

As a consequence of this new geometric definition the square x^2 is not to be resolved into two equal factors $(x \cdot x)$, but into two different factors. The one x represents the *jadhr* $(x \cdot 1^2)$ and the other x represents the pure number a.[3] Or, as it was worded in the writer's previous article: "In order to get the area

[1] Pp. 8–11; English text, pp. 13–16.

[2] P. 8, last line; p. 9, line 1, 15; English text, pp. 13, 14.

[3] Arabic text, p. 13, lines 9–11; English text, p. 19.

of any figure we must first multiply the one side by the square unit; this is then the basis to be multiplied by the other side. We do not multiply one side by one side, but one *jadhr*, representing the square basis, by one side representing the pure number."

It is now interesting to find that this rather strange process of squaring the area is preserved in the Rhind Mathematical Papyrus as the oldest method of computing the area used by the Egyptians as early as 1650 B.C. Thus it is quite possible that in formulating this geometric definition of the *jadhr* al-Khowârizmî was prompted, not only by mathematical reasons, but also by old historical reminiscences of the Egyptian geometry.

The Egyptian unit of length in the measurement of land was the *Khet* of 100 cubits. The commonest unit of area was the *setat* or square *Khet* which contained 10,000 square cubits. For practical purposes in land measuring they used a unit called the cubit-of-land or the cubit-strip. It was a narrow strip of land 100 cubits long and one cubit broad. Smaller portions of a *setat* were expressed in such cubit-strips.[1] To get the area of a rectangle they sometimes multiplied its length in cubits by its width in *Khet*. This gave them the correct answer in cubit strips.[2] Still, more clearly we see in Problem 48 that for the multiplication of 8 *Khet* into 8 *Khet* and of 9 *Khet* into 9 *Khet* "the working actually written looks like the multiplication of 9 *setat* by the pure number 9."[3]

Thus the Egyptian computed the square by first multiplying the one side (9 *Khet*) in the square unit. This was the square-basis (*jadhr*) to be multiplied by the other side, representing the pure number. The same process is also described by al-Khowârizmî[4] who says: "And in every square figure one of its sides multiplied into the square unit is its *jadhr* and we make the other side, namely, *hj*, three, and this is the number of its *jadhr's*."

This archaistic way of computing the square finds its justification not only in "the peculiar Egyptian system of multiplication"[5] but in the very nature of primitive computation. The ancient Egyptians did not compute areas and measure their fields according to abstract rules. They originally found the area in an experimental practical way by taking a small square measure and trying out how many times it was contained in the field to be measured, "as we today measure cloth by the yard."[6] For this purpose the mere length-

[1] J. E. Peet, *Rhind Mathematical Papyrus*, London, 1923, p. 24 et seq. A. B. Chace, *Rhind Mathematical Papyrus*, New York, 1927, vols. 1 and 2, examined in proof sheets, chapter on "Measures of Area."

[2] Chace, loc. cit. Chace does not quote expressly the examples but he probably refers to Problem 49.

[3] Peet, loc. cit., p. 89.

[4] P. 13, line 9 et seq.; English text, p. 19; quoted also this Monthly, vol. 33, p. 263, note 4.

[5] Peet, loc. cit.

[6] Chace, loc. cit.

measure, the *Khet* or cord, could not be of any use. They had, therefore, to create the square-unit, the cubit-strip, being a *Khet* long and one cubit broad. This was the first *jadhr*, square-basis, used in practical life.

This cubit-strip might also, in all probability, be the underlying idea for the Egyptian conception of the square root. The idea of the square root existed in Egypt and the technical term for it was *Ḳnbt*, literally "corner" or "angle." Peet[1] has the explanation for that, that the length of each of the two sides of a square containing any corner of it was its square root. Yet it is not the pure, one-dimensional side which contains a "corner" but only the *jadhr*, the side multiplied with the square unit, contains a "corner." "Corner" or "angle" is the primitive word for a small square unit. Apollonius (*c.* 225 B.C.) defines the angle as the contracting of a surface at one point under a broken line.[2] Since the word "corner," "cornerstone" usually implies also the meaning "lowest part," "basis, foundation" we might see in the Egyptian *Ḳnbt* the source and origin of the Greek *pythmenes* (bases), the Arabic *ass*[3] and *jadhr*, the Hebrew *iqqâr* and *ush* and the Hindu *mûla*.[4]

Thus we are enabled to pursue the origin of our term "root" to an algebraical term denoting the concrete number with a basis, the first power (x) and to a geometric term denoting the basic square unit ($x \cdot 1^2$). All the three conceptions are contained in the Arabic *jadhr* and are preserved in the definitions of al-Khowârizmî.[5]

[1] Loc. cit., p. 20.

[2] See Smith, *History of Mathematics*, vol. 2, p. 277. Compare also the definition of Heron (*c.* 250 A.D.) quoted by Tropfke, vol. 4, 2nd ed., p. 25.

[3] Al-Qalsadi, quoted above, says that the concrete number (*shay'*) has an *ass*; *i.e.*, it contains a basis or "corner."

[4] See the Article of Datta "*On Mûla, the Hindu Term for Root*," in the October issue of this Monthly (vol. 34, pp. 420–423). The writer is gratified to see that his paper inspired the distinguished Hindu scholar Datta to cooperate on the same subject and to give us an authoritative account of what the Hindu sources say about the square root. Since the term *mûla* is found already in the works of Āryabhaṭa (499) the writer will modify his statement and say that the Arabic *jadhr* is possibly due to the influence of the Hindu *mûla*. This is also the opinion of Woepcke (*La propagation des chiffres Indiennes*, p. 68) and Ruska (loc. cit., p. 68). But in consideration of the fact that al-Khowârizmî was not acquainted with the methods of Hindu algebra (see Leon M. Rodet, *L'Algèbre d'Al-Khârizmî*, Extrait du Journal Asiatique, Paris 1878) we may also refer to the Hebrew *iqqâr*. The latter term occurs in the oldest Hebrew geometry, *Mishnath ha-Middoth*, which, according to Shapiro's and the writer's opinion, was written *c.* 200 of the Christian era, and which was almost verbally copied by al-Khowârizmî. On the other hand the writer's opinion is supported by Datta, who informs us that Āryabhaṭa had in view also the concrete concept of the side for the term *mûla*, and that the terms *mûla* and *pada* imply also the meaning of "basis, foundation." Furthermore, the term *pada*, "part," "portion," "square on chessboard," throws more light on the writer's theory concerning the origin of the terms *taksîr*, *thishboreth*, for area; see this Monthly, vol. 34, p. 80 et seq.

[5] The writer wishes to express his thanks to Professor David Eugene Smith for the opportunity of examining the manuscript of Dr. Datta's article and the proof-sheets of Dr. Chace's edition of the Rhind Papyrus before publication, and for his critical help in the final preparation of this article.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. NOTE ON THE QUATERNARY QUADRATIC IDENTITY OF HERMITE.

By E. T. BELL, California Institute of Technology

In a recent number of this Monthly,[1] C. G. Latimer showed that Hermite's important identity is implied by its apparently very special case, namely Euler's theorem on four squares. The form concerned is

$$x_0^2 + \sum_{i,j=1}^{3} A_{ij}x_ix_j, \tag{1}$$

where A_{ij} is the cofactor of a_{ij} in the determinant $|a_{ij}|$, $a_{ij}=a_{ji}$. Hermite's identity exhibits the repetition of this form under multiplication. In previous papers[2] I gave a method for generalizing such multiplication theorems by the introduction, through irrational transformations of the variables, of parameters into the coefficients of the given forms. If in any particular case this method leads to a form which could have been obtained from the original by a rational integral transformation of the variables, with rational integral coefficients, it is clear that the final multiplication theorem is no more general than the original. For example, if $f(x, y)$ repeats under multiplication, we learn nothing new if we replace (x, y) by (ax, by), where a, b are rational. That the irrational transformations mentioned do not always lead to a mere restatement of the original theorem, is shown by the generalization of the eight square and other identities in the papers cited. To prove, therefore, in any case that such generalization by irrational substitution is impossible is not trivial.

A straightforward application of the method mentioned, precisely as in the generalization of the foursquare theorem, readily shows that Hermite's form (1) does not admit of generalization by the introduction of parameters into the coefficients.

[1] Vol. 34 (1927), pp. 363–4.

[2] Annals of Mathematics, vol. 27 (1925), pp. 99–104; Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 71–80.

II. Note on Stirling's Numbers

Jekuthiel Ginsburg, Teachers College, Columbia University

In Mr. Bennett's beautiful solution of problem 3161 (in the February, 1927, issue of this Monthly, p. 100), there appears a set of numbers given by the relationship

$$_{i+1}U_j = (j+1)\,{}_iU_j + {}_iU_{j-1}.$$

From these numbers, the writer adds, a triangular array may be formed which is similar to Pascal's triangle.

It is to be regretted that the author did not tabulate the values of the U's in the triangular form referred to. The beauty of the solution would have been shown to greater advantage and at least some of the readers would have identified the U's as that extremely interesting and very important set of numbers which have attracted the attention of some of the best mathematical minds for the last two centuries[1] and which have been named by Professor Nielsen of the University of Copenhagen "The Stirling Numbers," in honor of the man who first discovered them and conceived their importance.

This identification greatly simplifies the task of obtaining solutions of the problem referred to, since tables of the Stirling Numbers have been compiled by various writers.

We would have then

$$\phi_0 = e^x\,; \quad \phi_1 = \frac{dx\phi_0}{dx} = (x\phi_0)' = e^x(x+1)\,; \quad \phi_2 = (x\phi_1)' = e^x(x^2+3x+1);$$

$$\phi_3 = (x\phi_2)' = e^x(x^3+6x^2+7x+1)\,; \quad \phi_4 = (x\phi_3)' = e^x(x^4+10x^3+25x^2+15x+1)\,;$$

and the triangular array would then be of the form

1
1, 1
1, 3, 1
1, 6, 7, 1
1, 10, 25, 15, 1
1, 15, 65, 90, 31, 1

The Stirling Numbers are characterized by many very beautiful properties and have an interesting history. They have been discovered and rediscovered, cropping out again and again under various disguises, in almost every branch of mathematics. Even some of those who considered them so important as to compile and publish extended tables did not follow up their ramifications in branches outside of the one in which they were particularly interested.

Stirling[2] devised them for use as a tool by means of which he could write

[1] The list includes, among others, the names of Euler, Vandermonde, Cauchy, Sylvester, Cayley, Crelle, Grunert, Boole, Schlömilch, Nielsen.

[2] *Methodus differentialis*, London (1730), p. 6.

any given polynomial in the form of a series of factorials,—an object which was considered very important until Weierstrass proved its futility.[1]

A mathematician addicted to factorials would write

$$x^2 = x(x-1) + x,$$
$$x^3 = x(x-1)(x-2) + 3x(x-1) + x,$$
$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x, \quad \text{etc.},$$

the coefficients being the original Stirling Numbers, which may be seen to be identical with those in Mr. Bennett's solution.

The next appearance of the Stirling numbers was in the theory of finite differences, as submultiples of the differences of zero. But neither Boole,[2] who used De Morgan's tables, nor Cayley,[3] who computed extended tables of his own, realized that these numbers are certain multiples of the numbers which Stirling was using for his purposes. (The proof of their identity is too extensive to be given here, but the fact may easily be verified.)

The theory of finite differences has a kindred relationship to the practice of interpolation, and this is the reason for the attention bestowed on the Stirling numbers by writers of textbooks on interpolation and by astronomical observers. A very extended table of Stirling numbers was compiled by the astronomer, N. P. Bertelsen and was published in Thiele's *Interpolationsrechnung*,[4] although the most extended table is probably Glaisher's.[5]

Another event in the history of the Stirling numbers is seen in their appearance[6] as factors of coefficients of the expansion of $(e^x-1)^n$ in terms of x and as coefficients in the formulas expressing the sum of equal powers of the series of natural numbers.

For example, Professor I. J. Schwatt has, in his *Introduction to Operation with Series* (Philadelphia, 1924), the following formulas (p. 88):

$$\sum_1^n k^2 = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3},$$
$$\sum_1^n k^3 = \binom{n}{1} + 7\binom{n}{2} + 12\binom{n}{3} + 6\binom{n}{4},$$
$$\sum_1^n k^4 = \binom{n}{1} + 15\binom{n}{2} + 50\binom{n}{3} + 60\binom{n}{4} + 24\binom{n}{5},$$

[1] Crelle's Journal, vol. 51 (1860).
[2] *Finite Differences*, p. 20. Cf. De Morgan, *Differential Calculus*, p. 253.
[3] Transactions of the Cambridge Philosophical Society, 13 (1873), pp. 1–5.
[4] Leipzig, (1909), pp. 31–2.
[5] Quarterly Journal of Pure and Applied Mathematics, vol. 31 (1900), pp. 27–28.
[6] Cauchy, *Exercices*, III (1828), p. 346; Schlömilch, Crelle's Journal, vol. 44, p. 344.

$$\sum_1^n k^5 = \binom{n}{1} + 31\binom{n}{2} + 180\binom{n}{3} + 390\binom{n}{4} + 360\binom{n}{5} + 120\binom{n}{6}.$$

A comparison with the coefficients in the table previously given will help identify the above as the Stirling numbers, multiplied by the corresponding factorials (2!, 3!, etc.)

But $\sum k^p$ plays a very important part in the study of Bernoulli numbers. Hence the intimate connection between the numbers of Bernoulli and those of Stirling. In fact there is a way to show that from at least one point of view the Bernoulli numbers may be regarded as a species of the Stirling numbers.

The interesting career of the Stirling numbers is too long to be compressed in a note of the kind presented here. It will suffice to call attention to what seems to be the most convenient mode of approach. Accepting Professor Nielsen's division of the Stirling numbers into two classes,[1] the Stirling numbers of the first class corresponding to a number n may be defined as the sums, S, of the products of the first n integers taken as factors 1, 2, 3, $\cdots$, p at a time without repetitions, while the Stirling numbers of the second kind corresponding to a number n are the sums, S', of the products of the first n integers taken as factors 1, 2, 3, $\cdots$, p at a time with repetitions. With the help of these definitions the Stirling numbers may be derived without recourse to the numerous functions, in the development of which they play such an important part.

It may also be of interest to note, that, for any n, the sum S of the products of the first n integers taken two at a time may be directly obtained by performing the algebraic division of the numerator of $(2+x)/(1-x)^5$ by the expanded denominator. A similar fraction for the sums of products S', with repetitions, is $(1+2x)/(1-x)^5$. In fact, by actual division,

$$(2 + x)/(1 - x)^5 = 2 + 11x + 35x^2 + 85x^3 + \cdots$$

and

$$(1 + 2x)/(1 - x)^5 = 1 + 7x + 25x^2 + 65x^3 + 140x^4 + \cdots.$$

Similarly for the case of the factors taken 3 at a time, the Stirling numbers of the first kind will be expressed by the coefficients of the expansion of

$$(6 + 8x + x^2)/(1 - x)^7 = 6 + 50x + 225x^2 + \cdots,$$

while the Stirling numbers of the second kind will be expressed by the coefficients of

$$(1 + 8x + 6x^2)/(1 - x)^7 = 1 + 15x + 90x^2 + 350x^3 + \cdots.$$

[1] *Theorie der Gamma Funktion* (1904), p. 65.

Similarly the values of ${}_nS_4$ and ${}_nS_4'$ will be given by the coefficients of the expansions of

$$(24 + 58x + 22x^2 + x^3)/(1 - x)^9 \text{ and } (1 + 22x + 55x^2 + 24x^3)/(1 - x)^9$$

respectively.

The above relations give the following formulas for the Stirling numbers or, what is the same, for the corresponding combinatorial products of the first n integers taken 2, 3, and 4 at a time without or with repetitions.

$$
\begin{aligned}
{}_nS_2 &= 2\binom{n+2}{4} + \binom{n+1}{4} \\
{}_nS_3 &= 6\binom{n+4}{6} + 8\binom{n+3}{6} + \binom{n+2}{6} \\
{}_nS_4 &= 24\binom{n+6}{8} + 58\binom{n+5}{8} + 22\binom{n+4}{8} + \binom{n+3}{8} \\
{}_nS_2' &= \binom{n+2}{4} + 2\binom{n+1}{4} \\
{}_nS_3' &= \binom{n+4}{6} + 8\binom{n+3}{6} + 6\binom{n+2}{6} \\
{}_nS_4' &= \binom{n+6}{8} + 22\binom{n+5}{8} + 58\binom{n+4}{8} + 24\binom{n+3}{8}
\end{aligned}
$$

The proof of the above formulas, which the writer believes to be new, and the extension to sums of higher order are beyond the scope of this article.

III. Beginning Geometry and College Entrance

By Ralph Beatley, Harvard Graduate School of Education

In 1923 the College Entrance Examination Board published its Document 108 embodying a detailed statement of the revised requirements in plane and solid geometry. Those who were charged with the preparation of this document gave heed also to a suggestion of the National Committee on Mathematical Requirements[1] with respect to the desirability of introducing the more elementary notions of solid geometry in connection with related ideas of plane geometry, and prepared accordingly the syllabus for Geometry *cd*, the so-called Minor Requirement in Plane and Solid Geometry. Document 108 states that "this

[1] See p. 37 of their report published in 1923 under the auspices of the Mathematical Association of America, Inc.

requirement is designed to cover the most important parts of plane and solid geometry, in such a way that the preparation for it can be completed in the time usually devoted to the standard requirement in plane geometry," *i.e.*, one year.

It is possible that those who prepared this syllabus did not gauge with sufficient accuracy the amount of material which it was proper to include in a one-year course of demonstrative geometry. If that is so, it should not be difficult to make a suitable adjustment when more evidence is at hand on which to base a judgment. The requirement should probably stand as it is for the time being. Certain schools are interested to give it a fair trial, adequate texts are available, and the colleges are not unsympathetic.

It is common knowledge that solid geometry is less widely taught than formerly. That is, a smaller proportion of students completing plane geometry go on to solid geometry than heretofore. This may simply be the result of the students' preference, freely expressed under elective systems of one sort or another. Even a teacher of mathematics, while not discounting the value of the three dimensional experience, might well question the desirability of devoting a year and a half to demonstrative geometry in the secondary school. No one knows the relative value of eight months of geometry as compared with five; nor is there any information available as to just how many months the student should demonstrate and argue about geometric abstractions before he attains in reasonable measure to those disciplinary and cultural aims which we like to think our teaching of geometry achieves. There does seem to be widespread agreement, however, that the student should not bring his geometric experience to a close without having considered the three dimensional aspects of the world about him. Even before the recent decline in the popularity of solid geometry, our instruction in the mathematics of three dimensions—in colleges as well as in secondary schools—suffered in comparison with that of European schools. We do not need to remain forever complacent about this unfavorable comparison, simply because it is now so familiar to us.

So long, however, as the requirements of the College Entrance Examination Board permit of a choice between Plane Geometry (*C*) and Plane and Solid Geometry, Minor Requirement (*cd*), and so long as human nature remains human, teachers of geometry in secondary schools will tend to continue the same old round and will be slow to follow the lead of those more adventurous spirits who are trying to give in the first—and, for most students, the only—year of geometry not only an adequate training in close reasoning and argumentation, but also an appreciation of the geometry of three dimensions. The number of candidates presenting themselves each year for examination in the minor requirement in plane and solid geometry (*cd*) is pitifully small, and there is good reason to believe that most of these take the examination by

mistake. It is not out of place to suggest that the proposal to include a mite of trigonometry, and another mite of logarithms, in the revised requirement of elementary algebra would be today in equally low repute if the new requirement were merely an optional substitute for the old.

Should then this new minor requirement in plane and solid geometry be made the sole requirement, and should the old examination in plane geometry be discontinued? No: that would not be fair. This new requirement in geometry differs from the old far more than was the case in algebra. Furthermore, it is not certain that it is not a bit too long.

Should the new requirement in geometry be withdrawn because so few seek to prepare for it? No: that too would not be fair. The schools have not had time to give it a fair trial. Even in the case of the trigonometry and logarithms in the algebra requirement, the schools—though meeting the requirement fairly well by now—can hardly be said to have had sufficient time to give a thorough appraisal of the new material and the results of teaching it. No: we need this new requirement constantly before us as an ideal to measure up to. It may possibly prove desirable to modify it very slightly. But to modify it so drastically as to make it tomorrow a fair requirement for all schools, and no longer optional, would cut the heart out of it.

Why not rather modify requirement *C*, making changes gradually over a period of years, and replacing items in plane geometry by items in solid geometry no faster than the traffic can stand? For a beginning, why not expect the candidate to be familiar with simple mensurational work in three dimensions, so that he could meet with confidence a question on the examination paper involving the total area of a house, or cylinder, or silo? Such a question would be necessarily one of the easy originals on the paper. This slight addition to the requirement could be easily balanced by the omission of a few unstarred theorems, or by some other equitable adjustment. In any case this proposal is not intended to make the present requirement easier. The effect of this change would be to shift very slightly the emphasis from 'plane geometry only' to 'plane and solid geometry.' The present ratio of book theorems to originals could—and probably should—stand. At any rate, the emphasis on originals should not be reduced.

This is admittedly only a feeble beginning. But even this little can be very significant. The mensurational work mentioned above has commonly been taught under the head of arithmetic in the eighth grade. It is now appropriately included in the geometry of the junior high school. As such its status is similar to that of the numerical trigonometry now included in the algebra requirement, or of the spelling and grammar implicit in the requirement in English.

Later it ought to be possible to add to the requirement certain book theorems from solid geometry: only a few, if they are to be starred theorems; a dozen or more, if unstarred. In the latter case one of these unstarred theorems could

appear in the role of an original on the examination paper, just as can now occur in the case of unstarred theorems in plane geometry. However the change is effected, there should of course be a suitable reduction in the plane geometry part of the requirement.

The experience of the past few years with the requirements as they now stand in Document 108 has indicated the desirability of a few minor changes in the starring of theorems, and the like. When such changes are made it would be a simple matter to incorporate in the present requirement *C* (Plane Geometry) some slight modification in keeping with the suggestion made above. The title of the requirement might then appropriately be changed to "Demonstrative Geometry."

There will always be room for improvements in this and other syllabi also. As these periodic revisions are made, the change in emphasis in Requirement *C* toward the goal now set for us in Requirement *cd* can be gradually and painlessly accomplished. "In view of the changes taking place at the present time in mathematical courses in secondary schools, and the fact that college entrance requirements should so soon as possible reflect desirable changes and assist in their adoption, the National Committee recommends[1] that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on college entrance requirements in mathematics, such a committee to work in close cooperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools."

At present there is little to encourage the schools to make the change in their teaching of geometry which has been recommended above. It is practically necessary that any such change be accompanied by a simultaneous change in college entrance requirements if it is to be generally effective. This recommendation of a gradual modification of Requirement *C* to approach in character Requirement *cd* has for its main purpose, therefore, the improvement of instruction in geometry in our secondary schools. This is the important thing, and the question of college entrance is raised merely as a means to that end. It would be a shame were we teachers of geometry to witness the gradual disappearance of the extra half year's work in three dimensional geometry from the curriculum without doing anything to save some remnant for the bulk of our students, who seem destined from now on to have but one year to devote to demonstrative geometry.

[1] On p. 48 of its report.

IV. On Zeros and Pseudo Zeros

By Luise Lange, Chicago, Illinois

The following discussion is meant to draw attention to a well-known though rather neglected point of mathematical practice, notably school practice. The name under which this point usually goes is that of "the consideration (or rather lack of consideration) of the limits of accuracy." The neglect of this point frequently leads to the calculation of results to a fictitious degree of accuracy, that is to an accuracy greater than is warranted by the data used in the calculation. Yet in thinking over what is really at the bottom of this pseudo-accuracy we became aware that the whole issue turns on one very simple fact which we wish to call *abuses of the zero.* School mathematics, from the grades to the colleges, has made it a habit to abuse the zero. This contention we shall now endeavor to establish in the hope of bringing about a discussion on the question whether it may not be worth the effort—which it would doubtless cost—to rid school mathematics of this habit.

To state our case we first have to say a few words about the difference in the meaning of the term "number" according to whether one has to do with abstract or with denominate numbers. Granting the abstract integer numbers, each element of the continuous set of abstract numbers (we may limit ourselves here to real numbers) is defined by certain algebraic or transcendental operations on the integers. These operations may be regarded as rules to compute the particular number exact to *any* number of decimal places. In the case of rational numbers every further place is known, once its period has been found; for irrational numbers each place has to be computed individually. But, and this is the point, every place is defined by the one rule and can be found if desired.

Denominate numbers, on the contrary, are defined by concrete operations on given physical objects. If this operation is that of *counting* discrete, finite sets of things, a one-one correspondence exists between the so defined denominate numbers and the abstract integers. If, however, the operation is that of *measuring* a continuous magnitude, the so defined denominate number corresponds, *not to any one definite abstract number, but to a whole range in the continuous set of real abstract numbers.* For the "definition" of a denominate number extends necessarily only to a finite number of places, for the following reasons:

(1). Physical magnitudes cannot be *defined* beyond a certain limit of accuracy. They all are reducible to the measurement of three kinds of quantities, length, time, and mass. Now a length is undefined inasmuch as its extremes are not mathematical points. The time of an event is undefined inasmuch as every event occupies a duration, no matter how small. And unless the exact

moment (watch the limitation on this notion) of the measurement be given, a length or a mass is undefined inasmuch as it is subject to minute variations in time. Thus to give the distance between a ship and a light house exact to 10^{-4} inches, the earth's radius exact to yards, the time of a flash of lightning to 10^{-6} seconds, the weight of a certain piece of iron to 10^{-10} grams, would not only be practically difficult, but *devoid* of meaning.

(2) Even within the scope in which the magnitude can be defined the subjective and objective limitations in the accuracy of measuring reduces every denominate number to an essentially limited number of places.

(3) In the majority of cases denominate numbers are given only to a *voluntarily limited number of places*, exact "to the nearest" inch, gram, etc. The reasons for this may be twofold: (a) there may not be any interest in knowing the magnitude more accurately for the particular purpose; (b) if the number is wanted, not for its own sake, but only in connection with others for the computation of some other magnitude, then the relative accuracy of the various measurements should be adjusted in realization of the fact that "a chain is no stronger than its weakest link."

Any denominate number then, being intrinsically determined and defined only to a limited number of places, corresponds, as said before, to a whole range in the continuous set of abstract numbers. All abstract numbers are equally well suited to represent that denominate number if they agree with it in the known places though differing from each other in higher places. The width of the range depends on the accuracy of the measurement; in particular, if the measurement is "accurate to the nearest unit u," the width of the range is in general $\pm .5u$, unless specific error limits are given.

Now it is by not keeping in mind this essential peculiarity of denominate numbers that what we called the abuse of the zero occurs. First in its milder form it is this: In writing denominate numbers the zero symbol is being used in a twofold rôle. Like the other digits it indicates the number of units in a place saying that they are nought (as when we write 1001 nights); but unlike the other digits it is also written instead of unknown (because non-defined or non-measured) places in order to impart the proper place value to those which are known. For example, in giving the velocity of light as 300,000 km/sec, not all of these zeros stand for true noughts; some of the last ones are mere "dummies" put in the place of unknown numbers to assure the proper place value to the 3. This ambiguity of the zero symbol would be avoided if one would generally adopt the practice of the "exact scientist" and express the place value, not by means of dummy zeros, but by powers of ten, writing digits only in those places for which they are actually known. For example, $c=3.0\times10^5$ km/sec would mean that the value of c is known between the

limits $2.95\times10^5<c<3.05\times10^5$, etc. Yet, according to the practice of many, this form would be as ambiguous as the other because of a second and worse form of the habit of abusing the zero, namely the habit of dropping or adding zeros at will behind the last given decimal place. For example, the statements that a certain distance is 15 feet, 15.0 feet, 15.000 feet are treated as equivalent and interchangeable. Most people, it is true, may never have written 15.000 feet for 15 feet, though surely many have written 15 feet instead of 15.000 feet; but nevertheless they have *treated* the 15 as a 15.000 if they ever gave as its logarithm the value 1.17609 instead $1.17\pm.01$. The fact however is that if a place is not given it is *unknown*, not *zero*. If it were known to be zero it should be given as such. To put a zero into an unknown place really amounts to a falsehood, to the pretension of knowing more than one does know. Moreover, when the zeros are continued to places to which the magnitude cannot even be *defined*, they become downright nonsensical.

To illustrate the above we shall next give a few samples of typical problems in which, according to the aforementioned habit, zeros are being abused so that results of a fictitious accuracy are obtained. We shall indicate by the symbol x each instance where in thought or in fact a zero is usually substituted for an unknown number; by bold face type, a numerical value obtained by operations involving pseudo-zeros; also by a dot under a number we indicate that its value is uncertain due to insufficient information on the following places.

(1). A school room is 32 feet long, 26 feet wide, and 12 feet high. How many cubic feet does it contain?

32	$8\dot{3}2$
26	12
192	$16\dot{6}4$
$64\dot{x}$	$8\dot{3}2$
$8\dot{3}2$	9984

The answer is $v=9.9\times10^3$ cubic feet $\pm$ several hundred cubic feet. The pseudo-exact answer would be $v=9984$ cubic feet.

(2) The same problem computed by means of logarithms.

$\log 32.xxx=1.50\dot{5}15$
$\log 26.xxx=1.4\dot{1}497$
$\log 12.xxx=1.0\dot{7}918$
$\log v=3.99\dot{9}30$

$v=9\dot{9}84.0$ cubic feet.

(3) A lamp post is 9.0 feet high; it is casting a shadow 5.2 feet long. Find the angle of elevation of the sun at this moment.

$$\tan e = \frac{9.0}{5.2}$$

$\log 9.0xxx = .95424$
$\log 5.2xxx = .71\dot{6}00$
$\log \tan e = .2\dot{3}824$

The pseudo-exact answer is $59° 58' 54''$.

$e = 60° \pm$ about $1°$.

This symbol x for the pseudo-zero, I believe, makes it quite obvious how the error in the original measurement affects the result. At the same time it also serves well to demonstrate to what extent school mathematics is dealing with pseudo-zeros and with pseudo-accurate results. And it is not only the unaware teacher who accepts or even urges these unreasonably numerous decimal places; the textbooks, fairly alike for all types of schools, encourage this habit, (a) by disregard for the relative accuracy of the given data in a problem, (b) by the pseudo-accurate anwers printed in the back of the book. This contention hardly needs any demonstration, however I might mention that the above example (3) was taken from one of the most widely used college text books, only with this correction: in the book the post is given as 9 feet not 9.0 feet high, its shadow 5.2 feet. Though this would impair the result still further, it is nevertheless given exact to seconds, that is a result exact to $3 \cdot 10^{-3}\%$ is found on the basis of measurements exact to only about 10%.

We further wish to make this point: Aside from being mathematically objectionable, this failure to treat the limited number of places in a denominate number as an essential feature is, in our opinion, also regrettable from the pedagogical point of view. (1) One thereby misses a valuable chance to make the student use his own judgement; for questions like "May we use the slide rule?" or "Do you want us to interpolate?" would be self-prohibitive if the students were trained to regard the required accuracy in a problem not as something extraneous, to be imposed by the authority of the teacher, but as determined by the given data and the purpose. (2) If one would persistently emphasize this point, one would incidentally develop the much sought-after "functional thinking"; for the student would continually realize that the errors in the measurement affect the result, and to what degree. (3) The faulty habits acquired in high school and college training are most likely to be carried over into professional practice where they are objectionable not only from a theoretical but from an extremely practical point of view; for there they lead to such concrete evils as false information, and wasted time and money. This is a contention which anybody who ever worked as an "exact scientist" in an industry can illustrate from his own experience.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED

BROAD, C. D. *Sir Isaac Newton.* (Annual lecture on a master mind, Henriette Hertz trust of the British Academy.) London, Oxford University Press, 1927. 32 pages. $0.70.

DAVIS, H. T. *A survey of Methods for the Inversion of Integrals of Volterra Type.* (Indiana University Studies, Numbers 76, 77.) Bloomington, Indiana University, 1927. 72 pages. $1.00.

RAMANUJAN, SRINIVASA. *Collected Papers.* Edited by G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson. Cambridge, University Press, 1927. xxxvi+355 pages.

REVIEWS

Mathematical and Physical Papers, **1903–13**. By BENJAMIN OSGOOD PEIRCE. Cambridge, Mass., Harvard University Press, 1926. viii+444 pages. Price $5.00.

This volume contains a collection of twenty one papers which were published by Professor Peirce during the last ten years of his life; it will do much to give him a lasting reputation for mathematical scholarship. The collection does not represent the whole of Peirce's scientific work, for the bibliography at the end of the book gives a list of two books, published by Ginn & Co., and fifty six papers. The papers selected for republication represent Peirce's best work and together form a treatise on an important branch of electricity and magnetism which is inadequately treated in many textbooks. The original papers are inaccessible to many workers in this field and this volume may be comparable in usefulness with a volume of Heaviside's electrical papers. The eighth paper in the book might very well be read first. It is the great memoir entitled "On the determination of the magnetic behavior of the finely divided core of an electromagnet while a steady current is being established in the exciting coil." It commences with a useful historical account of the development of the ordinary mathematical theory of two neighboring circuits and then passes on to the consideration of circuits which contain iron. Some problems relating to eddy currents are then solved with the aid of linear partial differential equations and Bessel functions. Peirce shows in this paper that he has made himself a master of the methods of dealing with the partial differential equations of mathematical physics.

Another paper which is of particular interest to the mathematician is the nineteenth, which is entitled "The effects of sudden changes in the inductances of electric circuits as illustrative of the absence of magnetic lag

and of the von Waltenhoven phenomenon in finely divided cores. Certain mechanical analogies of the electrical problems." The mechanical analogies described in this paper are quite pleasing. One, of Peirce's own invention, is made from three weighted roller skates and some cheap pulleys. This paper contains many interesting diagrams and formulas and, like the other, ends with an elaborate mathematical investigation involving Bessel functions.

The reader will perhaps be anxious to know what is meant by "the von Waltenhoven phenomenon." The answer is given very clearly at the beginning of the next paper from which we take the following quotation:

"If an increasing current I, ending in the maximum value I', be sent through a long solenoid, the final value of the magnetic moment of a solid bar of originally demagnetized soft iron or steel within the solenoid frequently depends, not only upon the strength of the current, but also upon the manner of growth of the current in attaining this intensity. The moment will be greater if the current be suddenly applied in full strength than if it be made to grow slowly, either continuously or by short steps. If, after the current has remained steady for a short time, at the strength I', it be made to decrease to zero, the residual moment of the bar will usually be less if the circuit be suddenly opened than if the decrease be made slowly, by gradually introducing more and more resistance, and the demagnetizing factor of a given cylinder is considerably less when computed from observations made by the method of sudden reversals than if it is determined by the slow step-by-step changes in the exciting current."

This phenomenon greatly interested Peirce and he made a series of experiments to determine the conditions under which anomalous magnetization appears and discussed the matter later theoretically.

The papers do not deal wholly with the growth of currents. There are some papers devoted to the design, use, and improvement of physical apparatus. For instance, the eleventh paper of the set discusses the theory of the ballistic galvanometer of long period, while the first paper is devoted to the design of chilled and seasoned cast-iron magnets for the production of the permanent fields of d'Arsonval galvanometers. The second, third, and fourteenth papers belong to the subject of vector analysis and are of interest to geometers. In the last of these papers Peirce introduces the idea of the derivative of one scalar point function with respect to another.

The book is beautifully printed and illustrated and has as its frontispiece a picture of the author.

H. BATEMAN

Einführung in die Axiomatik der Algebra. By H. BECK. Berlin and Leipzig, Walter de Gruyter & Co., 1926. x+197 pages.

back of the book are more than usually adequate for class-room use. They include a six place table of logarithms, with proportional parts, for numbers from one to ten thousand. Answers are given to most of the problems but not to all of them.

The authors' policy of meting out large doses of the theory in one chapter and following this by a chapter consisting only of applications can be criticized from the pedagogical standpoint. However, this is perhaps more than compensated for by their frequent use of graphs and by the fact that many of the conventional formulae which must abound in such a treatise are followed by theorems which state in words the import of the formulae. Moreover the number of formulae to be memorized is somewhat reduced, it being shown explicitly that many auxiliary formulae are simply special cases of a general formula.

The statements and proofs of the theorems are precise and the wording of the problems leaves no room for ambiguity. The brief treatment of the theory of probability is rather better than the usual treatment in such texts.

B. F. KIMBALL

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin of at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3305. Proposed by Paul Wernicke, Washington, D.C.

(a) Prove that x^2+y^2, where x and y are integers, cannot be the square of an integer, unless x or y is divisible by 3.

(b) Modify the theorem for the case that both x and y are divisible by 3.

(c) Generalize the theorem so as to cover other exponents thereby proving a part of Fermat's greater theorem.

3306. Proposed by H. A. Mangan, Vandalia Public Schools, Mo.

A man bought a horse for A dollars and agreed to pay interest and principal in n equal monthly installments, interest r percent per annum. Show how to compute r and n.

As a practical example, a certain loan company will lend \$1200 to be repaid in 120 monthly installments of \$12.50 each. What is the rate of interest? Show how to compute it.

3307. Proposed by C. N. Mills, Normal, Illinois.

A pyramid whose faces are equilateral triangles and whose base is a square stands on a horizontal plane and faces the cardinal points. If α is the sun's altitude and β the distance from the meridian and θ the vertical angle of the shadow prove that

$$\tan\theta = \tan 2\alpha \cos(\tfrac{1}{4}\pi - \beta).$$

3308. Proposed by Victor d'Unger, Little Rock, Arkansas.

Given an ellipse, $a^{-2}x^2+b^{-2}y^2=1$, where a and b are the semi-axes; it is required to erect n perpendiculars to the major axis such that they intercept equal lengths of arc on the ellipse. What will be the distances between these perpendiculars along the major axis?

3309. Proposed by Otto Dunkel, Washington University.

If (1, 2, 3, 4, 5, 6) denotes the determinant of the 6th order whose ith row is

$$x_i^2,\ x_iy_i,\ y_i^2,\ x_i,\ y_i,\ 1, \quad \text{and if } (i,j,k) = \begin{vmatrix} x_i, & y_i, & 1 \\ x_j, & x_j, & 1 \\ x_k, & y_k, & 1 \end{vmatrix},$$

show that

$$(1,2,3,4,5,6) = (6,1,2)(2,3,4)(4,5,6)(1,3,5) - (1,2,3)(3,4,5)(5,6,1)(2,4,6).$$

is an identity in the 12 independent variables x_i, y_i.

UNSOLVED PROBLEMS

Readers are requested to send in solutions for the following unsolved problems which were proposed in the years 1918, 1919, 1920. The number of each problem is printed in bold face type, with the page number following.

1918

2663, 19; **2686**, 118; **2695**, 170; **2713**, 259; **2717**, 260; (1925, p. 316); **2721**, 303; **2725**, 303; **2727**, 396; **2730**, 397.

1919

2739, 35; **2743**, 36; **2762**, 170 (1925, p. 434); **2764**, 171; **2775**, 213 (1925, p. 520); **2778**, 268; **2786**,366; **2787**, 366; **2788**, 413; **2798**, 458. Note. Problem 2764 is incorrectly numbered 2740.

1920

2807, 80; **2808**, 80; **2810**, 81; **2817**, 134; **2821**, 134; (1925, p. 481); **2826**, 186; **2837**, 273; **2844**, 326, **2847**, 327; **2848**, 327; **2849**, 327; **2854**, 377; **2856**, 377; **2867**, 482.

SOLUTIONS

3244 [1927, 156]. Proposed by V. M. Spunar, Chicago, Illinois.

Divide a triangle by two straight lines into three parts, which when properly arranged shall form a parallelogram whose angles are of given magnitudes.

SOLUTION BY HARRY C. BRADLEY, Mass. Inst. of Technology

Divide the triangle into two parts by a line parallel to the base and bisecting each of the other two sides. Rotate the triangular piece cut off through 180°, and place against the quadrilateral piece so as to form a parallelogram. Make a cut across the quadrilateral piece at the required angle with the sides of this parallelogram. Transfer the piece cut off from one end of the parallelogram to the other, thus forming a second parallelogram having the required angles.

With certain triangles and given angles a solution in three pieces may not always be possible, since the second cut must be confined wholly within the quadrilateral piece left by the first cut. But there are, in general, three chances, as any side of the given triangle may be taken as the base.

Also solved by ALICE ANN GRANT, MICHAEL GOLDBERG, and A. PELLETIER.

3245 [1927, 156]. Proposed by A. A. Bennett, Lehigh University.

Let $r=f(x+y+z)$, $s=f(x-y-z)$, $t=f(-x+y-z)$, $u=f(-x-y+z)$. Obtain in symmetrical form with as little computation as possible the algebraic identity in r,s,t,u, for each of the three cases, $f(x) \equiv \sin x$, $\cos x$, $\tan x$.

SOLUTION BY J. A. BULLARD, U. S. Naval Academy

Since the sum of the four angles is zero, we have, in particular,

$$\sin[(x+y+z)+(x-y-z)] = -\sin[(-x+y-z)+(-x-y+z)],$$

which on expansion gives

$$r(1-s^2)^{1/2}+s(1-r^2)^{1/2}+t(1-u^2)^{1/2}+u(1-t^2)^{1/2}=0,$$

for $f(x)=\sin x$. Similarly,

$$s(1-t^2)^{1/2}+t(1-s^2)^{1/2}+r(1-u^2)^{1/2}+u(1-r^2)^{1/2}=0,$$
$$s(1-u^2)^{1/2}+u(1-s^2)^{1/2}+r(1-t^2)^{1/2}+t(1-r^2)^{1/2}=0.$$

Adding these equations we have

$$(1-r^2)^{1/2}(s+u+t)+(1-s^2)^{1/2}(r+u+t)+(1-t^2)^{1/2}(r+s+u)+(1-u^2)^{1/2}(r+s+t)=0.$$

If $f(x)=\cos x$, we have similarly

$$rs-(1-r^2)^{1/2}(1-s^2)^{1/2}=tu-(1-t^2)^{1/2}(1-u^2)^{1/2},$$
$$st-(1-s^2)^{1/2}(1-t^2)^{1/2}=ru-(1-r^2)^{1/2}(1-u^2)^{1/2},$$
$$su-(1-s^2)^{1/2}(1-u^2)^{1/2}=rt-(1-r^2)^{1/2}(1-t^2)^{1/2}.$$

If we rearrange terms, square, and add, we get

$$3(r^2+s^2+t^2+u^2)-6rstu \pm 6[(1-r^2)(1-s^2)(1-t^2)(1-u^2)]^{1/2}=6,$$

where the square root has the sign of the product of the sines of the angles involved.
Again, if $f(x)=\tan x$,

$$f[(x+y+z)+(x-y-z)] = -f[(-x+y-z)+(-x+y+z)],$$

and

$$(r+s)/(1-rs) = -(t+u)/(1-tu),$$

or $r+s+t+u=rst+rsu+rtu+stu$.

Also solved by MICHAEL GOLDBERG.

3246 [1927, 156]. Proposed by Daniel Kreth, Wellman, Iowa.

From a point within a square, lines drawn to three of the vertices are 30, 40, and 50. Required the side of the square. A geometrical solution is desired.

SOLUTION BY SMITH D. TURNER, Cambridge, Mass.

Let x be the side of the square with vertices A, C, D, B in counter clockwise order, and P a point such that $PA=a$, $PB=b$, $PC=c$ are given distances. Rotate in the plane of the figure the triangle APB through 270° about A and thus obtain a quadrilateral $PAP'C$ with $\angle PAP'=90°$ and the diagonal $PP'=a\sqrt{2}$. The second diagonal x will now be determined. The area A of a quadrilateral with sides d, e, f, g, in order, and diagonals u, v, is given by

$$16A^2 = 4u^2v^2-(d^2+f^2-e^2-g^2)^2.*$$

* See the article *The Area of a Quadrilateral* by R. C. Archibald in this Monthly, vol. 29 (1922), p. 32, for references to this formula.

Applying this to $PAP'C=\Delta PAP'+\Delta P'CP$, we have

$$[s(s-a\sqrt{2})(s-b)(s-c)]^{1/2}\pm\frac{a^2}{2}=\frac{1}{4}[8a^2x^2-b^4-c^4+2b^2c^2]^{1/2},$$

where $s=(a\sqrt{2}+b+c)/2$. Hence

$$x=\left[\frac{b^2+c^2\pm(4a^2b^2+4a^2c^2+2b^2c^2-4a^4-b^4-c^4)^{1/2}}{2}\right]^{1/2},$$

where the + sign is used when P is within the square, and the − sign when P is without. For $a=30$, $b=40$, $c=50$,

$$x=10\left(\frac{41\pm\sqrt{1071}}{2}\right)^{1/2}=60.7,\quad 20.4.$$

Also solved by NATHAN ALTSHILLER-COURT, H. C. BRADLEY, S. A. COREY, L. R. CHASE, L. A. DYE, MICHAEL GOLDBERG, DANIEL KRETH, R. M. MCFARLAND, A. PELLETIER, and PAUL WERNICKE.

3248 [1927, 156]. Proposed by F. A. Foraker, University of Pittsburgh.

Construct a triangle, given the sum of two sides, the third side, and the bisector of the angle opposite the third side.

SOLUTION BY PAUL WERNICKE, Washington, D. C.

In triangle ABC let the sides a, b, c be divided by the bisectors of the angles, t_a, t_b, t_c into the parts a', a''; b', b''; c', c'', respectively, the arrangement being counterclockwise throughout.

GIVEN: a, $b+c$, t_a.

ANALYSIS: We know that $b:a'=(b+c):a$. Calling F the foot of t_a, bisector t_c in triangle ACF will meet the side $t_a=AF$ in a point M dividing AF in the known ratio $b:a'=b+c:a$. This determines the intersection, M, of the bisectors t_a, t_b, t_c.

CONSTRUCTION: On a line lay off $AF=t_a$. Divide AF internally at M and externally at N in the ratio $b+c:a$. Draw a circle with the diameter MN. This is a locus of both B and C. Through F draw a chord $BC=a$. ABC will be the required triangle, to be obtained, of course, in two positions.

DELIMITATION: In order that a may be a chord through F, we must have

$$\frac{2a(b+c)t_a}{(b+c)^2-a^2}>a>\frac{2at_a}{[(b+c)^2-a^2)]^{1/2}},$$

the greater quantity on the left being the diameter MN, and the smaller one on the right, a chord perpendicular to MN at F. We may write,

$$2(b+c)t_a>(b+c)^2-a^2>2t_a[(b+c)^2-a^2]^{1/2}$$

Also solved by NATHAN ALTSHILLER-COURT, THEODORE BENNETT, A. PELLETIER, and J. Q. MCNATT.

3249[1927, 156]. Proposed by Otto Dunkel, Washington University.

If $i!f_i=d^if(x)/dx^i$ and

$$D=\begin{vmatrix} f_r & f_{r+1}\cdots f_{r+n-1}\\ f_{r+1} & f_{r+2}\cdots f_{r+n}\\ \vdots & \vdots\quad\vdots\\ f_{r+n-1} & f_{r+n}\cdots f_{r+2n-2}\end{vmatrix},\quad\text{prove that}\quad\frac{dD}{dx}=(r+2n-1)D',$$

where D' is the determinant D with the subscripts of the last row increased by unity.

SOLUTION

Set $a_{ij}=f_{r+i+j-2}$, $(j=1, 2, \cdots, n,\ i=1, 2, \cdots, n+1)$.

Then $D=\sum_{j=1}^{j=n} a_{ij}A_{ij}$ $(i=1, 2, \cdots, n)$ where A_{ij} denotes the cofactor of a_{ij} in D; also $da_{ij}/dx=(r+i+j-1)a_{i+1,j}$. Hence

$$\frac{dD}{dx}=\sum_{i=1}^{i=n}\left[\sum_{j=1}^{j=n}(r+i+j-1)a_{i+1,j}A_{ij}\right].$$

Subtract from each bracket above

$$(r+2i-1)\sum_{i=1}^{i=n}a_{i+1,j}A_{ij}=0, \quad i=1,2,\cdots,n-1$$

$$=(r+2n-1)D', \quad i=n,$$

and there results

$$\frac{dD}{dx}=\sum_{i=1}^{i=n}\sum_{j=1}^{j=n}(j-i)a_{i+1,j}A_{ij}+(r+2n-1)D'$$

$$=(r+2n-1)D',$$

for to each term in the double sum there is a corresponding term $(i-j)a_{j+1,i}A_{ji}$, except where $i=j$, and the two cancel since $a_{i+1,j}=a_{j+1,i}$ and $A_{ij}=A_{ji}$.

Also solved by RICHARD S. BURINGTON and PAUL WERNICKE.

3252[1927, 217]. Proposed by the late Laenas G. Weld.

Find the equation of the plane curve along which a point, moving with a velocity in constant ratio to its ordinate, will pass from (x_1, y_1) to (x_2, y_2) in the least possible time.

SOLUTION BY RAYMOND GARVER, University of Rochester.

The integral to be minimized is obviously $I=\int_{x_1}^{x_2} f dx$, $f=(1+y'^2)^{1/2}y^{-1}$. We must have $y\neq 0$, and may agree to take y positive. To determine the extremals for this problem we may apply the condition $f-y'f_{y'}=$constant, (where subscripts represent formal partial differentiation), which follows from Euler's differential equation when f does not contain x explicitly, and when a second derivative exists along the minimizing arc. (See Bliss, *Calculus of Variations*, page 48). That a second derivative does exist in this case may be shown by a method exactly like that used by Bliss in his discussion of the brachistochrone problem (Bliss, page 50).

Since $f_{y'}=y'y^{-1}(1+y'^2)^{-1/2}$, the above condition reduces to

$$y^{-1}(1+y'^2)^{1/2}-y'^2y^{-1}(1+y'^2)^{-1/2}=y^{-1}(1+y'^2)^{-1/2}=c^{-1}$$

and the solution is $(x+k)^2+y^2=c^2$. The extremals are thus a two-parameter family of semi-circles (since $y>0$), with centers on the x-axis. One and only one circle of this family will pass through (x_1, y_1) and (x_2, y_2), if $x_1\neq x_2$ and this actually will be the minimizing arc. (If $x_1=x_2$, the minimizing arc will be a straight line.) To complete the discussion, sufficiency conditions must be applied. These are relatively simple when the problem is "regular." (See Bliss, pages 154 and 159). The Jacobi condition is satisfied here; the test given in Bliss, 149, may be applied. And if we cover the half-plane in which y is positive by the one-parameter family of extremals obtained by putting $k=k_1$, we have a field (in the Weierstrass sense) in which $f_{y'y'}=y^{-1}(1+y'^2)^{-3/2}>0$ for $y>0$. These form sufficient conditions that the semi-circle actually be a minimizing arc.

Also solved by MICHAEL GOLDBERG and F. L. WILMER.

3254[1927, 217]. Proposed by C. M. Sparrow, Rouss Physical Laboratory, University of Virginia.

ABC is a triangle with angles α, β, γ. Pairs of lines are drawn through the vertices, symmetrical with respect to the bisectors and making angles $\lambda\alpha$, $\lambda\beta$, $\lambda\gamma$ with the sides (λ any real multiple). The two

lines adjacent to any side determine a point. Prove that the three points so determined form a triangle which is perspective to the original triangle.

SOLUTION BY PAUL WERNICKE, Washington, D.C.

Over the base BC we have the given triangle with the sides $2R \sin C$, $2R \sin B$, where R is the radius of the circumscribed circle, and a second triangle $A'BC$ with the sides $2R' \sin \lambda C$, $2R' \sin \lambda B$. Taking the origin at B and the x-axis along BC, we have for the coördinates of A, ($2R \sin C \cos B$, $2R \sin C \sin B$) and for A', ($2R' \sin \lambda C \cos \lambda B$, $2R' \sin \lambda C \sin \lambda B$). Writing the equation of the straight line AA' in determinant form and then setting $y=0$, we obtain the intercept BA_1 from

$$\begin{vmatrix} x & 2R \sin C \cos B & 2R' \sin \lambda C \cos \lambda B \\ 0 & 2R \sin C \sin B & 2R' \sin \lambda C \sin \lambda B \\ 1 & 1 & 1 \end{vmatrix} = 0$$

or

$$BA_1 (R \sin C \sin B - R' \sin \lambda\, C \sin \lambda\, B) = 2\, RR' \sin C \sin \lambda\, C \sin (1 - \lambda) B\,.$$

If in this result we interchange C and B we obtain A_1C. Hence

$$BA_1/A_1C = \sin C \sin \lambda C \sin (1 - \lambda)B / \sin B \sin \lambda B \sin (1 - \lambda)C.$$

Multiplying the three ratios thus obtained for the three sides, we evidently obtain unity for the product. Hence by Ceva's theorem the three lines AA', BB', CC' are concurrent.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Miss ALICE K. BELL is studying at the University of Michigan while on leave of absence from the State Teachers College, Fresno, California.

Professor C. M. CONWELL of Albany State College for Teachers has resigned to accept a mastership at St. Paul's School, Concord, N.H.

Mr. B. F. DOSTAL of the University of Michigan has been appointed assistant professor of mathematics at the University of Florida.

Dr. LESTER S. HILL has been appointed assistant professor of mathematics at Hunter College.

Professor R. L. JEFFREY of Acadia University is completing work for his doctorate at Cornell University.

Professor GERALD E. MOORE of Hanover College is completing work for his doctorate at the University of Illinois.

Miss MARGUERITE REINERT, formerly instructor at the University of California at Los Angeles, is now instructor at Pasadena Junior College.

Dr. GLENN F. ROUSE of Lehigh University has been appointed assistant professor of physics and mathematics at American University. Professor F. SHENTON is chairman of the department of mathematics.

Associate Professor LAO G. SIMONS has been promoted to a full professorship and head of the department of mathematics at Hunter College.

Mr. C. W. STROM of Luther College returned this year from three years of study at Oxford University.

Dr. D. V. WIDDER has been appointed associate professor of mathematics at Bryn Mawr College.

Professor F. E. WOOD of Northwestern University is spending the year in research in Bologna and Rome.

Dr. H. S. WALL and Mr. K. G. FULLER have been appointed instructors in mathematics at Northwestern University.

The following courses in mathematics are announced for the summer of 1928:

University of Chicago, first term, June 18 to July 25; second term, July 26 to August 31. In addition to the usual courses in trigonometry, plane analytical geometry and differential and integral calculus, the following advanced courses are announced. By Professor L. E. DICKSON: Topics in the theory of numbers. By Professor E. T. BELL: Theory of functions of a complex variable; Applications of analysis to the arithmetic of quadratic forms. By Professor E. P. LANE: Synthetic projective geometry; Projective differential geometry I. By Professor F. D. MURNAGHAN: Modern hydrodynamical theory. By Professor M. I. LOGSDON: Solid analytic geometry; Higher plane curves. By Professor L. M. GRAVES: Modern theories of integration. By Professor R. W. BARNARD: Advanced calculus; Introduction to higher algebra. By Professor WALTER BARTKY: Introduction to celestial mechanics II. By Dr. F. R. BAMFORTH: Differential equations.

University of Colorado, first term, June 18 to July 21; second term, July 23 to August 24. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, calculus, the following courses will be offered. First term—By Professor LIGHT: Teacher's course in mathematics; History of mathematics; Calculus of variations. By Professor KEMPNER: Advanced teacher's course in mathematics; Differential equations; Group theory and introduction to Galois theory. Second term—By Professor LIGHT: Statistics; Theory of equations; Calculus of variations (continued). By Professor KEMPNER; Advanced teacher's course (repeated); Differential equations (continued); Group theory and introduction to Galois theory (continued).

Columbia University, July 9 to August 17. In addition to courses in trigonometry, solid geometry, college algebra, analytic geometry, and calculus, and a series of courses for teachers of secondary mathematics, the following advanced courses are offered. By Professor W. B. FITE: Differential equations; Theory of

infinite series. By Professor J. F. RITT: Theory of functions of a complex variable. By Professor K. W. LAMSON: Differential geometry. By Dr. B. O. KOOPMAN: Introduction to modern geometry.

University of Illinois, June 18 to August 11. In addition to the usual courses in college albegra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered. By Professor G. A. MILLER: Introduction to higher algebra, based on Bôcher; Critical study of the history of mathematics. By Professor J. B. SHAW: Functions of a real variable. By Professor H. R. BRAHANA: Projective geometry. By Dr. L. L. STEIMLEY: Differential equations. By Dr. H. W. BAILEY: Teachers' courses in elementary mathematics. By Mr. G. L. EDGETT: Mathematical statistics for teachers.

University of Iowa, first term, June 7 to July 20—In addition to courses in algebra, solid geometry, trigonometry, and calculus, the following subjects are offered. By Dr. M. A. NORDGAARD: Subject matter and teaching of mathematics. By Dr. CONKWRIGHT: Ordinary differential equations; Theory of functions of a complex variable. By Professor WYLIE: Elementary mechanics; Mathematics of finance; Descriptive astronomy. By Professor WOODS: Advanced coordinate geometry; Elliptic integrals. By Professor REILLY: Algebra for high school teachers; Disability theory applied to life insurance. By Professor CHITTENDEN: Advanced calculus; Functions of infinitely many variables. Second term, July 23 to August 24—By Dr. NORDGAARD: The history of mathematics. By Dr. WARD: Elementary mechanics; Theory of functions of a complex variable. By Professor BAKER: Differental equations; Geometry of forces. By Professor REILLY: The numerical solution of equations; Frequency distributions and correlations.

Johns Hopkins University, June 25 to August 3. In addition to elementary courses the following graduate course will be given. By Professor J. R. MUSSELMAN: Theory of algebraic equations and forms.

University of Kansas, first term, June 5 to July 20—By Professor C. H. ASHTON: Modern synthetic geometry; Series. By Professor E. B. STOUFFER: Seminar. By Professor G. W. SMITH: Higher plane curves; College algebra. By Professor J. J. WHEELER: Differential calculus; Analytical geometry. Second term, July 23 to August 17—By Professor H. E. JORDAN: Integral calculus; Plane trigonometry. By Professor G. W. SMITH: Analytical mechanics; Seminar.

University of Nebraska, June 4 to July 13. In addition to the usual introductory courses up to and including the calculus, the following courses will be offered. By Professor BRENKE: Differential equations. By Professor GABA: Advanced Euclidean geometry; Foundations of algebra and geometry. By

Professor CAMP: Advanced algebra. By Professor PIERCE: Theory of equations. By Mr. HARPER: Elements of the mathematical theory of statistics.

Ohio State University, June 18 to August 31. In addition to the usual courses in trigonometry, analytic geometry, and calculus, the following courses are offered. By Professor C. L. ARNOLD: The teaching of mathematics; Differential equations. By Professor C. C. MORRIS: Theory of equations; Mathematical statistics. By Professor J. H. WEAVER: Differential geometry; Introduction to higher algebra.

University of Pennsylvania, July 2 to August 11. In addition to courses in trigonometry, college algebra, analytic geometry, and calculus, the following courses are offered. By Professor H. H. MITCHELL: Probability. By Professor M. J. BABB; College geometry. By Professor ARNOLD DRESDEN, of Swarthmore College: Elliptic functions; Differential equations. By Professor J. R. KLINE: Theories of integration. By Professor J. M. THOMAS: Introduction to tensor analysis.

University of Wisconsin, six weeks session, June 25 to August 3. By Professor J. H. TAYLOR: Differential equations; Differential geometry. By Professor E. B. VAN VLECK: Modern geometrical concepts; The location of the roots of an algebraic equation. Special nine weeks course for graduates, June 25 to August 24. By Professor R. E. LANGER: Partial differential equations; Calculus of variations. By Professor WARREN WEAVER: Theory of potential; Introduction to statistical mechanics.

Professor F. J. DICK, of the Theosophical University, Point Loma, died May 25, 1927.

Captain D. M. GARRISON, head of the department of mathematics at St. John's College, Annapolis, Md., died suddenly on December 30, 1927. He had been a member of the Association since 1923.

Mr. RAYMOND L. MODESITT of the Eastern Illinois State Teachers College, a member of the Association since September, 1921, died suddenly on December 16, 1927.

Professor H. E. RUSSELL, of the University of Denver, died May 31, 1927.

CORRIGENDA

The following corrections should be made in the December (1927) issue of the Monthly:

On page 499, in line 21, replace the phrase "a branch of the American Academy of Science" by "The Louisiana Academy of Sciences."

On page 518, in line 25 and in line 26, replace z by r.

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BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, April 27-28.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 9-10.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 3.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, May.

OHIO, Columbus, Ohio, April 6.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Durham, N. C., March.

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TEXAS, Texas A. and M. College, Jan. 28.

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5. Underline theorems with blue pencil (avoid ink).
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8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: 1, 2, $\cdots$, n.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, O means "zero"; to indicate "cap O," backspace and overprint period; thus: O.
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* Boldface German—$\mathfrak{b\ A\ B\ D}$

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O. F. H. BERT, Washington and Jefferson College.
WILLIAM BETZ, East High School, Rochester, N. Y.
R. V. BLAIR, Vanderbilt University.
H. F. BLICHFELDT, Stanford University.
J. D. BOND, University of Tennessee.
L. H. BOWEN, Furman University.
P. P. BOYD, University of Kentucky.
E. W. BROWN, Yale University.
E. T. BROWNE, University of North Carolina.
H. E. BUCHANAN, Tulane University.
R. S. BURINGTON, Case School of Applied Science.

W. D. CAIRNS, Oberlin College.
R. D. CARMICHAEL, University of Illinois.
I. S. CARROLL, Syracuse University.
J. B. COLEMAN, University of South Carolina.
JULIA T. COLPITTS, Iowa State College.
J. L. COOLIDGE, Harvard University.
LENNIE P. COPELAND, Wellesley College.
A. R. CRATHORNE, University of Illinois.

JULIA DALE, Mississippi Delta State College.
H. T. DAVIS, Indiana University.
MARY S. DAY, Salem College.
L. A. V. DE CLEENE, St. Norbert's College.
W. W. DENTON, University of Michigan.
ELEANOR C. DOAK, Mount Holyoke College.
B. F. DOSTAL, University of Florida.
ARNOLD DRESDEN, Swarthmore College.

W. W. ELLIOTT, Duke University.
G. C. EVANS, Rice Institute.
G. W. EVANS, Lynn, Mass.

B. F. FINKEL, Drury College.
L. R. FORD, Rice Institute.
W. B. FORD, University of Michigan.
TOMLINSON FORT, Lehigh University.

C. A. GARABEDIAN, University of Cincinnati.
H. M. GEHMAN, Yale University.
EMMA M. GIBSON, High School, Springfield, Mo.
J. L. GIBSON, University of Utah.
J. S. GOLD, Bucknell University.

J. A. HARDIN, Centenary College.
MARGARET E. HARRIS, Grenada College.
E. R. HEDRICK, University of California at Los Angeles.
ARCHIBALD HENDERSON, University of North Carolina.
GERTRUDE A. HERR, Iowa State College.
G. W. HESS, Howard College.
R. E. HILL, University of Louisville.
F. J. HOLDER, Mercer University.
H. M. HOSFORD, Southern Methodist University.
ANNA M. HOWE, Newcomb Memorial College.
JEWELL C. HUGHES, University of Arkansas.
E. V. HUNTINGTON, Harvard University.
MABEL HUTCHINS, Blue Mountain College.
EMMA HYDE, Kansas State Agricultural College.

LOUIS INGOLD, University of Missouri.
M. H. INGRAHAM, University of Wisconsin.

DUNHAM JACKSON, University of Minnesota.
E. H. JONES, Southern Methodist University.
S. I. JONES, Life and Casualty Ins. Co., Nashville.

C. D. KILLEBREW, Alabama Polytechnic Inst.
L. C. KNIGHT, College of Wooster.
F. W. KOKOMOOR, University of Florida.
H. W. KUHN, Ohio State University.

E. P. LANE, University of Chicago.
GILLIE A. LAREW, Randolph-Macon Woman's College.
J. W. LASLEY, Jr., University of North Carolina.
C. G. LATIMER, University of Kentucky.
HARRY LEVY, University of Illinois.
F. A. LEWIS, University of Alabama.
J. J. LUCK, University of Virginia.
E. B. LYTLE, University of Illinois.

C. C. MACDUFFEE, Ohio State University.
SISTER MARY-OF-THE-PRESENTATION MCNELLY, Our Lady of the Lake College.
E. L. MACKIE, University of North Carolina.
A. C. MADDOX, Louisiana State Normal College.
ISRAEL MAIZLISH, Centenary College.
YETTA V. MAIZLISH (Mrs. I. Maizlish), Centenary College.
WILLIAM MARSHALL, Purdue University.
T. E. MASON, Purdue University.
LIDA B. MAY, Kidd-Key College.
J. F. MESSICK, Emory University.
A. D. MICHAL, Ohio State University.
G. A. MILLER, University of Illinois.
G. R. MIRICK, Lincoln School, Teachers College.
W. L. MISER, Vanderbilt University.
NELLIE P. MISER (Mrs. W. L. Miser), Ward-Belmont School.
B. E. MITCHELL, Millsaps College.
U. G. MITCHELL, University of Kansas.
CLARA P. MIZE, Martin College.
E. C. MOLINA, Tel. and Tel. Co., New York, N. Y.
C. N. MOORE, University of Cincinnati.
R. L. MOORE, University of Texas.
W. A. MOORE, Birmingham Southern College.
RICHARD MORRIS, Rutgers College.
A. B. MORTON, Georgia School of Technology.
THIRZA A. MOSSMAN, Kansas State Agricultural College.
ARRIA MURTO, High School Carthage, Mo.

J. H. NEELLEY, Carnegie Inst. of Technology.

G. D. OLDS, Amherst College.
JESSE OSBORN, Harris Teachers College, St. Louis.
W. P. OTT, University of Alabama.

C. I. PALMER, Armour Inst. of Technology.
G. Y. RAINICH, University of Michigan.
S. E. RASOR, Ohio State University.
P. L. REA, Marietta College.
B. P. REINSCH, Southern Methodist University.
C. N. REYNOLDS, West Virginia University.
R. G. D. RICHARDSON, Brown University.
P. R. RIDER, Washington University.
H. L. RIETZ, University of Iowa.
W. J. RISLEY, SR., Colorado School of Mines.
H. A. ROBINSON, Agnes Scott College.
MRS. W. F. ROBINSON, Mars Hill College.
G. M. ROBISON, Duke University.
DOUGLAS RUMBLE, Emory University.

F. H. SAFFORD, University of Pennsylvania.
S. T. SANDERS, Louisiana State University.
FAITH SAUNDERS, Maysville, Mo.
J. H. SCARBOROUGH, Warrensburg (Mo.) State Teachers College.
H. C. SHAUB, Washington and Jefferson College.
J. A. G. SHIRK, Pittsburg (Kans.) State Teachers College.
H. A. SIMMONS, Northwestern University.
T. M. SIMPSON, University of Florida.
H. E. SLAUGHT, University of Chicago.
L. L. SMAIL, Lehigh University.
CLARA E. SMITH, Wellesley College.
D. M. SMITH, Georgia School of Technology.
E. R. SMITH, Iowa State College.
I. W. SMITH, North Dakota Agricultural College.
VIRGIL SNYDER, Cornell University.
R. P. STEPHENS, University of Georgia.
J. M. STETSON, Pennsylvania State College.
GUY STEVENSON, University of Louisville.
JOSEPHINE STONE, Athens College.
E. B. STOUFFER, University of Kansas.
CORA STRONG, North Carolina College for Women.

A. HELEN TAPPAN, Western College.

L. E. WARD, University of Iowa.
WARREN WEAVER, University of Wisconsin.
W. P. WEBBER, Louisiana State University.
J. F. WELCH, Louisiana State University.
E. W. WHITE, Carson and Newman College.
G. T. WHYBURN, University of Texas.
F. B. WILEY, Denison University.
K. P. WILLIAMS, Indiana University.
A. H. WILSON, Haverford College.
R. M. WINGER, University of Washington.
FREDRICK WOOD, Wesleyan College, Macon, Ga.
META A. WOOD, Synodical College.
F. L. WREN, Peabody College.

C. N. WUNDER, University of Mississippi.
C. C. WYLIE, University of Iowa.
JESSICA M. YOUNG, Washington University.
E. I. YOWELL, University of Cincinnati.
C. M. LOWRY, Official Representative, John B. Stetson University.

The sessions of the American Association for the Advancement of Science began on Monday evening with the opening session in the Nashville War Memorial Building. After welcoming addresses by Dr. J. H. Kirkland, Chancellor of Vanderbilt University, and Judge Grafton Greene, Chief Justice of the Supreme Bench of Tennessee, Dr. A. A. Noyes, president of the American Association, gave an extended address in reply to the addresses of welcome. The retiring president, Dr. L. H. Bailey, was unable to be present because of serious illness but a letter from him was read at that time; this was followed by an illustrated lecture by Dr. S. G. Morley on "Recent excavations in Yucatan and Guatemala." A number of semi-popular lectures were given during the week under the auspices of the American Association, including an address on "E. E. Barnard, His life and work" by Dr. R. G. Aitken, of the Lick Observatory. On Wednesday afternoon Professor E. W. Brown of Yale University gave the fifth Josiah Willard Gibbs Lecture under the Auspices of the American Mathematical Society as a general lecture of the American Association; his subject was "Resonance in the solar system." This address is to appear in the BULLETIN of the American Mathematical Society.

Professor R. C. Archibald, Brown University, was chosen vice-president of Section A of the American Association. Professor C. N. Moore, University of Cincinnati, was elected secretary of the section, and Professor E. T. Bell, California Institute of Technology, a member of the section committee.

The mathematicians were housed in the dormitories of Ward-Belmont School in very comfortable fashion. They had their meals in the dining-room of the School, and the parlors afforded very convenient meeting places for general social purposes. Nearly two hundred attended the joint dinner of the mathematicians which was held Thursday evening under the eloquent toastmastership of Professor Archibald Henderson. Professor Snyder as president of the Society spoke on the esthetic side of mathematics as developed by the activities of the Society. Professor Slaught as a representative of the Mathematical Association spoke of the men prominent in the history of the Monthly and the Association, of the part in the development of expository work which is possible from the Association publications, and of the arousing of active interest in the many groups of its sections over the country. Miss Jewell Hughes told of the opportunities for women in the teaching of mathematics in the United States, affirming and substantiating the statement that women have an opportunity equal with that of men in college and university positions in mathematics unless it be in the higher university positions. Dr. C. F. Roos

told of the National Research Fellowships with their opportunity for uninterrupted studies, largely under one's own initiative but identifying one's work with some institution; he spoke particularly of the benefit which comes from a general discussion of one's problem with members of the faculty and of desirable help upon the bibliography of the subject. Professor R. M. Winger of the University of Washington spoke as a representative of the Western mathematicians and told of various notable accomplishments in that region. Professor R. L. Moore of the University of Texas spoke as a representative of the higher institutions of learning in the South. After raising the question whether it is desirable to extend the number of departments of mathematics in the South conducting extensive graduate work and hence conferring the degree of Ph.D., he concluded that there may very well be departments in the South that are outstanding in some particular field and that may discover young men of talent who would otherwise be unknown, and expressed his belief that the number of students going North for their doctor's degrees may well be increased. Professor J. L. Coolidge who spoke "for mathematicians everywhere" said that we come to these meetings to meet kindred spirits and to bring together the teachers from the larger and smaller colleges to consider our common ideals; our highest usefulness is to maintain the ethical etiquette and standards of our students, to meet the increasing competition between the best brains in America and Europe with the highest honesty and with warm, sympathetic cooperation.

The convenience of the arrangements and the many pleasant features which surrounded the stay in Ward-Belmont School were recognized by a motion offered by Professor Rietz and adopted by a rising vote at the last session, whereby the Association expressed its thanks to the Committee on Arrangements under the chairmanship of Professor Miser and in particular to the authorities of Ward-Belmont School for the hospitality which was extended during the sessions.

The American Mathematical Society held its thirty-fourth annual meeting on Tuesday morning and afternoon and Wednesday morning for the reading of papers. At the beginning of the Wednesday morning session Professor James Pierpont gave an address on "Mathematical rigor, past and present" by invitation of the Society; this address appears in the Bulletin of the Society for January-February 1928.

The program of the sessions beginning on Thursday afternoon is given herewith, accompanied by abstracts of part of the papers. The program was prepared by the following committee: Professors David Barrow, H. J. Ettlinger, S. W. Reaves, and H. E. Buchanan, Chairman.

JOINT SESSION OF THE ASSOCIATION WITH THE SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION

1. "The notion of probable error in elementary statistics" by Professor E. V. HUNTINGTON, Harvard University, Retiring vice-president of Section A of the American Association.

2. "The human significance of mathematics" by Professor DUNHAM JACKSON University of Minnesota, Retiring president of the Mathematical Association.

3. "Some philosophic aspects of mathematics" by Professor ARNOLD DRESDEN, Swarthmore College, representing the Mathematical Society.

Abstracts are not given for these three addresses for the reason that they are published in full in journals readily available to any reader. The address by Professor Huntington appeared in Science for Dec. 30, 1927; that by Professor Jackson will appear in an early issue of the Monthly and that by Professor Dresden in an early issue of the Bulletin of the American Mathematical Society.

FIRST SESSION OF THE ASSOCIATION

1. "Observations on simultaneous equations" by Professor ARCHIBALD HENDERSON, University of North Carolina.

2. "Mathematics in the junior colleges" by Professor JEWELL C. HUGHES, University of Arkansas.

3. "The reorganization of secondary mathematics in theory and practice" by Vice-principal WILLIAM BETZ, East High School, Rochester, New York.

Abstracts of these papers follow:

1. The usual methods employed for the solution of simultaneous quadratic equations, to which subject the present paper is restricted, lead to the solution of a quartic equation in either variable, which lends itself to solution according to familiar methods. Equations of the type $x^2+y=a^2+b$, $x+y^2=a+b^2$, where a and b are rational, may however be reduced to the solution of a cubic representing the points of intersection in either coordinate of either of the original parabolas with the hyperbola $(x+a)(y+b)-1=0$.

In the present paper a general method, depending upon the solution of a cubic equation, is presented for effecting the solution of the pair of quadratic equations in two variables, in the most general form:

$$U_1 \equiv a_0x^2 + a_1y^2 + 2a_2xy + 2a_3x + 2a_4y + a_5 = 0, \tag{1}$$

$$U_2 \equiv b_0x^2 + b_1y^2 + 2b_2xy + 2b_3x + 2b_4y + b_5 = 0. \tag{2}$$

Employing homogeneous coordinates, we seek the condition that the conic through the four points of intersection of (1) and (2), viz., $u_1+\lambda u_2=0$ reduce to a pair of lines. The solutions of the cubic

$$\begin{vmatrix} a_0 + \lambda b_0 & a_2 + \lambda b_2 & a_3 + \lambda b_3 \\ a_2 + \lambda b_2 & a_1 + \lambda b_1 & a_4 + \lambda b_4 \\ a_3 + \lambda b_3 & a_4 + \lambda b_4 & a_5 + \lambda b_5 \end{vmatrix} = 0,$$

give rise to three pairs of lines $p_1=0$, $p_2=0$; $q_1=0$, $q_2=0$; $r_1=0$, $r_2=0$. The solutions of equations (1) and (2) are then $(p_1q_1r_1)$, $(p_1q_2r_2)$, $(p_2q_1r_2)$, $(p_2q_2r_1)$. All the different cases, depending upon particular collocations of the lines, in the progressive degeneration of the complete quadrangle, are analyzed. The method outlined is applicable to all cases, save the trivial one when all four intersections are coincident, which point is obviously the point of tangency of the common tangent to $U_1=0$, $U_2=0$, at the intersection.

2. The address by Miss Hughes will be published in the Monthly.

In the discussion which followed Professor Slaught spoke of the importance to teachers of junior college mathematics of their becoming members of the Association and of our having the cooperation of members of the Association in inviting to membership those in our immediate neighborhoods. Professor Rietz said that the question with regard to credit to be given to work in junior colleges by the four-year colleges and universities is as yet vague and must be controlled by proper inspection of the junior colleges, and by a knowledge of the competence of the teacher and the character of the course. Professor T. M. Simpson urged that the state representatives present carry the cause of the Association to the junior colleges in our own states.

3. After a brief analysis of the present educational situation, Mr. Betz showed that a crisis has resulted from the conflicting views of specialists and educational theorists. It is of paramount importance, in his opinion, that these opposing groups should work in greater harmony toward a tangible goal. The speaker offered a critical review of the progress made in reorganizing secondary mathematics during the past quarter of a century. He pointed out that neither the hopes of the reformers of twenty-five years ago nor the demands of the general educator had been fully realized. The compartment system is still in full vogue, and the "unification of pure and applied mathematics" is still a dream. The curriculum expert is complaining of the purely "academic" character of mathematical instruction and of the total disregard of many necessary readjustments. Mr. Betz insisted that more attention must be given to the human significance of mathematics, to scientific classroom procedures, and to improved professional resources for the training of teachers. In conclusion, the speaker offered a tentative six-year curriculum. He also submitted to the Association a memorandum on behalf of the curriculum committees appointed by the New York State Department of Education, asking for cooperation in the preparation of new syllabi for New York State.

In the discussion of this paper, Professor Slaught said that we offer no apology for our interest in secondary mathematical education; that one of our

first plans was to form the National Committee, and that it is perfectly logical that we continue our active interest in the improving of the teaching of mathematics. He commended, too, the yearbooks of the National Council of Teachers of Mathematics, the mathematical federation of the hundred groups which were brought together by the National Committee. (The Trustees adopted a resolution on the cooperation requested by Mr. Betz, as is indicated in the account of the meeting of the Trustees). Professor Lytle asked why we do not have in America the same cooperation and active interest in the secondary associations which are given by eminent mathematicians in England in connection with the Mathematical Association of England. Professor Lane replied that our Mathematical Association represents exactly the answer to Dr. Lytle's question, for many of the men in America interest themselves sympathetically in secondary education and in the training of teachers even though they are themselves research men. As an instance of the sort there was mentioned the joint session and the joint meetings of such organizations as the Louisiana-Mississippi Section and the branch of the National Council.

SECOND SESSION OF THE ASSOCIATION

1. "The fundamental mathematical requirements of Biology" by Professor J. ARTHUR HARRIS, Head of the Department of Botany, University of Minnesota. (By title.)

2. "Present tendencies in projective geometry" by Professor E. P. LANE, University of Chicago.

3. "The law of small numbers" by Professor A. R. CRATHORNE, University of Illinois.

4. "Application of vectors to the formation of the roots of the cubic" by Professor J. W. LASLEY, Jr., University of North Carolina.

Abstracts of the papers follow:

1. On account of Professor Harris's unavoidable absence from the meeting, his paper was read by title only, but it will be published in the Monthly, much to the gratification of a considerable number who expressed their interest in this special topic.

2. The present tendencies in projective geometry can best be understood after a brief survey of the origin and development of this science. Although there existed isolated instances of theorems of a projective nature before 1800, still there was no organized projective theory before the fundamental work of Poncelet in the first quarter of the nineteenth century.

Synthetic projective geometry grew most rapidly in the first half of the last century, and analytic projective geometry about the middle. In the latter part of the century Halphen began the first systematic projective differential investigations.

In the first quarter of the present century, Wilczynski founded the American school of projective geometry, and Fubini founded an Italian school, using very different methods.

The present tendency is to correlate the two methods used by Americans and Italians, and to attempt to develop a systematic theory of the projective differential geometry of a general variety in hyperspace.

3. Professor Crathorne's paper is an expository and critical discussion of the "Law of Small Numbers," giving its history, its application in statistics, and a resumé of the objections which have been raised against it.

Professor Rietz in commenting upon the objections made to the name "Bortkiewitsch's Law of Small Numbers," said that the authors of the *Handbook of Mathematical Statistics* chose this term as against the term "Poisson's Formula" supported by certain French mathematicians. Mr. Molina stated that the Bell Telephone Company is determining the number of trunks required for definite forms of service by the use of this formula.

4. This paper concerns itself with two pedagogical questions relative to the general solution of the cubic equation: first, the solution itself as to technic, second, the formation of the solution from the roots of the resolving quadratic. A solution is here presented based on the binomial theorem and the relation between the roots and coefficients of a quadratic equation. It is believed to be free from the artificiality and arbitrariness of the usual text-book solution of the cubic. The notion of vectors is applied to the formation of the roots of the cubic from those of the resolving quadratic. For this purpose vectors called "symmetric" are defined. Two vectors are said to be symmetric if the sum of their angles is an integral multiple of π. Certain properties of symmetric vectors are deduced, and on the basis of these a method is outlined for forming by means of the symmetric vectors the roots of the cubic from those of the quadratic. This method, it is believed, will impress the student with the fact that one need not always begin with the principal cube root of the roots of the resolving quadratic, but may begin the formation at will with any cube root of either root of that equation. Moreover, the solution is obtainable with equal ease whether or not one begins with the principal cube roots.

MEETINGS OF THE BOARD OF TRUSTEES

Eleven members of the Board of Trustees were present at the sessions.

The following twenty persons and five institutions were elected to membership on applications duly certified:

To Individual Membership

EUGENE ALLIOT, Lic. ès Sc. (Paris). Prof., St. Michael's Coll., Winooski, Vt.

O. K. BOWER, A.M. (Illinois). Asst. Instr., Univ. of Illinois, Urbana, Ill.

MARY S. DAY, Ph.D. (Columbia). Head of Dept., Salem Coll., Winston-Salem, N.C.

GORDON FULLER, A.B. (W. Tex. State Teachers Coll.). Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

GERTRUDE A. HERR, M.S. (Iowa State Coll.). Asso. Prof., State Coll., Ames, Ia.

R. E. HILL, A.M. (Louisville). Head of Dept., Univ. of Louisville, Louisville, Ky.

LYLAH KRYDER, A.M. (Columbia). Instr., Hunter Coll., New York, N. Y.

W. N. MEBANE, JR., B.S. (Davidson). Asst. Prof., Davidson Coll., Davidson, N. C.

W. L. MURFEE, Col., A.M. (Virginia). Pres., Marion Inst., Marion, Ala.

W. B. ORANGE, A.M. (California). 5071 Whittier Blvd., Los Angeles, Calif.

LELA OXSHEER, A.M. (Teachers Coll., Columbia). Asso. Prof., Stephen F. Austin Teachers Coll., Nacogdoches, Tex.

W. O. PENNELL, B.S. in E.E. (Mass. Inst. of Tech.). Chief Engineer, So. Western Bell Tell. Co., St. Louis, Mo.

J. F. RANDOLPH, B.S. (W. Tex. State Teachers Coll.). Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

H. H. RAPHAEL, A.M. (California). Instr., Jr. Coll., Glendale, Calif.

HERBERT REBARKER, Ph.D. (Peabody). Head of Dept., Demonstration School, and Teacher of the Teaching of Math., Peabody. Coll., Nashville, Tenn.

H. P. ROGERS, A.M. (Illinois). Instr., James Millikin Univ., Decatur, Ill.

I. D. STEWART, A.B. (Muskingum). Instr., Bryson Coll., Fayetteville, Tenn.

W. H. TAYLOR, Ph.D. (Iowa). Registrar and Prof. Hamline Univ., St. Paul, Minn.

L. W. TURNER. Supervisor Agency, Statistical Dept., Hartford Acc. and Ind. Co., Hartford, Conn.

C. H. WHEELER III, B.S. (Washington and Jefferson). Student Asst., Johns Hopkins Univ., Baltimore, Md.

G. T. WHYBURN, Ph.D. (Texas). Adj. Prof., Pure Math., Univ. of Tex.

To Institutional Membership

ALABAMA COLLEGE, Montevallo, Ala.

JOHN B. STETSON UNIVERSITY, DeLand, Fla.

LOUISIANA STATE UNIVERSITY, Baton Rouge, La.

ST. JOHNS COLLEGE, Annapolis, Md.

NORTH TEXAS AGRICULTURAL COLLEGE, Arlington, Tex.

The following were appointed associate editors of the Monthly for the year 1928:

H. E. BUCHANAN	H. S. EVERETT	C. N. MILLS
ELIZABETH CARLSON	B. F. FINKEL	F. D. MURNAGHAN
OTTO DUNKEL	R. A. JOHNSON	H. L. OLSON
H. J. ETTLINGER	H. W. KUHN	D. E. SMITH

The financial statement for the year 1927 was presented by Professor Cairns. The report had been examined by Professor Slaught for the Finance Committee and by Professor Crathorne for the Association. It was accepted and approved by the Trustees.

The Committee on Carus Monographs requested that any suggestions as to possible future monographs be given in writing to the committee and they requested that the committee be enlarged so as to number at least five. This appointment will be made by the president at an early date.

The By-Laws of the recently organized Philadelphia Section were approved.

As an expression of loyalty to the American Mathematical Society and of the unique relation which the Association bears to the Society, the Trustees voted to apply for a sustaining membership in the Society. Five members of the Society will be nominated under this membership.

An invitation was received from the president and the department of mathe-

matics of the University of Minnesota to hold the summer meeting there in 1931, in connection with a similar meeting of the Mathematical Society.

The Trustees adopted the following resolution:

"The Association recognizes the benefit accruing from membership in mathematics clubs and honor mathematical societies, and commends the favorable consideration of these to colleges and universities as a means of encouraging and stimulating students in the early part of their career and of furnishing a stepping stone toward membership in the Association and the society."

In response to the request of the New York State Syllabus Committee on Algebra, submitted by Mr. William Betz, for the Association's active cooperation in dealing with the program of requirements in algebra, the following resolution was adopted:

"Resolved that the Trustees of the Mathematical Association of America wish to express their interest in the efforts of the Algebra Syllabus Committee of the New York State Department of Education to revise its syllabus for the eleventh and twelfth years of the high school course in the spirit of the recommendations of the National Committee on Mathematical Requirements, and that a committee of three members of the Association be appointed to act in an advisory capacity when so requested by the Algebra Syllabus Committee."

Professor J. W. Young, E. R. Hedrick and Ralph Beatley were appointed as this committee.

The president was empowered to appoint delegates to the International Mathematical Congress.

The Finance Committee was empowered to transfer another thousand dollars to the General Endowment Fund if this seems wise.

Other matters were considered concerning the formation of sections of the Association and concerning possible provision for an extension of facilities for the publication of certain types of mathematical articles.

Annual Business Meeting of the Association

Professor Cairns reported the death of the following members:

R. F. Borden, Assistant professor of mathematics, George Washington University, Washington, D. C. (March 15, 1927).

W. G. Bullard, Professor of mathematics, Syracuse University (February 16, 1927).

F. H. Loud, Professor emeritus of mathematics, Colorado College (March 2, 1927).

G. A. Osborne, Professor emeritus of mathematics, Massachusetts Institute of Technology (November 20, 1920).

H. E. Russell, Professor of mathematics, University of Denver (May 31, 1927).

A. C. Bose, Deputy Magistrate, Bengal Provincial Civil Service, Calcutta, India (December 11, 1926).

C. D. Garlough, Professor of mathematics, Wheaton College (November 25, 1927).

R. L. Modesitt, Teacher, Eastern Illinois State Teachers College (December 16, 1927).

J. L. Riley, Professor of mathematics, Ouachita College (July 12, 1927).

H. D. Thompson, Professor of mathematics, Princeton University (August 30, 1927).

The election of officers for the year 1928 was conducted by mail and in

person at this meeting; the tellers, Professors J. D. Bond and G. W. Hess, reported the result of the balloting as follows:

For Vice-Presidents: A. J. Kempner, 253 votes; R. E. Moritz, 205 votes; F. D. Murnaghan, 251 votes; Miss Clara E. Smith, 217 votes.

For additional members of the Board of Trustees, to serve until January 1931: R. C. Archibald, 301 votes; D. Buchanan, 218 votes; L. P. Eisenhart, 303 votes; B. F. Finkel, 206 votes; J. R. Kline, 125 votes; E. P. Lane, 224 votes; W. L. Miser, 134 votes; H. L. Rietz, 332 votes.

The following were accordingly declared elected:

Vice-Presidents: A. J. KEMPNER, University of Colorado; F. D. MURNAGHAN, Johns Hopkins University.

Additional members of the Board of Trustees: R. C. ARCHIBALD, Brown University; L. P. EISENHART, Princeton University; E. P. LANE, University of Chicago; H. L. RIETZ, University of Iowa.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 12, 1927

RECEIPTS			EXPENDITURES		
Balance Dec. 16, 1926		$7,514.66	Publisher's bills (Nov. '26–Oct. '27)		$4,899.01
1925 indiv. dues	$ 87.00		Manager's office		32.01
1926 indiv. dues	364.70		Editor-in-Chief's office 1926		51.47
1926 instit. dues	31.00		Editor-in-Chief's office 1927		576.77
1927 indiv. dues	6,301.62		*Register* expense		95.75
1927 instit. dues	769.30		Committee on Membership		145.00
1927 subscriptions	681.75		Joint Com. on Membership		89.06
Initiation fees	280.00		Expense, Report, Com. on Math. Requirements		34.09
Advertising	778.95		Information Bur. for Appointments		8.50
Sale copies of MONTHLY	88.65		Secretary-Treasurer's office:		
Sale reprints	1.35		Postage	$383.20	
Sale First Carus Mon.	63.75		Bond	5.00	
Sale Second Carus Mon.	111.25		Ins. back issues MONTHLY	6.40	
Sale Third Carus Mon.	1,095.25		Safety Deposit	4.00	
From Mon. Fund for expenses	139.75		Office supplies	45.70	
Recd. for subscriber's Monographs	2.00		Express, tel., etc.	42.30	
For *Annals* subscriptions.	3.00		Clerical work	528.59	
Recd. for expense La.–Miss. Section	18.00		Printing	436.50	
Life memb. payments	118.62		Library expense	7.25	
Sale Rhind Papyrus	1,020.00		Pd. copies MONTHLY	14.25	
From matured bond	500.00		Philadelphia meeting	80.00	
Contributions A. B. Chace	5,898.66		Madison meeting	69.41	
Int. Oberlin Savgs. Bk.	145.87		Nashville meeting	15.10	
Int. Peoples Bkg. Co.	121.67		Refd. subscriptions	17.10	
Int. Liberty Bonds	85.00		Refd. Third Mon.	1.25	
Int. Hardy Fund	120.00		Forwarded for subscribers' Monograph	2.00	
Int. on Chace contributions	40.20		Forwarded init. fee to *Math. Teacher*	2.00	
		18,867.34			1,660.05
Total 1927 receipts		$26,382.00			

		Annals subvention	250.00
		Pd. *Annals* subscriptions	4.50
		Forwarded *Annals* subscriptions	3.00
		Pd. to sections from initiation fees	175.52
		Cost of new bond	523.47
		Pd. int. Hardy Fd. to B. F. Finkel	120.00
		Honorarium Carus Monograph	300.00
		Transferred to Mon. Fd.	1,218.90
		Expense acct. Mon. Fd.	85.00
		Cost of bonds for Genl. Endowment Fd.	2,091.26
		Printing Rhind Papyrus	5,771.98
		Pd. expense La.-Miss. Sec.	18.00
Total expenditures	18,153.34	Total expenditures	$18,153.34
Balance to the end of 1927 business	$8,228.66	Cash on hand	4.00
		Checking account	584.66
Received on 1928 business	811.50	Oberlin Savgs. Bk. acct.	3,608.53
		Peoples Bkg. Co. acct.	1,754.11
		Liberty Bonds	1,000.00
		Certifs. of deposit	2,088.86
Book balance Dec. 12, 1927	$9,040.16	Bank balance Dec. 12, 1927	$ 9,040.16

CARUS MONOGRAPH FUND

Balance Dec. 16, 1926 ($2,824.42; $201.40)		$3,025.82
Receipts: Sales	$1,270.25	
Interest	149.49	1,419.74
		$4,445.56
Expenditures: Honorarium	$ 300.00	
Expense account	85.00	385.00
		$4,060.56
Certificate of deposit	$4,053.06	
Cash in bank	7.50	
		$4,060.56

ARNOLD BUFFUM CHACE FUND

Receipts from advance sales	$1,020.00
Balance Dec. 12, 1927, certificate of deposit	$1,020.00

CHAUVENET PRIZE FUND

Balance Dec. 16, 1926	$20.00
Amount set aside for 1927	20.00
Balance Dec. 12, 1927, cash in bank	$40.00

LIFE MEMBERSHIP FUND

Liability on life memberships Dec. 16, 1926 (Corrected)	$336.65	
Receipts for life memberships	118.62	
		$455.27
To be transferred to current funds, surplus		11.43
Liability on life memberships Jan. 1, 1928		$443.84

GENERAL ENDOWMENT FUND

Gift from previous owners of the MONTHLY	$1,000.00
Transferred from current funds in 1926	1,000.00
Transferred from current funds in 1927	1,000.00
Interest, U.S. Treasury Notes	150.00
Interest, Liberty Bonds	42.50
	$3,192.50

U.S. Treasury Notes, present value	$1,150.00	
Liberty Bond	1,000.00	
Land Trust Certificate	1,000.00	
Cash in bank	42.50	
		$3,192.50

Of the funds on hand, as indicated above, $7.50 belongs to the CARUS MONOGRAPH FUND (not yet transferred); $1,020.00 belongs to the ARNOLD BUFFUM CHACE FUND; $126.68 remains for the expense of printing the Rhind Papyrus; $40.00 is set aside as the first two payments toward the Chauvenet Prize Fund; $443.84 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1928; $42.50 belongs as interest to the GENERAL ENDOWMENT FUND. Aside from these, the sum of $3,000 is held in reserve as a GENERAL ENDOWMENT FUND; and the CARUS MONOGRAPH FUND, now amounting to $4,053.06, is carried as a separate fund in the form of a certificate of deposit which bears 4%, compounded quarterly.

When the accounts were closed Dec. 12, 1927, there remained on the total business for the year 1927 the following items:

BILLS RECEIVABLE (partly estimated)		BILLS PAYABLE (partly estimated)	
1927 individual dues	$ 130.00	Publisher's bills (Nov., Dec. '27)	$1,150.00
Advertising	70.00	President's office	40.00
	$ 200.00	Manager's office	40.00
		Editor-in-Chief's office	100.00
		Other editors' postage	50.00
		Committee on Membership	50.00
		Register	600.00
		Secretary-Treasurer's office	250.00
		Annals subvention	200.00
		Initiation fees due to sections	600.00
		Carus Monograph Fund	7.50
		Chace Fund	1,020.00
		Rhind Papyrus funds	126.68
		Chauvenet Prize Fund	40.00
		Life Membership Fund	443.84
		General Endowment Fund	42.50
			$4,760.52

If to the balance on 1927 business shown in this report, $8,228.66, there be added the bills receivable, $200.00, and there be subtracted the estimated bills payable, $4,760.52, there results an estimated final balance on 1927 business of approximately $3,700, a satisfactory advance in the Association's resources for the year.

W. D. CAIRNS, *Secretary-Treasurer*

THE MECHANICAL COMBINATION OF LINEAR FORMS

By D. H. LEHMER, University of California, Berkeley.

The "method of exclusion" introduced by Gauss, although a tentative method, is still one of the most powerful weapons for attacking many problems in the theory of numbers. Perhaps the most familar of such problems is the solution of the congruence $x^2 \equiv D \pmod{m}$. The method is also of great use in the problem of representing a number by a given form such as

$$a^2 - b^2, \quad a^2 + b^2, \quad a^2 \pm Db^2, \cdots .$$

Suppose, for instance, we wish to represent a given number N as the difference of two squares. We may choose 5 as an "excluding number." Let $N \equiv 3 \pmod 5$. We construct the following table

$$\begin{aligned} a &\equiv 0\ 1\ 2\ 3\ 4 \\ a^2 &\equiv 0\ 1\ 4\ 4\ 1 \pmod 5 \\ a^2 - N = b^2 &\equiv 2\ 3\ 1\ 1\ 3 \end{aligned}$$

But 2 and 3 are non-residues of 5 so that the cases $a \equiv 0$, 1, and 4 (mod 5) are excluded. We have then $a = 5n+2, 3$. Other small prime moduli may be used as excluding numbers with the result that a is restricted to $(p \pm 1)/2$ values modulo p. The problem that now presents itself is the combination of the linear forms thus obtained. If the range of a is large, it is necessary to use many excluding numbers and the labor of combining directly these linear forms is prohibitive.

A graphical method has been suggested[1] to solve this difficulty in which use is made of a table ruled in squares, each cell representing a possible value of a. "Movable strips" are made, the length of each being some excluding number p, and on which impossible values of a modulo p are indi-

[1] Kraitchik, *Theorie des Nombres* (Paris, 1922), Chapter 2.

cated by crosses. These strips are moved down the columns of the table and the cells adjacent to the crosses on the strip are then ruled out. As many strips as are necessary are computed and applied until the number of empty cells is reasonably small for actual trial.

But even this method has certain disadvantages. In the first place it requires very close attention and much care. Also the strips are short in comparison with the length of the table and have to be shifted frequently, which is a possible source of error. Moreover the method makes it necessary to fix the limit of the table before commencing the work, so that the whole range must be entirely covered before any definite information can be obtained.

These difficulties are overcome by a machine constructed by the writer for the combination of linear forms. It is the object of this paper to give a brief description of the machine and to exhibit some of the results obtained by it.

Here instead of strips we have chains. The number of links in each chain is a convenient multiple of some small prime. In fact the chain lengths are: 64, 27, 25, 49, 22, 26, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, and 67. Each chain hangs in a loop from a sprocket having 10 teeth. All the 19 sprockets are fixed rigidly to a single shaft driven by a motor. The possible values modulo p are indicated by small pins fastened to the appropriate links of the chain (the zero link being indicated by red paint.) Whenever a link provided with a pin arrives at the top of the shaft a small spring with a tungsten point is lifted by the projecting pin. This breaks for the moment the electric contact between the spring and a brass bar running parallel to the shaft. By means of a relay in the circuit, the motor is shut off and the machine stops itself. When several chains are provided with springs the machine will not stop unless all the springs are lifted, so that every time the machine stops it means that a number satisfying all the imposed conditions has appeared. This number can be read directly by means of a revolution counter connected to the shaft. The shaft revolves 300 r. p. m. so that the machine canvasses 3000 numbers per minute. When all chains are provided with springs a "solution" occurs once in several hours during which time the machine runs without any attention.

The use of the machine is best explained by means of examples. Let us first consider the representation of

$$N = (10^{20} + 1)/(10^4 + 1) = 9999000099990001 = a^2 - b^2$$

as the difference of two squares. A representation besides

$$a \equiv (N + 1)/2, \quad b = (N - 1)/2$$

is possible since N has been shown to be the product of two primes.[1] The factors of N are of the form $40n+1$. The expansion of the square root of N in a regular continued fraction seems to indicate that, although 21 is a residue of N, 3 and 7 are both non-residues. The assumption that 3 is a non-residue gives the following form for the factors of N:

$$\left.\begin{matrix} 3n+2 \\ 40n+1 \end{matrix}\right\} \quad 120n+41.$$

Let us then write

$$a+b = 120n+41, \tag{1}$$

$$a-b = 120m+41. \tag{2}$$

Multiplying (1) and (2) we get

$$a^2-b^2 = N = 120^2mn+120\cdot 41(m+n)+41^2.$$

Now $N \equiv 10801 \pmod{120^2}$. Therefore $120\cdot 41(m+n) \equiv 9120 \pmod{120^2}$, or

$$41(m+n) \equiv 76 \pmod{120}.$$

Solving this congruence we have

$$(m+n) \equiv 116 \pmod{120}. \tag{3}$$

Adding (1) and (2) we get $a=60(m+n)+41$. Substituting from (3) we get

$$a = 60(120k+116)+41 = 7200k+7001. \tag{4}$$

The range for a is given by the formula

$$\sqrt{N} < a < \frac{1}{2}\left(W+\frac{N}{W}\right),$$

where W is the limit to which the number has been searched for factors. In this case[2] $W=120{,}000$ so that a has the range

$$99995000 < a < 41662560416 \text{ with } 13888 < k < 5786466.$$

Now let $x=k-13888$ so that $0<x<5772578$. Then (4) gives

$$a = 7200(x+13888)+7001 = 7200x+100000601, \tag{5}$$

so that a is restricted to one case in 7200.

[1] Bulletin of the American Mathematical Society, vol. 33, p. 336.

[2] This limit has been established by Shanks' table of the number of digits in the repetend of the reciprocal of every prime <120000.

We proceed now to exclude x with small prime moduli. Let us take for example the excluding number 11. We have from (5)

$$a \equiv 6x + 8 \pmod{11} \qquad N \equiv 1 \pmod{11}.$$

We now construct the following table:

$$\begin{aligned} x &\equiv 0\ 1\ 2\ 3\ \ 4\ 5\ \ 6\ 7\ 8\ 9\ 10 && \pmod{11} \\ a \equiv 6x + 8 &\equiv 8\ 3\ 9\ 4\ 10\ 5\ \ 0\ 6\ 1\ 7\ \ 2 \\ a^2 &\equiv 9\ 9\ 4\ 5\ \ 1\ 3\ \ 0\ 3\ 1\ 5\ \ 4 \\ a^2 - N = b^2 &\equiv 8\ 8\ 3\ 4\ \ 0\ 2\ 10\ 2\ 0\ 4\ \ 3 \end{aligned}$$

Excluding those values of x which correspond to non-residues in the last line, we get the following linear forms for x modulo 11:

$$x = 11n + 2,\ 3,\ 4,\ 8,\ 9,\ 10.$$

Other excluding numbers may be dealt with in like manner.

Kraitchik[1] has given tables of the possible values of a for any N and for excluding numbers $\leqq 47$. The possible values of x can easily be obtained from the tabulated values of a by means of the transformation

$$x \equiv \frac{1}{k}a - \frac{1}{k}l \pmod{p} \text{ whenever } a \equiv kx + l \pmod{p}.$$

These tables have been recalculated and the errors noted are given at the end of the present paper. In order to serve the needs of the machine these tables have been extended to excluding numbers $\leqq 67$. The use of the tables is to be preferred to the method illustrated by the above example. The calculation is much simpler and the results can easily be checked by symmetry. For instance the table for 11 gives in our case

$$a \equiv 1,\ 2,\ 4,\ 7,\ 9,\ 10 \pmod{11}.$$

Making the transformation

$$x \equiv \frac{1}{6}a - \frac{1}{6}8 \equiv 2a + 6 \pmod{11},$$

we get $x \equiv 8, 10, 3, 9, 2, 4 \pmod{11}$ as before.

We can also make good use of the condition that 7 is a non-residue of N. The factors of N are then of the form $7n+3$, 5, or 6. Since $N \equiv 2 \pmod{7}$, we have the following two cases to consider.

[1] Kraitchik, loc. cit., pp. 187–199.

CASE I. $a+b=7n+3$; $a-b=7m+3$.

$$N = a^2 - b^2 = 49mn + 21(m + n) + 9 \equiv 9 \pmod{7^2}.$$

$$\therefore m + n \equiv 0 \pmod 7 \; ; \quad a \equiv 7\left(\frac{m + n}{2}\right) + 3 \; ; \quad a \equiv 3 \pmod{49}.$$

CASE II. $a \pm b = 7n+5$; $a \mp b = 7m+6$.

$$N = 49mn + 7(5m + 6n) + 30 \equiv 9 \pmod{7^2}.$$

$$\therefore 5m + 6n \equiv 4 \pmod 7 \; ; \quad m + n \equiv 0, 1, 2, 3, 4, 5, 6 \pmod 7.$$

$$a \equiv 7\left(\frac{m + n}{2}\right) + \frac{11}{2}.$$

$$a \equiv 7K + 30 \pmod{49} (K = 0, 1, 2, 3, 4, 5, 6).$$

$$a \equiv 30, 37, 44, 2, 9, 16, 23 \pmod{49}.$$

Considering both cases we have

$$a = 49n + 2, 3, 9, 16, 23, 30, 37, 44.$$

Transforming to x we get

$$x = 49n + 2, 9, 16, 23, 25, 30, 37, 44.$$

We have thus restricted x to 8 cases modulo 49 instead of the usual 28 cases. The result is that the machine will run longer between solutions.

Linear forms for x were thus calculated for every excluding number $\leqq 67$ and the appropriate links of each chain were supplied with pins. The machine was then set in motion. After running about 2 hours it stopped itself at $x = 400453$,

$$a = 2983262201 \; ; \quad a^2 - N = 8889854359815374400 = 2981585880^2 = b^2$$

This gives us $N = (a-b)(a+b) = 1676321 \times 5964848081$.

If the entire range had been covered without finding the factors of N, it would have indicated that 3 and 7 were residues contrary to our previous assumption. This case could have been taken care of, without changing the position of the pins, by simply shifting the origin on each chain as explained later. The machine would then have to be set back to zero and another run made. Even if two runs were necessary the consideration of the quadratic character of 3 reduces the number of values of x to 2/9.

Another example of the representation by the difference of two squares is furnished by $N = 20191210335106439$, a large factor of $3^{111}+1$. Every factor of the number is of the form $111n+1$. Hence, as before, a can be restricted to one case in $111^2 = 12321$. Also $a = 4n+2$. Consequently we have $a = 49284k+6550$. The range for a is $44935627 < a < 10096101675$ corresponding to $W = 100{,}000$ as set by congruence tables. Hence

(5) $$a = 49284x + 44978200 \text{ with } 0 < x < 203943 .$$

The linear forms for x were calculated as above with the exception of those for the excluding numbers 3 and 37 which are not prime to the modulus 49284. The first of these was computed as follows. The form (5) may be written

$$a \equiv 36x + 34 (\text{mod } 81), \quad a^2 \equiv 18x + 22 (\text{mod } 81),$$
$$N \equiv 4 (\text{mod } 81), \quad b^2 \equiv 18x + 18 (\text{mod } 81),$$
$$r \equiv 2x + 2 (\text{mod } 9),$$

where r designates all the quadratic residues of 9, which are 0, 1, 4, 7. Putting in these values we obtain

$$x = 9n + 1, 4, 7, 8.$$

The linear forms for 37 were calculated in a similar way. All the linear forms were set on the chains and the machine, after running only 4 seconds, stopped at $x = 145$ giving

$$a = 52124380 \quad b = 26414781.$$

Therefore $N = 20191210335106439 = 25709599 \times 78539161$.

If the same problem had been attacked by the "movable strip" method, the entire range would have been excluded before this solution was discovered.

The machine was also used to show that the other large factor of $3^{111}+1$, namely 64326272436179833 is a prime which gives the complete factorisation

$$3^{111} + 1 = 2^2 \times 7 \times 223 \times 18427 \times 107671 \times 25709599 \times 56737873$$
$$\times 78539161 \times 64326272436179833.$$

The 5th and 7th of these primes are due to Poulet.

We consider next the representation of a number as the sum of two squares. The indeterminate equation $x^2 - 1721y^2 = -1$ has for its fundamental solution

$$x = 31738680901536260 \qquad y = 76506518214341.$$

We have then $x^2+1 = 1721y^2$. Thus y is a divisor of the sum of two squares and therefore every factor of y is the sum of two squares. We have

$$y = 17 \times 4500383424373 = 17 \times N.$$

We propose to represent N as the sum of two squares. Using 5 as an excluding number we get $a = 5n+2, 3$. To save time we consider the two separate cases $a = 5x+2$ and $a = 5x+3$. Taking the first case we proceed to exclude values of x with all the excluding numbers except 5. The tables constructed for the difference of squares may be used for the sum of squares. For an excluding number

of the form $4n+1$ the entries are identical. For the case $4n+3$ the entries ***not*** tabulated are those desired, with the exception of the pair of entries corresponding to $b^2 \equiv 0 \pmod{p}$, which must be included.

The range of a in either case is

$$0 < a < \sqrt{(N/2)}, \quad 0 < a < 1500063, \quad 0 < x < 300013.$$

The machine covered the range for the first case with but a single stop at

$$x = 196113, \quad a = 980567, \quad N - a^2 = 3538871782884.$$

This number is not a square although it is a residue of every prime $\leqq 67$.

To consider the second case, in which $a = 5x+3$, no further calculation was necessary. The values of a in each case must be congruent modulo p. That is

$$5x_1 + 2 \equiv 5x_2 + 3 \pmod{p} \text{ or } x_1 - x_2 = 1/5 \pmod{p}, \text{ for } x_2 \equiv 0, \; x_1 \equiv 1/5 \pmod{p}.$$

On the second start therefore, instead of setting all the chains on their zero positions we set them on the link corresponding to the value of $1/5 \pmod{p}$. This time the machine stopped only once giving

$x = 157044$, $a = 785223$, $b = 1970738$. Hence $N = (785223)^2 + (1970738)^2$.

Since this is a unique representation, it follows that N is a prime. This also gives us the complete factorisation of $(31738680901536260)^2+1$.

By separating the work into two cases, results were obtained in 2/5 of the time required for a single run.

The machine has been used to study the following problem previously considered by Kraitchik,[1] namely to find the least non-square which is a quadratic residue of all primes $\leqq p$.

Following Kraitchik, we consider 0 as a non-residue and the number 8 instead of the prime 2. The numbers sought belong to the forms

$$\left.\begin{matrix} 8n+1 \\ 3n+1 \end{matrix}\right\} \quad 24n+1.$$

Letting $N = 24x+1$ we proceed to exclude values of x with moduli $\geqq 5$. To obtain the necessary linear forms we transform a table of residues by means of the relation $x \equiv (r-1)/24 \pmod{p}$, where r represents the residues of p. All the chains were supplied with pins before starting. To start with, the chain for 5 was provided with a spring and the machine was set in motion. The first non-square solution obtained was 241. Then the next prime was introduced by setting on the spring of the chain for 7. The machine was then run to the first non-square solution after which the spring for the prime 11 was put on,

[1] *Recherches sur la Théorie des Nombres* (Paris, 1924), pp. 41–46.

etc. The squares appearing as solutions are of course squares of primes or products of primes $>p$. These squares were predicted in advance so as to avoid unnecessary reading of the machine. The work was carried up to $p=61$. The results are summarized in the following table, which gives the least non-square N_p that is a residue of all primes $\leqq p$.

$N_3=17$ Prime	$N_{17}=18001$ 47×383	$N_{41}=3206641$ 643×4987
$N_5=73$ Prime	$N_{19}=53881$ Prime	$N_{43}=3818929$ Prime
$N_7=241$ Prime	$N_{23}=87481$ Prime	$N_{47}=9257329$ Prime
$N_{11}=1009$ Prime	$N_{29}=117049$ 67×1747	$N_{53}=22000801$ Prime
$N_{11}=2641$ 19×139	$N_{31}=515761$ Prime	$N_{59}=48473881$ Prime
$N_{13}=8089$ Prime	$N_{37}=1083289$ Prime	$N_{61}=48473881$ Prime

It is seen that most of the N's are primes. If N_p is a prime it has for residues all the primes $\leqq p$ and also -1. If it is composite there exists some prime $\leqq p$ which is a non-residue of an even number of factors of N. The interesting case of $N_{59}=N_{61}$ is the first of its kind.

The above results confirm and extend the work of Kraitchik. He used the movable strip method with a table having 180 columns and 84 lines to carry the work to N_{47}. A full account of his work is given in his book. A few errors are to be noted which however do not happen to influence the final results. It should be pointed out that to extend Kraitchik's work by his method would imply the construction of an entirely new table the size of which must be determined in advance. Using the machine the work may be extended indefinitely and is complete as far as it goes.

ERRORS IN KRAITCHIK'S TABLE III.

Théorie des Nombres, pp. 190–9

p	N	for	read
31	5	$a = 8$	$a = 11$
31	5	$t = 11$	$t = 8$
47	6	$t = 10$	$t = 11$
47	34	$t = 19$	$t = 22$

Since writing the above, the author's attention has been directed to Kraitchick's new second volume (Théorie des Nombres, vol. 2, Paris, 1926) which contains tables of $a \pmod p$ for $p\leqq67$. The above errors have been corrected. Only two new errors appear, namely:

p	N	for	read
59	23	$x=14$	$x=15$
59	44	$x=14$	$x=17$

STRONG AND WEAK INEQUALITIES INVOLVING THE RATIO OF TWO CHORDS OR TWO ARCS OF A CIRCLE; CHAINS OF INEQUALITIES

By HARVEY A. SIMMONS, Northwestern University

1. Introduction. It is known that if two arcs OA and OB, with arc $OA <$ arc $OB \leqq \pi a$ be taken on a circle of radius a (c.f. Fig. 1), then[1]

$$(1) \qquad OA/OB > \text{arc } OA/\text{arc } OB$$

Furthermore it is easy to prove that $(OA/OB)^2$ does not exceed arc OA/arc OB for all chords OA, OB, with $OA < OB$, that can be taken on a semi-circumference.[2] Owing to this fact, the inequality (1) might be called *strong*. In this paper an inequality $p^n > q$ will be called *strong* if it is true for all pairs of values of p, q on the interval in which they are considered while the inequality $p^{n+1} > q$ is not true for some choice of p, q on this interval, n being a positive integer.

One purpose of this paper is to establish several *strong* inequalities, each of which involves either the ratio of two chords of a circle, or that of two arcs, and to obtain other inequalitites which it seems fitting to call *weak*, from the *strong* ones.[3] Another purpose is to develop two interesting chains of inequalities one of which contains as its last link the inequality that Professor Uspensky stated in Theorem III of his article referred to above (c.f. Theorem VII below). Incidentally, it seems possible that the numerous new trigonometric inequalities in this paper may stimulate some reader to make another geometric proof by "curious" induction.

When unspecified, the units of length, area, and volume in this paper will be understood to be arbitrary.

Let a semi-circle in the xy-plane (see Fig. 1) be represented by the equation

$$(2) \qquad r = 2a \sin\theta, \quad 0 \leqq \theta \leqq \pi/2,$$

[1] A geometric proof of (1) was given by J. V. Uspensky, this Monthly, vol 34, May (1927), p. 249.

[2] An example in which $(OA/OB)^2 <$ arc OA/arc OB is afforded by taking θ in equation (2) to be 30° for OA and 90° for OB. For $(a^2/4a^2) < [(\pi a \div 3)/\pi a]$. An example in which $(OA/OB)^2 >$ arc OA/arc OB is afforded by taking θ in equation (2) to be 60° for OA and 90° for OB. For $(3a^2/4a^2) > [(2\pi a \div 3)/\pi a]$.

[3] These terms "strong" and "weak" are not to be understood in a sense which is often used in connection with inequalities, namely that $z > y$ is a stronger inequality than $x > y$ if $z > x$. Our meaning is in accord with the usage by which a first theorem is said to be stronger than a second theorem if the conclusions in both theorems are the same, while the hypotheses of the first are true for a larger class than that for which the hypotheses of the second hold. For example, of the two theorems (I) if $x > 1$, $x > 0$ and (II) if $x > 2$, $x > 0$ the former is the stronger. In this paper strong inequalities are those that lead to, or express strong theorems.

and denote the coördinates of A and B, two points on the circumference, by (r_1, θ_1) and (r_2, θ_2), respectively, where $0 < \theta_1 < \theta_2 \leqq \pi/2$, and denote by a_1, a_2 the lengths of the arcs OA, OB, respectively. Let k_1 and k_2 denote the areas of the segments OmA and OmB, respectively, of the circle. Let v_1 and v_2 denote the volumes of the segments of the sphere (obtained by rotating the semi-circle about the y-axis) which lie below the planes ZOA and ZOB, respectively, OZ being perpendicular to the xy-plane. Let s_1 and s_2 denote the areas of the spherical portions of the former and latter segments, respectively. Using these definitions we can prove the following six theorems.[1]

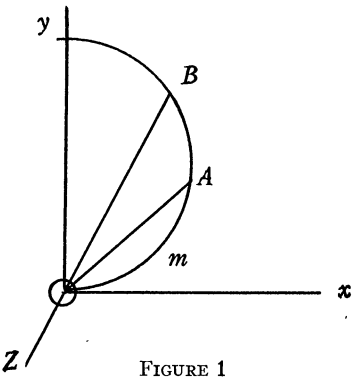


FIGURE 1

THEOREM I. *The chords r_1, r_2 and the areas k_1, k_2 satisfy the inequality $(r_1/r_2)^3 > k_1/k_2$, which is strong.*

THEOREM II. *The arcs a_1, a_2 and the areas k_1, k_2 satisfy the inequality $(a_1/a_2)^2 > k_1/k_2$, which is strong.*

THEOREM III. *The chords r_1, r_2 and the areas s_1, s_2 satisfy the inequality $(r_1/r_2)^2 > s_1/s_2$, which is strong.*

THEOREM IV. *The arcs a_1, a_2 and the areas s_1, s_2 satisfy the inequality $a_1/a_2 > s_1/s_2$, which is strong.*

THEOREM V. *The chords r_1, r_2 and the volumes v_1, v_2 satisfy the inequality $(r_1/r_2)^4 > v_1/v_2$, which is strong.*

THEOREM VI. *The arcs a_1, a_2 and the volumes v_1, v_2 satisfy the inequality $(a_1/a_2)^2 > v_1/v_2$, which is strong.*

2. Proofs of Theorem I and Theorem II. To prove Theorem I, we first observe that $(r_1/r_2)^3 = \sin^3\theta_1/\sin^3\theta_2$, by (2), and that the areas k_1, k_2 have the values

$$k_1 = \frac{1}{2}\int_0^{\theta_1} 4a^2 \sin^2\theta \, d\theta = a^2(\theta_1 - \sin\theta_1 \cos\theta_1), \quad k_2 = a^2(\theta_2 - \sin\theta_2 \cos\theta_2).$$

[1] In this paper, we prove only the first two theorems, which are the ones that we use in §4. Our method of proof is essentially the same for all six of the theorems and a good deal of repetition is avoided by the omission.

We need to prove, then, that

$$(3)\qquad \frac{\sin^3\theta_1}{\sin^3\theta_2} > \frac{\theta_1 - \sin\theta_1\cos\theta_1}{\theta_2 - \sin\theta_2\cos\theta_2}, \quad \text{or} \quad \frac{\sin^3\theta_1}{\theta_1 - \sin\theta_1\cos\theta_1} > \frac{\sin^3\theta_2}{\theta_2 - \sin\theta_2\cos\theta_2}.$$

Since $0<\theta_1<\theta_2\leqq\pi/2$, the inequality (3) will be established if it is shown that $y=\sin^3x/(x-\sin x\cos x)$ decreases as x increases from 0 to $\pi/2$, the value $x=0$ itself being excluded. Differentiating y with respect to x and simplifying the result by means of the identity $\sin^2x+\cos^2x=1$, we find

$$y' = \frac{\sin^2 x(3x\cos x - 3\sin x + \sin^3 x)}{(x - \sin x\cos x)^2}.$$

In this equation, $y'<0$ if $3x\cos x-3\sin x+\sin^3x<0$, or if

$$(4)\qquad x < \tfrac{2}{3}\tan x + \tfrac{1}{3}\sin x\cos x,\ 0 < x \leqq \pi/2.$$

To prove (4), consider the curves

$$y = x,\ y = \tfrac{2}{3}\tan x + \tfrac{1}{3}\sin x\cos x.$$

Each of these curves passes through the origin with slope 1, but the latter curve has at every point the slope $\frac{2}{3}\sec^2x+\frac{1}{3}\cos2x$, which is greater than 1 throughout the interval $0<x\leqq\pi/2$; for

$$\frac{2\sec^2 x}{3} + \frac{\cos 2x}{3} \equiv \frac{2 + 2\cos^4 x - \cos^2 x}{3\cos^2 x} > 1$$

since $(\cos^2x-1)^2>0$ on the interval $0<x\leqq\pi/2$. Consequently, on this interval the latter curve lies above the former. Hence (4) holds, $y'<0$, and the function $\sin^3x/(x-\sin x\cos x)$ decreases, as was to be proved. This proves $(r_1/r_2)^3>k_1/k_2$. That this inequality is strong, and hence that Theorem I is true, now follows from the example below in which $(r_1/r_2)^4<k_1/k_2$.

Example. Take $\theta_1=30°$ and $\theta_2=60°$ in (2). Then $(r_1/r_2)^4<k_1/k_2$ since

$$\frac{\sin^4 30°}{\sin^4 60°} < \frac{(\pi/6) - \sin 30°\cos 30°}{(\pi/3) - \sin 60°\cos 60°}, \qquad [c.f.\ (3)].$$

Before proving Theorem II, we pause to give a geometric interpretation of inequality (4). Let C (see Fig. 2) be the center of a circle of radius $CA=1$, AQ a tangent to the circle, P the point where CQ intersects the circumference, M the foot of the perpendicular dropped from P upon CA, and the angle $ACP=x$, where $0<x\leqq\pi/2$. Then the inequality

$$(3x)/2 < \tan x + \tfrac{1}{2}\sin x\cos x,$$

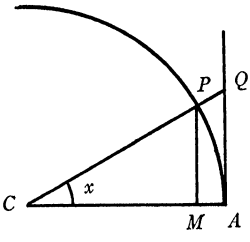


FIGURE 2

which is equivalent to (4), reads: (three times the area of the sector CAP) is less than (twice the area of the triangle CAQ) plus (the area of the triangle CMP).

To prove Theorem II, we first observe that

$$\frac{a_1^2}{a_2^2} = \frac{\theta_1^2}{\theta_2^2}, \quad \frac{k_1}{k_2} = \frac{\theta_1 - \sin\theta_1\cos\theta_1}{\theta_2 - \sin\theta_2\cos\theta_2}.$$

We need to prove, then, that

$$(5) \qquad \frac{\theta_1^2}{\theta_2^2} > \frac{\theta_1 - \sin\theta_1\cos\theta_1}{\theta_2 - \sin\theta_2\cos\theta_2}, \quad \text{or} \quad \frac{\theta_1^2}{\theta_1 - \sin\theta_1\cos\theta_1} > \frac{\theta_2^2}{\theta_2 - \sin\theta_2\cos\theta_2}.$$

To establish (5), we need only to show that $y = x^2/(x - \sin x\cos x)$ decreases as x increases from 0 to $\pi/2$, the value $x=0$ itself being excluded, since $0 < \theta_1 < \theta_2 \leqq \pi/2$. Differentiating y with respect to x and simplifying the result by means of the identity $\sin^2 x + \cos^2 x = 1$, we find

$$y' = \frac{2x\cos x\,(x\cos x - \sin x)}{(x - \sin x\cos x)^2} < 0,$$

since[1] $x < \tan x$ on the interval $0 < x \leqq \pi/2$. Hence (5) is true. This proves that $(a_1/a_2)^3 > k_1/k_2$. That this inequality is strong, and hence that Theorem II is true, now follows from the

Example. Take $\theta_1 = 30°$ and $\theta_2 = 60°$. Then $(a_1/a_2)^3 < k_1/k_2$ since

$$\frac{(\pi/4)^3}{(\pi/3)^3} < \frac{(\pi/6) - \sin 30°\cos 30°}{(\pi/3) - \sin 60°\cos 60°}, \qquad [\text{see (5)}].$$

3. Corollaries to the above theorems. By definition $r_1 < r_2$ (see §1), so that $r_1/r_2 < 1$. Hence if $(r_1/r_2)^n > N$, where N is a positive number and n is an integer greater than 1, the powers $1, \cdots, n-1$ of r_1/r_2 exceed N. These statements about r_1/r_2 apply also to a_1/a_2. Hence inequalities, which it seems proper to call *weak*, follow from five of the above theorems. These inequalities are expressed in the following corollaries.

COROLLARY I. $(r_1/r_2)^2 > k_1/k_2$, $r_1/r_2 > k_1/k_2$, from Theorem I.

COROLLARY II. $a_1/a_2 > k_1/k_2$, from Theorem II.

COROLLARY III. $r_1/r_2 > s_1/s_2$, from Theorem III.

COROLLARY IV. $(r_1/r_2)^3 > v_1/v_2$; $(r_1/r_2)^2 > v_1/v_2$; $(r_1/r_2) > v_1/v_2$; from Theorem V.

COROLLARY V. $a_1/a_2 > v_1/v_2$, from Theorem VI.

[1] This known inequality may be seen from the fact that the area (in Fig. 2) of the sector CAP is less than the area of the triangle CAQ.

4. Chains of inequalities. In this section we apply the first two theorems above to obtain two chains of inequalities which it seems proper to include here. The proof of the third inequality in the first chain [see (6)] is the most difficult part of this paper, the difficulty coming from the fact that $\sin^2 x/x$ increases on part of the interval from 0 to $\pi/2$ and decreases on the rest of it.

Let P and Q be two polygons inscribed in the same circle of radius a. Denote by $p_1, p_2, \cdots, p_n$ the lengths of the sides of P in increasing order of magnitude and by $u_1, u_2, \cdots, u_n$ their respective intercepted arcs (each $\leqq \pi a$) on the circumference, which will be taken as the unit of arc. Denote by $q_1, q_2, \cdots, q_m$ the lengths of the sides of Q in decreasing order of magnitude, and by $v_1, v_2, \cdots, v_m$ the lengths of their respective intercepted arcs, and suppose $q_1 < p_1$. Denote by j_i $(i=1, 2, \cdots, n)$ the area of the segment of the circle which is bounded by the i-th side of P and its intercepted arc, and by k_t $(t=1, 2, \cdots, m)$ the area which is bounded by the t-th side of Q and its intercepted arc. Let $J=j_1+j_2+\cdots+j_n$ and $K=k_1+k_2+\cdots+k_m$. Using these definitions, we shall prove

THEOREM VII. *The numbers* p_i, u_i, j_i $(i=1, 2, \cdots, n)$, q_t, v_t, k_t $(t=1, 2, \cdots, m)$, J, K *satisfy the inequalities*

$$(6)\quad \frac{J}{K} = \frac{j_1+j_2+\cdots+j_n}{k_1+k_2+\cdots+k_m} > \frac{p_1^3+p_2^3+\cdots+p_n^3}{q_1^3+q_2^3+\cdots+q_m^3} > \frac{p_1^2+p_2^2+\cdots+p_n^2}{q_1^2+q_2^2+\cdots+q_m^2} > 1 > \frac{p_1+p_2+\cdots+p_n}{q_1+q_2+\cdots+q_m},$$

$$(7)\quad \frac{J}{K} > \frac{u_1^2+u_2^2+\cdots+u_n^2}{v_1^2+v_2^2+\cdots+v_m^2} > \frac{u_1+u_2+\cdots+u_n}{v_1+v_2+\cdots+v_m} = 1.$$

PROOF OF (6). The equality sign holds by definition. To prove the first inequality, we recall that $p_1 \leqq p_2 \leqq \cdots \leqq p_n$ and $q_1 \geqq q_2 \geqq \cdots \geqq q_m$, by definition, so that $j_1 \leqq j_2 \leqq \cdots \leqq j_n$ and $k_1 \geqq k_2 \geqq \cdots \geqq k_m$. Therefore, by Theorem I,

$$\frac{p_1^3}{j_1} \geqq \frac{p_2^3}{j_2} \geqq \cdots \geqq \frac{p_n^3}{j_n} \text{ and } \frac{q_1^3}{k_1} \leqq \frac{q_2^3}{k_2} \leqq \cdots \leqq \frac{q_m^3}{k_m}.$$

Consequently

$$\frac{p_1^2}{j_1} \geqq \frac{p_1^3+p_2^3+\cdots+p_n^3}{j_1+j_2+\cdots+j_n} \text{ and } \frac{q_1^3}{k_1} \leqq \frac{q_1^3+q_2^3+\cdots+q_m^3}{k_1+k_2+\cdots+k_m}.$$

Now $q_1 < p_1$, by definition, and it follows from Theorem I that $(q_1^3/k_1) > (p_1^3/j_1)$. Hence

$$\frac{q_1^3 + q_2^3 + \cdots + q_m^3}{k_1 + k_2 + \cdots + k_m} > \frac{p_1^3 + p_2^3 + \cdots + p_n^3}{j_1 + j_2 + \cdots + j_n},$$

which is equivalent to the first inequality of (6).

The second inequality of (6) follows from the facts

$$p_1^3 + p_2^3 + \cdots + p_n^3 \geqq p_1(p_1^2 + p_2^2 + \cdots + p_n^2),$$
$$q_1^3 + q_2^3 + \cdots + q_m^3 \leqq q_1(q_1^2 + q_2^2 + \cdots + q_m^2),$$

so that

$$\frac{p_1^3 + p_2^3 + \cdots + p_n^3}{q_1^3 + q_2^3 + \cdots + q_m^3} \geqq \frac{p_1(p_1^2 + p_2^2 + \cdots + p_n^2)}{q_1(q_1^2 + q_2^2 + \cdots + q_m^2)} > \frac{p_1^2 + p_2^2 + \cdots + p_n^2}{q_1^2 + q_2^2 + \cdots + q_m^2},$$

the last inequality holding because $q_1 < p_1$ by definition.

The third inequality of (6) is equivalent to

$$\frac{p_1^2 + p_2^2 + \cdots + p_n^2}{u_1 + u_2 + \cdots + u_n} > \frac{q_1^2 + q_2^2 + \cdots + q_m^2}{v_1 + v_2 + \cdots + v_m},$$

since $u_1+u_2+\cdots+u_n=v_1+v_2+\cdots+v_m=1$ by our choice of the unit of arc; and this inequality, in turn, is equivalent to the following one which we desire to prove

$$(8) \qquad \frac{\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n}{\theta_1 + \theta_2 + \cdots + \theta_n} > \frac{\sin^2\phi_1 + \sin^2\phi_2 + \cdots + \sin^2\phi_m}{\phi_1 + \phi_2 + \cdots + \phi_m},$$

where the $\theta_i (i=1, 2, \cdots, n)$ and $\phi_t (t=1, 2, \cdots, m)$ are the angles at the center of the circle which intercept on the circumference arcs of lengths $u_i/2$ and $v_t/2$ respectively. The definitions of the u_i, v_t imply that

$$(9) \qquad \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_n \leqq \pi/2,\ \phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_m > 0,\ \phi_1 < \theta_1,$$

$$\theta_1 + \theta_2 + \cdots + \theta_n = \phi_1 + \phi_2 + \cdots + \phi_m = \pi,\ m > n \geqq 3.$$

Now since ϕ_1 is the largest ϕ_t, (8) will follow if we prove that

$$(10) \qquad (\sin^2\theta_i)/\theta_i > (\sin^2\phi_1)/\phi_1$$

for every i. To do this, we need the two properties (P_1) and (P_2), stated below, of the function $y=\sin^2 x/x$; these properties of y following from an interpretation of it derivative

$$y' = [\sin x(2x\cos x - \sin x)]/x^2.$$

(P_1). On the interval $0<x<X$, where X is the smallest positive solution of the equation $2x\cos x-\sin x=0$ and lies between 66° and 67°,[1] y increases with x. When $x=\pi/4$, $y=2/\pi$ and when $x=X$, $y=\sin^2 X/X$.

[1] If $x=66°$, $2x=2.3038>\tan x=2.2460$; if $x=67°$, $2x=2.3387<\tan x=2.3559$.

(P_2). On the interval $X<x\leqq\pi/2$, y decreases as x increases. When $x=\pi/2$, $y=2/\pi$.

Since $n\geqq 3$, by (9), $\theta_1\leqq 60°<X$ and consequently $\phi_1<60°$. Hence, by (P_1) and (9), (10) holds unless $\theta_i>X(>66°)$; and since $\theta_1+\theta_2+\cdots+\theta_n=\pi$, by (9), there are at most two values of i for which $\theta_i>X$. Hence we only need to prove (10) in the following cases: (a) when $\theta_{n-1}\leqq X<\theta_n$; (b) when $\theta_{n-2}<X<\theta_{n-1}\leqq\theta_n$.

(a). If $n\geqq 4$, $\theta_1<[(\pi-X)/3]<38°$, and consequently $\phi_1<38°$ by (9). Then if $i<n$, (10) holds, by (P_1). When $i=n$, the left hand side of (10) has a value $\geqq 2/\pi$, by (P_2), while the right hand side is less than $2/\pi$, by (P_1). Hence (10) holds when $n\geqq 4$. The case $n=3$ we argue as follows. If $i<3$, (10) holds, by (P_1). If $i=3$, let $\theta_3=\frac{1}{2}\pi-z$. As $\theta_3>X$, it follows that

$$\theta_1\leqq\frac{\pi-(\frac{1}{2}\pi-z)}{2}=\frac{\pi}{4}+\frac{z}{2},\quad 0<z<\left(\frac{1}{2}\pi-X\right).$$

so that $\phi_1<\frac{1}{4}\pi+\frac{1}{2}z$ on the specified interval. By (P_2), the ratio

$$\sin^2(\tfrac{1}{2}\pi-z)/(\tfrac{1}{2}\pi-z)$$

increases with z on its interval. Hence we shall prove (10) if we show that

$$(11)\qquad\frac{\sin^2(\frac{1}{4}\pi+\frac{1}{2}z)}{\frac{1}{4}\pi+\frac{1}{2}z}<\frac{\sin^2(\frac{1}{2}\pi-z)}{\frac{1}{2}\pi-z},\quad 0<z<\left(\frac{1}{2}\pi-X\right).$$

But (11) is equivalent to

$$z>\frac{\pi\sin z}{2(2-\sin z)},\quad 0<z<\left(\frac{1}{2}\pi-X\right),$$

which follows from the fact that on the specified interval $\sin z<z$. We leave the proof to the reader. Hence (10) holds in case (a).

(b). The method used in (a) to handle the case $n\geqq 4$ also applies here. If $n=3$, since θ_2 and θ_3 both exceed $X(>66°)$, $\theta_1=\pi-\theta_2-\theta_3<48°$, and consequently $\phi_1<48°$. Hence if $i=1$, (10) holds by (P_1). By (P_2) and the definitions of θ_2 and θ_3, (10) holds for $i=2$ if it holds for $i=3$. If $\phi_1<\pi/4$, $(\sin^2\phi_1/\phi_1)<2/\pi$ by (P_1), and $2/\pi\leqq\sin^2\theta_3/\theta_3$, by P_2; therefore (10) holds. Hence we only need to consider the case $\pi/4<\phi_1<48°$. Then $\theta_3<69°$, by (9) and the inequalities $66°<X<\theta_2<\theta_3$. To obtain the desired result, we put $z=6°$ in (11), and find

$$\frac{\sin^2(4\pi/15)}{4\pi/15}<\frac{\sin^2(7\pi/15)}{7\pi/15}.$$

The left member of this inequality exceeds any value that $\sin^2\phi_1/\phi_1$ can have on the interval $\pi/4<\phi_1<48°$, by (P_1); and the right member is less than any

value that $\sin^2 \theta_3/\theta_3$ can have on the interval $X<\theta_3<69°$, by (P_2). Hence (10) holds for every i.

One can now prove (8), and thus complete the proof of the third inequality in (6), by arranging the ratios $\sin^2 \theta_i/\theta_i$ $(i=1, 2, \cdots, n)$ in increasing order of magnitude and using the method which was employed above in proving the first inequality of (6).

The fourth, and last, inequality of (6) can be proved by use of inequality (1) and the method employed above in proving the first inequality of (6), as was shown by Professor Uspensky[1]. Hence (6) is true.

PROOF OF (7). As our proof of (7) is not difficult, we merely describe it. The first inequality follows from Theorem II in the same way that the first inequality of (6) followed from Theorem I; u, v, and the exponent 2 playing the rôle here which p, q, and the exponent 3, respectively, played there. The method which was used in proving the second inequality of (6) can be employed to prove the second inequality of (7). The equality sign in (7) holds because $u_1+u_2+\cdots+u_n=v_1+v_2+\cdots+v_m$. Hence Theorem VII follows.

The corollaries below can be obtained by a small amount of inference from (6), (7), and the definitions of the symbols involved.

COROLLARY VI. If $\theta_i(i=1, 2, \cdots, n)$ and $\phi_t(t=1, 2, \cdots, m)$ are positive angles, each $\leqq \pi/2$, such that $\theta_1+\theta_2+\cdots+\theta_n=\phi_1+\phi_2+\cdots+\phi_m=\pi$, and if the smallest θ_i is greater than the largest ϕ_t, and if λ is a positive integer, the following inequalities hold:

$$\frac{\sum_{i=1}^{n}(\sin\theta_i)^{\lambda+1}}{\sum_{t=1}^{m}(\sin\phi_t)^{\lambda+1}} > \frac{\sum_{i=1}^{n}(\sin\theta_i)^{\lambda}}{\sum_{t=1}^{m}(\sin\phi_t)^{\lambda}}; \quad \frac{\sum_{i=1}^{n}(\theta_i)^{\lambda+1}}{\sum_{t=1}^{m}(\phi_t)^{\lambda+1}} > \frac{\sum_{i=1}^{n}(\theta_i)^{\lambda}}{\sum_{t=1}^{m}(\phi_t)^{\lambda}} \geqq 1.$$

COROLLARY VII. If the polygons P, Q, described above, are regular and have sides of lengths p, q, respectively, and if J, K are as defined above, the following inequalities hold:

$$\frac{J}{K} > \frac{np^3}{mq^3} > \frac{np^2}{mq^2} > 1 > \frac{np}{mq}.$$

If m, or n and m, are sufficiently large, the first inequality of Corollary VII gives results which no existing tables of logarithms are accurate enough to detect. We give an example in which four-place tables are inadequate.

[1] Loc. cit., p. 250.

Example. Suppose the regular polygons P, Q have 60, 120 sides, respectively, and that they are inscribed in the same circle. Then the first inequality of Corollary VII says

$$J/K > (\sin^3 3°)/(2 \sin^3 1°30').$$

COROLLARY VIII. Denote by A and A' the areas of two polygons P and Q, respectively, which are inscribed in the same circle. If the shortest side of P is greater than the longest side of Q, then $A < A'$.

Remark. The type of reasoning used in proving the third inequality of (6) can be used to prove directly that $[(p_1^3 + p_2^3 + \cdots + p_n^3)/(q_1^3 + q_2^3 + \cdots + q_m^3)] > 1$, the function $\sin^3 x/x$ playing the rôle here which $\sin^2 x/x$ played there. To prove directly that $(np^3/mq^3) > 1$, (see Corollary VII), we can replace this inequality by the equivalent one: $n \sin^3 (\pi/n) > m \sin^3 (\pi/m)$, $m > n \geqq 3$, and then obtain the desired result by considering the function $y = x \sin^3 (\pi/x)$, where x has a continuous range over all real numbers $\geqq 3$.

In all three of the proofs just mentioned, the case $n = 3$ requires special consideration, while the other cases, $n = 4, 5, \cdots$, ad infinitum, naturally call for simultaneous treatment. We are inclined to call this procedure an analog in analysis to Professor Uspensky's geometric induction.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. THE LOWEST COMMON MULTIPLE OF n POLYNOMIALS

By ARNOLD DRESDEN, Swarthmore College

1. It is shown in elementary algebra that the product of two polynomials is equal to the product of their highest common factor (H.C.F.) and their lowest common multiple (L.C.M.). Since the H.C.F. of two polynomials can always be found by means of the Euclidean algorithm, this theorem furnishes a direct method for determining the L.C.M. of two polynomials. For the case of three or more polynomials, the determination of the H.C.F. and L.C.M. could be carried out by a step by step process, on the basis of the recursion formulas developed in §2. While this is not extensively laborious for the H.C.F. it involves more work than is necessary for the L.C.M. The purpose of the

present note is to generalize the formula for the case of two polynomials referred to above and thus to obtain another method for finding the L.C.M. of n polynomials.

2. We shall denote the H.C.F. and the L.C.M. of two polynomials A_1 and A_2 by $H(A_1, A_2)$ and $L(A_1, A_2)$ respectively, or, briefly, by H_{12} and L_{12}. The H.C.F. and the L.C.M. of n polynomials $A_1, \cdots, A_n$ will be denoted by $H_{12\ldots n}$ and $L_{12\ldots n}$ respectively.

It is readily seen that $H_{12\ldots n}=H(H_{12\ldots,n-1}, A_n)$. For if $A_k=H_{12\ldots,n-1}B_k$, $k=1, \cdots, n-1$, then $B_1, B_2, \cdots, B_k$ have no factor in common. If, moreover, $H_{12\ldots,n-1}=\alpha H$, and $A_n=\alpha A$, where A and H are relatively prime, then $H(H_{12\ldots,n-1}, A_n)=\alpha$. Furthermore $A_k=\alpha HB_k$; since A and HB_k are without common factor (because none of the factors of A occur in H, and if a factor of A were present in every B_k these polynomials would have a factor in common), it follows that $H_{12\ldots n}$ is also equal to α.

To show that a similar relation holds for the L.C.M., we observe that a necessary and sufficient condition that L is the L.C.M. of the polynomials $A_1, A_2, \cdots, A_n$ is that there exist a set of polynomials $C_1, C_2, \cdots, C_n$, without common factor, such that $L=A_kC_k$, $k=1, \cdots, n$, (the polynomials C_k are then necessarily unique). For, if these polynomials exist, L is clearly a common multiple of $A_1 \cdots, A_n$; and if L_1 is an arbitrary common multiple of $A_1, \cdots, A_n$, there must exist polynomials $D_1, \cdots, D_n$ such that $L_1=A_kD_k$ and hence such that $LD_k=L_1C_k$, $k=1, \cdots, n$. If there were then any factor of L which was not a factor of L_1, it would have to be a factor of every C_k, so that these polynomials would not be relatively prime; therefore every multiple of $A_1, \cdots, A_n$ is divisible by L, so that L is indeed the L.C M. of these polynomials.

We see now readily that $L_{12\ldots n}=L(L_{12\ldots,n-1}, A_n)$. For there exist polynomials $B_1, \cdots, B_{n-1}$ without common factor, such that $A_kB_k=L_{12\ldots,n-1}$, $k=1, \cdots, n-1$. Suppose furthermore that $L_{12\ldots,n-1}=\alpha L$ and that $A_n=\alpha A$, where A and L are relatively prime. Then $A_nL=A_kB_kA=\alpha AL$, $k=1, \cdots, n-1$ and the polynomials $L, B_1A, \cdots, B_{n-1}A$ are without common factors, because none of the factors of L occurs in A, and none of these factors can be present in every B_k, $k=1, \cdots, n-1$, which were supposed to be relatively prime. Hence $L(L_{12\ldots,n-1}, A_n)=\alpha AL=L_{12\ldots,n}$.

3. We note the obvious fact, that for every polynomial p, $H(pA, pB)=pH(A, B)$.

We define a set of polynomials $H_1, \cdots, H_n$ as follows:

$$H_1 = H, \quad H_2 = H_{12}, \quad H_k = H(A_1A_2 \cdots A_{k-1}, \quad H_{k-1}A_k), \quad \text{for } k = 3, \cdots, n.$$

The desired formula is then obtained as follows:

$$L_{123} = L(L_{12}, A_3) = \frac{L_{12}A_3}{H(L_{12}, A_3)} = \frac{A_1A_2A_3/H_2}{H(A_1A_2/H_2, A_3)}$$

$$= \frac{A_1A_2A_3}{H(A_1A_2, H_2A_3)}\ ;\ \text{i.e. } A_1A_2A_3 = L_{123}H_3.$$

Suppose now that $A_1A_2 \ldots A_{n-1} = L_{12\ldots,n-1} \cdot H_{n-1}$. Then

$$L_{12\ldots n} = L(L_{12\ldots,n-1},\ A_n) = \frac{L_{12\ldots,n-1}A_n}{H(L_{12\ldots,n-1}, A_n)} = \frac{A_1A_2\ldots A_n}{H_{n-1}\cdot H(A_1\ldots A_{n-1}/H_{n-1},\ A_n)}$$

$$= \frac{A_1A_2\ldots A_n}{H(A_1A_2\ldots A_{n-1},\ A_nH_{n-1})}\ ;\ \text{i.e. } A_1A_2 \cdots A_n = L_{12\ldots n}\cdot H_n.$$

This formula, of which the validity has thus been established, is the generalization of the familiar result of elementary algebra referred to in the opening paragraph; it furnishes a simple method for the calculation of the L.C.M. of n polynomials.

II. Note on the Calculation of Standard Deviations

By Vladimir Rojansky, University of Minnesota

The object of this note is to describe a simple device which can be effectively used to shorten the work of systematic calculation of standard deviations of sets of numbers. The practicability of the device depends on the range of the sets; the smaller the range, the handier the device. The limitations due to range will be apparent after the method has been described.

The standard deviation of a set of n numbers $x_1, x_2, \cdots, x_n$, is ordinarily computed by direct substitution into the formula:

$$\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - M_a)^2 - (M - M_a)^2\right]^{1/2},$$

where M is the arithmetic mean of the set, and M_a is a conveniently chosen "assumed mean." The cumbersome part of the calculation is the evaluation of the first term within the brackets.

The present method can best be described by an illustration, which, to save space, will be extremely simple. Let it be desired, for example, to calculate the standard deviations of a series of sets of integers, the integers in each set ranging, say from 350 to 360. Two strips of paper are then prepared. On one, which will be referred to as A, the consecutive integers from 350 to 360 are written in a vertical column. On the second, which will be referred to as B, a column of squares of consecutive integers, say, from -5 to $+5$ is written. The alignment of the entries on each strip is uniform, and is the same on both strips.

Let a particular set of numbers whose standard deviation is to be found consist of the following entries: 360, 351, 360, 352 and 360. The procedure is then as follows:

A strip of paper, which will be referred to as W, is placed beside A, and check marks are made on it opposite the numbers on A which correspond to those read from the records. The numbers on A, marked W, are added on an adding machine, care being taken of the frequencies. Division by the number of entries gives the mean, which is written out on W. An assumed mean is chosen and recorded on W, and the corresponding number on A is conspicuously marked on W. The work in this particular example appears as follows:

$$M = 1783 \div 5 = 356.6\,; \quad M_a = 356\,; \quad M - M_a = .6.$$

A	W	B	W
350		25	—
351	—	16	—
352	—	9	
353		4	
354		1	
355		0	*
356	*	1	
357		4	
358		9	
359		16	—\|\|
360	—\|\|	25	
1783	5	89	5

Strip W is then removed from A and placed beside B in such a way that the mark of the assumed mean on W falls opposite the zero on B. The numbers on B, checked on W, are added, care being taken of the frequencies, and thus the sum of the squares of the deviations of the entries from the assumed mean is obtained. The work appears as shown.

The standard deviation is then found to be:

$$\sigma = [89 \div 5 - (.6)^2]^{1/2} = 4.176.$$

The example just given describes the essence of the method, the chief advantages of which are: it practically eliminates tabulation of intermediate steps, and makes it unnecessary actually to evaluate the deviations of the entries from the assumed mean, and their squares. For these reasons it makes the work less tedious and much more rapid than does the direct substitution into the formula. The main disadvantages of the method are: it leaves no comprehensible record of the steps taken, and for large ranges the strips become long and difficult to handle, while using smaller figures to shorten the strips increases the possibility of misplaced check-marks.

Particular conditions are, of course, to determine the relative importance of the merits and the disadvantages. With respect to the last objection it may be said, however, that, first, it does not apply in a good part of the cases arising in the use of statistical methods in natural sciences, where the ranges are usually not great, and, secondly, it may be partly obviated by placing the columns A and B on two thin circular drums mounted on a common axle, and by putting the frequency marks on a third drum, placed between the first two. (This drum may be a part of drum A.) Careful construction may then

extend the applicability of the device to ranges as great as from 1 to 1000 without prohibitive bulkiness. Several simplifications may also be used in particular cases. For instance, if it is of no advantage to calculate the means from column A it becomes unnecessary to write out the numbers on A in full, so that the flexibility of the device increases.

The method was devised while the writer was temporarily employed at the Institute of Anatomy of the University of Minnesota, and it is a pleasure for him to acknowledge the encouragement and helpful advice he received from Dr. Richard E. Scammon.

III. A Corollary to Cauchy's Integral Formula

By Norman Miller, Queens University

A well known theorem in the theory of functions of a complex variable states that if $f(z)$ is continuous on a curve C open or closed, then the integral $(1/2\pi i)\int_C f(t)dt/(t-z)$ denotes an analytic function of z in any connected region of the plane which does not include points of C. If, in particular, C is a closed curve and f is analytic in the closed region bounded by C then the integral represents the function $f(z)$ at all points within C and the function zero outside C. The purpose of the present note is to prove the following extension of this theorem.

Theorem: *Suppose that $f(z)$ is a function which in a certain region R is single valued and analytic wherever defined and has no removable singularities; suppose also that C is a closed curve within R, on which $f(z)$ is analytic and within which it has at most a finite number of isolated singular points; if $\phi(z)$ is defined within C by the formula $(1/2\pi i)\int_C f(t)dt(t-z)$ and defined on and outside C by analytic continuation of this function wherever possible, and if $\psi(z)$ is defined outside C by the same formula and defined on and within C by analytic continuation wherever possible, then $f(z)$ is identical with $\phi(z)-\psi(z)$ at all points of R where f is defined.*

Let $f(z)$ have, within C, k singular points having the principal parts

$$P_1(z), P_2(z), \cdots, P_k(z) \text{ and let } P(z) \equiv P_1(z)+P_2(z)+\cdots+P_k(z).$$

Then $f(z) = \bar{f}(z)+P(z)$ where $\bar{f}(z)$ has only removable singularities within C, which we shall suppose removed. If z is a point within C we have

$$\phi(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} = \bar{f}(z) + \frac{1}{2\pi i}\int_C \frac{P(t)dt}{t-z},$$

of which the latter term is the sum of k terms of the form $(1/2\pi i)\int_C P_i(t)dt/(t-z)$.

If $P_i(z)$ is the principal part of a pole at a point z_0 it consists of a finite number of terms of the form $a_n/(z-z_0)^n$, n being a positive integer. We have then to consider

$$\int_C \frac{dt}{(t-z_0)^n(t-z)} \tag{1}$$

which may be written

$$\int_C \frac{dt}{(t-z_0)^n(t-z_0-\overline{z-z_0})} = \int_C \frac{1}{(t-z_0)^{n+1}}\left\{1+\frac{z-z_0}{t-z_0}+\left(\frac{z-z_0}{t-z_0}\right)^2+\cdots\right\}dt.$$

Restrict z to lie within a circle Γ of center z_0, within C. The series of functions of t which constitutes the integrand is then uniformly convergent in a region including the curve C and may be integrated term by term around this curve. Since $n \geqq 1$, it follows that the integral of each term is zero and hence (1) is zero within the circle Γ. But (1) represents an analytic function of z at all points within C. This function is therefore zero.

Again if $P_i(z)$ is the principal part arising from an essential singular point z_0 it is an infinite series of the form $\sum_{n=1}^{\infty} a_n/(z-z_0)^n$, which converges at all points of the plane except z_0 and converges uniformly in any region exterior to a circle of center z_0. Consequently the integral $\int_C P_i(t)dt/(t-z)$ may be found by termwise integration. By the foregoing proof the integral of each term vanishes. We conclude finally that $\int_C P(t)dt/(t-z)=0$ for all points within C. Hence $\phi(z)\equiv\bar{f}(z)$ within C and $\bar{f}(z)$ furnishes the analytic continuation of $\phi(z)$ as far as this is possible outside C.

Consider next a point z outside C. Since $\bar{f}(z)$ is analytic within C it follows that $\int_C \bar{f}(t)dt/(t-z)=0$. Hence $\psi(z)=(1/2\pi i)\int_C P(t)dt/(t-z)$. The reasoning already used shows that this integral may be found by termwise integration, a typical term being

$$\frac{a_n}{2\pi i}\int_C \frac{dt}{(t-z_0)^n(t-z)} = \frac{a_n}{2\pi i}\int_C \frac{dt}{(t-z_0)^n(t-z_0-\overline{z-z_0})}$$
$$= -\frac{a_n}{2\pi i(z-z_0)}\int_C \frac{1}{(t-z_0)^n}\left\{1+\frac{t-z_0}{z-z_0}+\left(\frac{t-z_0}{z-z_0}\right)^2+\cdots\right\}dt.$$

If, for the moment, we restrict z to lie outside a circle Γ' of center z_0 including C, then the series which forms the integrand converges uniformly and may be integrated term by term. The integral is zero for every term except one, viz.,

$$-\frac{a_n}{2\pi i(z-z_0)}\int_C \frac{dt}{(t-z_0)(z-z_0)^{n-1}} = -\frac{a_n}{2\pi i(z-z_0)^n}\cdot 2\pi i = -\frac{a_n}{(z-z_0)^n}.$$

This value holds at all points outside the circle Γ'. But since the integral from which it arises denotes an analytic function at all points outside C the same expression gives its value at all these points. It follows that for all points outside C, $\psi(z) = -P(z)$. Moreover $-P(z)$ furnishes the analytic extension of $\psi(z)$ as far as this is possible within C.

We have finally that for all points in R at which f is defined, $f(z) = \bar{f}(z) + P(z) = \phi(z) - \psi(z)$, as the theorem states.

When the function $f(z)$ and the curve C are given, the method of the foregoing proof serves to determine the functions $\phi(z)$ and $\psi(z)$. Thus, if the curve C be taken as the unit circle and the region R as the complete z-plane, simultaneous values of f, ϕ, and ψ are given in the following examples.

(1) f analytic within C, $\quad \phi = f, \quad \psi = 0.$

(2) $f = z + \frac{1}{z}, \quad \phi = z, \quad \psi = -\frac{1}{z}.$

(3) $f = \frac{1}{z(z-2)}, \quad \phi = \frac{1}{2(z-2)}, \quad \psi = \frac{1}{2z}.$

(4) $f = \frac{z}{4z^2-1}, \quad \phi = 0, \quad \psi = \frac{-z}{4z^2-1}.$

(5) $f = \frac{z^2+1}{(4z^2+1)(z^2+4)}, \quad \phi = \frac{1}{5(z^2+4)}, \quad \psi = \frac{-1}{5(4z^2+1)}.$

(6) $f = e^z + e^{1/z}, \quad \phi = e^z + 1, \quad \psi = 1 - e^{1/z}.$

(7) $f = 2z^2e^{1/z}, \quad \phi = 2z^2 + 2z + 1, \quad \psi = 2z^2 + 2z + 1 - 2z^2e^{1/z}.$

(8) $f = \sin\frac{1}{z}, \quad \phi = 0, \quad \psi = -\sin\frac{1}{z}.$

(9) $f = \cos\frac{1}{z}, \quad \phi = 1, \quad \psi = 1 - \cos\frac{1}{z}.$

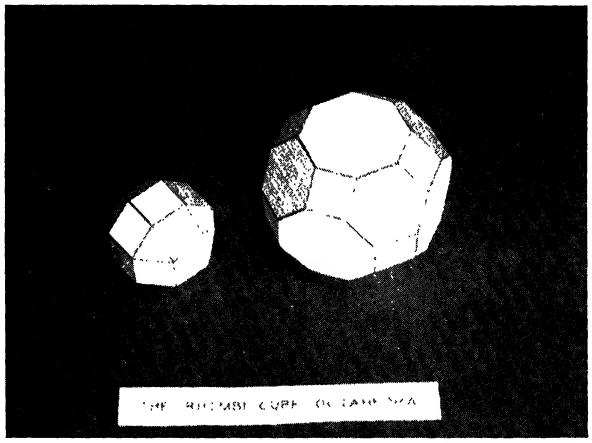
(10) $f = z\cos\frac{1}{z}, \quad \phi = z, \quad \psi = z - z\cos\frac{1}{z}.$

IV. Note on an Error in the Encyclopedia Britannica

By Michael Goldberg, Washington, D.C.

A list and description of the thirteen Archimedean solids (not including the prisms and prismoids), which are defined as polyhedra bounded by regular

polygons not all of the same species, and which have all the solid angles equal, is given on page 28, vol. 22 of the Encyclopedia Britannica (11th edition) under



the article "Polyhedron." The rhombi-cube-octahedra are incorrectly described. The small one is given as a polyhedron bounded by 12 *pentagons*, 8 triangles and 6 squares, while the actual figure is bounded by 18 *squares* and 8 triangles. The large one is described as a polyhedron bounded by 12 *decagons*, 6 *squares* and 8 *triangles*, while the actual figure possesses 12 *squares*, 6 *octagons* and 8 *hexagons*.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems. By GRIFFITH CONRAD EVANS. New York, The American Mathematical Society, Colloquium Publications, Volume VI, 1927. viii+150 pages. Price $2.00.

"The work gives a unified treatment of the basis of the theory of Laplace's equation in two dimensions, suitable, it is hoped, for graduate students of a moderate degree of advancement, and is intended to be of service in the development of the theory of partial differential equations of elliptic type. These developments are generating a compound of two of the most important elements of modern analysis—the concepts of Lebesgue on the one hand, and of Volterra on the other."

Algebraic Arithmetic. By ERIC T. BELL. New York, The American Mathematical Society, Colloquium Publications, Volume VII, 1927. iv+180 pages, Price $2.50.

"There is a large, growing and somewhat uncoordinated body of results, with many aims and methods peculiar to itself, which falls into neither the classic theory of numbers nor the modern developments of the analytic and algebraic theories, and this region, which we have called algebraic arithmetic,

* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

offers many suggestive opportunities for systematic exploration. It is the purpose to outline a few promising directions in which progress may be made toward classifying, extending, and generalizing the methods and results of algebraic arithmetic."

Spherical Harmonics, An Elementary Treatise on Harmonic Functions with Applications. By T. M. MACROBERT, M.A., D.Sc., Professor of Mathematics in the University of Glasgow. New York, E. P. Dutton and Company, 1927. xii+302 pages. Price $4.50.

"..... A text-book on the elements of the theory of the spherical harmonics, with applications to mathematical physics, so far as this could be done without employing the method of contour integration, [and including] discussions on similar lines of Fourier series and Bessel functions, with corresponding applications."

The Pythagorean Proposition, its Proofs Analyzed and Classified, and Bibliography of Sources. By ELISHA S. LOOMIS. Given by the Author to the Masters and Wardens Associations of the 22nd Masonic District of the Most Worshipful Grand Lodge of Free and Accepted Masons of Ohio and Published by said Association. 214 pages. Price $2.00.

"The object of this work is to present to the future investigator, under one cover, simply and concisely, what is known relative to the Pythagorean Proposition, and to set forth certain established facts concerning the proof and the geometric figures pertaining thereto." Some 230 proofs of the theorem are given.

Análisis Matemático, por el Dr. HUGO BROGGI, Professor titular de las Universidades de La Plata y de Buenos Aires. Volumen II, Teorías Generales, Funciones de Más de Una Variable. La Plata, Facultad de Ciencias Físico-Matemáticas, 1927. Price 6 pesos.

This South American text deals with the following topics: planes and straight lines, determinants, quadric surfaces, complex numbers, simple differential equations, vector analysis, series, functions of more than one variable, curvilinear and multiple integrals.

Sir Isaac Newton. By C. D. BROAD, Fellow of the British Academy. Oxford University Press, American Branch, New York, 1927. 32 pages. $0.70.

This is the 1927 "Annual Lecture on a Master Mind" of the British Academy, and presents a survey of the life and the scientific work of Newton.

REVIEWS

History of the Sciences in Greco-Roman Antiquity. By ARNOLD REYMOND, translated by Ruth Gheury de Bray. New York, E. P. Dutton, n.d. [1927] viii+245 pages. Price, $2.50.

This is a translation of a work which appeared in 1924 under the longer title *Histoire des Sciences Exactes et Naturelles dans l'Antiquité des Écoles et des Principes.* Each edition has the honor of having a preface by M. Brunschvicg, "membre de l'Institut" and a scholar of standing. The author holds the chair of professor of philosophy in the University of Lausanne, and this work is based upon his lectures there and at the University of Neuchâtel. By reason of its sponsor, its author, and its translator it should rank as an authority, or at least be a safe guide for beginners. Reluctantly it must be said, however, that it is neither. What the French edition needed was not primarily a translator; it needed an editor. M. Reymond is a philosopher, but he is far from being a reliable historian; he has a general view of the development of the sciences, but his vision of details is imperfect.

As to the history of mathematics, his footnotes show only a slight acquaintance with secondary authorities, and in no case does he give evidence of having consulted any sources. He mentions Eisenlohr's edition of the Rhind Papyrus, but the text leads to the belief that he never read it; his translator has inserted a reference to Peet's edition, but it appeared after the French edition was written. He mentions the Moscow Papyrus, but expresses a doubt as to its existence at the present time, although his translator could easily have found that its contents are now known and that it is still accessible to scholars. He makes the assertion that the Egyptians were ignorant of the art of calculating an angle,—an ambiguous expression at best, and certainly untrue at its worst.

With respect to the accuracy of statement, both the French and the English versions show too many defects to allow them to pass unnoticed. To select one of a considerable number for example, Parmenides is given as having been born in 516 B.C. of which we are by no means sure; Zeno is given as born in 489, of which we are even more uncertain; and the latter is stated to have been twenty-five years younger then the former, which does not harmonize with the dates which M. Reymond gives as precise. If these were isolated errors, the matter would be unworthy of mention, but to express doubt as to the dates for writers like Ptolemy and to assign exact years for most of the other Greek scientists (generally quite uncertain), to give the dates 330–270 to Euclid, and to have Archimedes born in 257 B.C. instead of 287 (an error in the English edition only), is to go beyond the bounds of a historian's patience.

As to the translation, this is in the main very satisfactory. It is difficult, however, to see why "l'ancienne *philosophie naturelle* de l'Ionie" should appear in the much less acceptable form "the ancient *philosophy* of Ionia," or why "le rapport étroit" should have the uncompromising rendering of "inseparable connection." Such minor slips of the pen or of the typist as "Dox, 476" instead of "Dox., 476" are of course, not to be charged severely against any translator, however careful. Whether the French "On désigne souvent sous le nom d'hylozoïsme" should appear as "is often given the name Hylozoism," with the emphasizing capital and with no indication of the etymology of the "name" is a matter that may well be left to the judgment of the readers of this review. Since the Greek is not infrequently given in the text, it would seem to be appropriate to give the roots of such a word as this,—a word not so familiar in the history of science, no doubt, as in the history of philosophy.

It is quite unnecessary to dwell further upon particulars. Suffice it to say that the cases cited are, unfortunately, so typical as to warrant the judgment that, so far as it relates to mathematics, the work is unreliable and cannot be recommended for reference purposes or as a handbook to be used by students.

DAVID EUGENE SMITH

The Byzantine Astrolabe at Brescia. By O. M. DALTON. London and New York, Oxford University Press, American Branch. 14 pages+3 plates. Price 70 cents.

This memoir, communicated to The British Academy on July 7, 1926, describes a Byzantine astrolabe which was presented to the Museo dell' Età Cristiana at Brescia, Italy, by Signora Sajler, in 1844. The instrument bears a date equivalent to 1062 of our era. It is therefore contemporary with that celebrated Neoplatonist, Michael Constantine Psellus (1020–1110), whose works on mathematics were often printed in the sixteenth century and which, up until now, gave us the only knowledge that we had of this science in Constantinople in his time. The instrument, indeed, might have been conjecturally ascribed to him had it not been for the fact that one of the inscriptions relates that it was made by "Sergius the Persian," a man of consular rank, and that another states that it was made under the "decree and command of Sergius."

The author adds to the value of his study by giving a brief description of astrolabes in general and a reference to earlier Greek writers upon whose works Sergius or his craftsman may have depended,—chiefly Joannes Philoponus[1] and Severus Sabokt[2] (Sebokht, Sabocht), bishop of Kennesrin (Quenšerin, Kenneshre), on the Euphrates each of whom depended upon writers like Ptolemy, Proclus, and Ammonius, and upon the poet-orator Synesius, who had studied under Hypatia.

The inscriptions are all in Greek and although Sergius speaks of himself as a Persian the instrument gives evidence of only the Byzantine influence. The significance of the piece is thus summarized by the writer:

"Before the examination of the Brescia astrolabe, it was possible to question the activity of medieval Byzantine astronomers, but this is a position which can no longer be maintained. Sergius cannot have been the sole ἐπιστήμων[3] of his age and city, nor can the craftsman who worked for him have been the single Greek maker of mathematical instruments. The astrolabe of Sergius proves that Hellenistic science was remembered as late as A.D. 1062 and makes it probable that astronomical studies continued until the time of the Fourth Crusade. This instrument, inscribed and precisely dated, takes its place as a historical document; it has all the evidential value of a manuscript."

DAVID EUGENE SMITH

[1] The author says that he "lived in the first half of the seventh century," which is probable, but the date commonly assigned (*c.* 640) is quite uncertain, as is also the century.

[2] He lived in the first half of the seventh century.

[3] In the inscription the word is EIIICTINOC, which the writer conjectures to stand for ἐπιστῆμονος, a savant.

Modern Plane Geometry. By J. R. CLARK and A. S. OTIS. Yonkers, World Book Company, 1927. x+310 pages.

Plane Geometry. By D. M. BERNARD. Richmond, Johnson Publishing Company, 1927. xv+334 pages.

The first of these books, unlike English books with similar titles, does not deal with topics in plane geometry developed in recent times. The book is modern in method rather than in content, and contains the standard topics of the recognized secondary school course. But it is emphatically not just another plane geometry text-book. "In this book," the authors claim, "you will find for the first time, carefully planned and fully prepared, that development of each topic which psychology teaches and which your experience has shown you to be essential to the complete understanding of the statement and proof of a proposition." The authors further assert that tests in some twenty schools, in which experimental editions of their book were brought into competition with old style texts giving complete synthetic proofs, showed their book "significantly superior in developing *power*" and "slightly superior in developing *information*." In the hands of skillful, wise and sympathetic teachers, it seems probable that this record will be maintained in the larger field now open.

This greater efficiency has been secured by giving the student a different attitude than is common towards the work before him. The authors consistently place the student in the position of a discoverer, not only of proofs but of facts. "Do you think that if a triangle has two equal angles, the sides opposite those angles are equal?" "Ernest reasoned as follows Do you see any error in the reasoning?" The student is also met throughout by a series of *challenges*. These usually ask the student to prove a fact whose truth is suspected without the use of the next section in the book. If he succeeds he may give himself a grade of A in a table provided for that purpose at the back of the book. If he fails, he may read the next section for suggestions; if, with the help of these hints, he succeeds in devising a proof, he marks himself B. This system of self-classification is extended also to work with originals. Other distinctive features are found in this text including an excellent "introductory course" leading up to the demonstrative geometry. But perhaps enough has been said to indicate that the book possesses great merit and originality and deserves a thorough study by teachers. Not all of them will approve of all its features or find it adapted to their own methods of teaching the subject; but all can profit by its consideration.

The merits of this progressive text are so great, that overstatement in the preface may be forgiven. For the book, though a pioneer in modernism, is not entirely without precursors. For example, among books known to the reviewer, the *Elementary Geometry* by the English authors Godfrey and Siddons,

published in 1903 by the Cambridge University Press, featured a thing,—"systematic training in the discovery of geometric facts"—that Clark and Otis say "has not appeared in any previous text book;" and E. R. Smith in his *Plane Geometry developed by the Syllabus Method*, published in 1909 by the American Book Company, taught students to discover proofs, if not facts, for themselves.

The *Plane Geometry* by Bernard will appeal to the more conservative teacher. Its chief departure from the traditional book is that in its developed proofs, after the statement of what is given and what is to be proved, a brief outline of the proof is inserted, labelled *method*, before the proof is set out in detail. In many cases, the method is given without the detailed proof. While this seems like a small thing, the reviewer believes that here we have a real improvement. No doubt numbers of teachers have been encouraging their pupils to analyse the given facts and to see the reason for the steps used before learning the proof. This brief outline will be of great help here. Curiously enough, in those proofs which are given at length in the Clark-Otis book, exactly the same thing is done, the outline here being labelled *plan* instead of *method*. Bernard's book is characterized by a clear statement of proofs, good diagrams and an excellent selection of originals, including groups of supplementary excercises taken from entrance examinations at a number of colleges and credited to their source. Wherever possible the author has simplified his proofs by using single letters for angles and for line-segments. The theorems are not assigned numbers, other than those of the sections. Both books are attractively manufactured, meet every requirement as to content, and deserve well of teachers of mathematics in secondary schools.

J. W. CLAWSON

Integral Bases. By W. E. H. BERWICK. Being number 22 of Cambridge Tracts in Mathematics and Mathematical Physics, edited by G. H. Hardy and E. Cunningham. Cambridge University Press, 1927. 5+95 pages.

The algebraic integers in a field of algebraic numbers of degree n form an aggregate of the type

$$x_0\omega_0 + x_1\omega_1 + \cdots + x_{n-1}\omega_{n-1},$$

where $x_0, x_1, \cdots, x_{n-1}$ are rational integers, provided that $\omega_0, \omega_1, \cdots, \omega_{n-1}$ is a suitably chosen set of algebraic integers in the field. The n integers which form such an *integral basis* of the field are not unique but every two integral bases are connected by linear homogeneous relations of a certain particular type. A quadratic field can always be defined by an equation of the form $\theta^2 - m = 0$, where m is an integer which is divisible by no square factor greater than unity; and then an integral basis is of the form

$$(1, \tfrac{1}{2} + \tfrac{1}{2}\theta) \text{ or } (1, \theta) \text{ according as } m \equiv 1 \bmod 4 \text{ or } m \equiv 2 \text{ or } 3 \bmod 4.$$

Methods of finding an integral basis of a cubic field have been given independently by G. B. Mathews and G. T. Woronoj. Kummer has obtained an integral basis of a cyclotomic field, while Hilbert's detailed investigations of quadratico-quadratic fields include a discussion of the algebraic integers belonging to them. Beyond these cases little appears to have been done heretofore in developing methods for constructing integral bases for algebraic fields.

In this tract the author develops a method of finding all the algebraic integers of the field $[\theta]$ defined by a rational integral irreducible equation

$$a(\theta) \equiv \theta^n + a_1\theta^{n-1} + a_2\theta^{n-2} + \cdots + a_n = 0.$$

His researches arose from his investigation of the special case $\theta^n - a = 0$, the problem to which he devotes his final chapter XV which requires about one-third of the whole monograph for its treatment. In this particular problem there are twenty-three cases which demand discussion. In treating this special case the author was led to the methods which he has applied to the general problem named.

The first five chapters (pp. 1–16) are devoted to preliminaries, including (among other things) a summary of the tract and an account of Dedekind's theory of algebraic numbers. The new methods are discussed in chapters VI to VIII (pp. 17–35) and an integral basis of the field defined by the equation $a(\theta) = 0$ is treated in chapter IX (pp. 35–39). The rule for putting down an integral basis follows a process of approximation, the first three stages of which are worked out in the tract. In his preface the author says, "Failing cases exist but the approximations given are sufficient to cover nearly any numerical equation not specially constructed to defy them." The author states that he has carried through a fourth stage in the approximation but that it was not thought necessary to print this since it appears to lead no nearer to finality than the three stages which are developed in the tract. Many of the methods apply equally to a relative field, a fact of which use has been made in the treatment of the special case of chapter XV. In the course of the tract a few numerical examples are worked out to illustrate the power of the methods when applied to a concrete case. An integral basis of $[a^{1/n}]$ is obtained for all values of a and n.

R. D. CARMICHAEL

A Table of Interpolation Multipliers. By A. S. LITTLE. Boston, Financial Publishing Company, 1927. 29 pages.

If $A, B, C, \cdots$, are tabulated values of a function for equal increments of its argument, then

$$A_1 = B - A, \quad A_2 = C - 2B + A, \quad A_3 = D - 3C + 3B - A, \cdots.$$

are known as the first, second, third, $\cdots$, differences and in terms of them the Gregory-Newton interpolation formula is written

$$A + kA_1 + \frac{k(k-1)}{2!}A_2 + \frac{k(k-1)(k-2)}{3!}A_3 + \cdots.$$

If it be assumed that all differences above the third vanish, this formula may be rearranged in the form

$$a(k)A + b(k)B + c(k)C + d(k)D.$$

When so written the multipliers containing k are independent of the function and so if tabulated would be usable for any interpolation problem involving only third differences. The author of this highly useful book has done precisely this work of tabulating these multipliers for $k = .002, .004, \cdots, 998$ and has arranged the whole in a very convenient form for use on a computing machine.

Unfortunately the book is published in a form which assumes its use only in the evaluation of bond purchase prices, and so is addressed to a non-mathematical public. A distinct gain would be made if a proper mathematical introduction were added and also the actual values of k were given in the tables.

C. F. CRAIG

Plane Trigonometry and Tables. By H. E. BUCHANAN and PAULINE SPERRY. Richmond, Johnson Publishing Co., 1926. xi+116 pages, tables 112 pages.

The subject matter in this book is presented in a clear, concise manner. There is no duplication of material and no essential facts are overlooked. The exercises are excellent.

Radian measure is introduced on the first page and is applied in problems throughout the book. The logarithmic solution of right triangles is taken up before the functions of the general angle. A clear explanation of logarithms and of the tables is given in the text, together with enough facts and illustrations to enable the average student to comprehend what is meant by accuracy of results. The tables are quite satisfactory though we wonder why the three place table of logarithms of numbers is carried only to 500 instead of to 1000. There is, of course, a five place table of logarithms of numbers 1 to 10,000.

In dealing with general angles, identities, etc., the book follows the usual procedure. The symbol ∞ is used to indicate "a state of affairs, and is not a symbol for a number." One always hopes in vain that a new text in trigonometry will state the simple truth, that tan 90° does not exist. The proof of the addition formulas by projection seems to offer some advantages over the usual method, especially in the case of obtuse angles.

The chapter on the solution of oblique triangles is short and to the point. The student is not burdened with a mass of unnecessary formulas. All cases

are covered. The applications to polar coordinates and to wave motion are well presented. We would prefer to see the curves in cartesian coordinates drawn for negative values of the abscissa as well as for positive values.

P. A. FRALEIGH

Plane Trigonometry with Tables. By M. A. KEASEY, G. A. KLINE and D. A. MCILHATTEN. Philadelphia, P. Blakiston's Son and Co., 1927. 63 pages +tables. Price $1.28.

This text has been written with emphasis on the use of Trigonometry in Engineering and "contains all the necessary material for college preparation."

The problems in the text are numerous and of sufficient variety from the point of view of usefulness to impress the beginner with the value of the subject. They constitute , in the opinion of the reviewer, the best part of the text. The definitions of the first few pages are a good introduction to the subject. The tables consist of a five place table of logarithms of numbers from 100 to 1000, a five place table of the logarithms of the trigonometric functions at one minute intervals, and a four place table of the natural functions. There are no tables of proportional parts. The book is provided with a good index.

There is, however, a great lack of mathematical exactness, expecially in the treatment of the new functions from the point of view of analytics. The usual discussion of directed lines is omitted, the definitions of coordinates are incomplete, and in the figures lines are read indiscriminately in either direction. There are also many careless forms of expression which detract from the accuracy of thought.

If there is no desire to teach anything of trigonometry beyond the numerical solution of triangles, the book may prove of advantage; but unless one can be sure that the mathematics learned therefrom will not be required in later work, especially in analytics, the presentation seems unmathematical.

A. W. SMITH

Die Vektoranalysis und ihre Anwendung in der theoretischen Physik. By W. V. IGNATOWSKY. Third Edition. Teil I: Die Vektoranalysis, X+110 pp. Teil II: Anwendung der Vektoranalysis in der theoretischen Physik, IV+123 pp. Leipzig-Berlin, Teubner, 1926.

This is the third edition of Ignatowsky's book, which first appeared in 1909-10. Reviews of the first edition can be found in the Bulletin of the American Mathematical Society (November, 1910, p. 100) and in the Jahrbuch über die Fortschritte der Mathematik (bd. 41, h. 1, 1910). No substantial changes are made in this new edition, except a few minor additions.

In chapter I, an article on reciprocal systems of vectors is added, which is

utilized in proving that the definitions of gradient, divergence and curl are independent of the form of the volume.

In chapter IV, a change is made in the proof that a vector function can be represented as a sum of a curl and a gradient. In the chapter on dyadics, the sections on the relations between polar and axial vectors are replaced by detailed discussions of linear vector functions, tensors, etc.

The theory of linear vector functions is then applied, in vol. II, to the discussions of the properties of elastic bodies. On the whole, the book ought to make profitable reading for students in applied mathematics.

S. D. ZELDIN.

Die Ebene Vektorrechnung und ihre Anwendungen in der Wechselstromtechnik. By H. KAFKA. Teil I: Grundlagen, VII+132pp. Leipzig-Berlin. Teubner, 1926.

This book is mainly intended for students in electro-technics. It treats the theory of complex quantities from a geometrical standpoint and their applications to electrical problems. The book ought to appeal to the electrical engineer, since it gives him something concrete to work with. In the first chapter plane vectors are defined, the operations of addition, subtraction and multiplication (scalar and vector) of vectors are discussed and amply illustrated by diagrams. The second chapter studies the sinusoidal time functions and their vectorial representation. The remaining chapters are devoted to the vector representation of principles in electro-magnetic theory.

S. D. ZELDIN.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activites should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB TOPICS

THE EVOLUTION OF NUMBERS—AN HISTORICAL DRAMA IN TWO ACTS*

By H. E. SLAUGHT, University of Chicago

ACT I: INTERNAL DISSENSION IN THE SECRET SOCIETY OF NUMBERS.
Time: 1625 A.D. Place: Vestibule of the Secret Chamber of the Society in Paris.
Dramatis Personae:
The Cardinal: (Representing Whole Numbers, the original Aristocrats).
The Barbarian: (Representing Negative Numbers, the First Interlopers).
The Arab: (Representing Zero, the Second Interloper).
The Hoi Polloi: (Representing Common Fractions).
The General: (René Descartes in Charge of Personnel).

Prologue to Act I. (*A young speaker dressed in heraldic costume stands between the parted curtains.*)
My fellow devotees of Mathematics, you will agree that the Number concept is as old as the human race. Let me recall to you some of the struggles through which this concept has come in order to

* This playlet may be found helpful in arranging entertainment features for undergraduate mathematics clubs.

reach its present development. In primitive times there were only a few whole numbers, used for counting, and a few special fractions each with numerator unity. The idea of negative numbers was the outgrowth of the experience of many centuries. It was Descartes in the seventeenth century who conceived the notion of picturing numbers by points on a line extending in both directions from a zero point and thus giving negative numbers and zero a concrete and natural representation. The aristocratic whole numbers then became Descartes' positive integers. They had maintained a secret organization, so to speak, for some 3000 years and had persistently black-balled the negative numbers every time they had sought admission. Likewise zero had been an outcast, a nobody, since the childhood of the race. Even the resourceful Greeks and the mighty Romans were deprived of renown which might have been theirs if they had only been wise enough to admit the humble zero to their clumsy number orgainzations. What they failed to do the wiser Hindus and Arabs accomplished later on when they brought back to Europe the Hindu-Arabic number symbols and the decimal notation—possibly the greatest invention of all time. Well might Descartes be awarded martial honors for his services as "general in charge of personnel" when he stationed the positive and negative integers and zero as sentinels on his Number Line and then filled in the intervening spaces with the lowly fractions, now recognized as swarming in countless hordes, thus forming the battle front of the mathematical warfare which was to conquer the world.

(*The speaker withdraws from the stage and the curtains open. The Barbarian enters and tries to open the door to the adjoining room, the Secret Chamber of the Society of Numbers. The Cardinal then enters and finds the Barbarian trying to break in. Each actor wears a placard on his breast showing his name in large print.*)

The Cardinal (*Dressed in a cardinal robe*): Is this you, the Barbarian, trying again to gain admittance to the secret chamber of our Number Society?

The Barbarian (*Dressed in a rakish costume*): Yes, General Descartes has told me that we are fully entitled to a place coordinate with you on his new Number Line.

The Cardinal: This is an outrage. We, the Cardinals, have held the major positions in our Society for 3,000 years and it is preposterous now for General Descartes to assign us to positions on the right wing and you to equally important positions on the left.

The Barbarian: But you Cardinals cannot hold the modern battle line alone. It may have been simple enough a few centuries ago when your Number Society was small and there was no need for an extended battle front.

General Descartes (*Entering in a costume appropriate to the time. See the Open Court Portraits of Famous Mathematicians*): The Barbarian is quite right. The number army has been only half organized heretofore and many a battle in mathematics has been lost solely because the army had no left wing. What would you think of an army in the military world that should try to do its fighting with one wing while the enemy was free to execute a flank movement on the unprotected side?

The Barbarian: As an example, there was Captain Cardan a hundred years ago, who needed us Negative Numbers the very worst way in order to complete his victory over the Cubic Equation. We were there ready and anxious to do our part but he spurned our assistance and called us "fictitious" and "absurd."

The General: You Cardinals and Barbarians must quit your quarreling and divide the responsibility of sentinel duty on the Number Line, the Cardinals on the right wing and the Barbarians on the left.

The Arab (*Now entering clad in typical Arab costume*): Yes, but that leaves an unguarded sentinel post between the wings and I alone, the long despised and rejected Zero, am the only one qualified to guard this point. I am absolutely the center of things, the *King Pin*, so to speak, in this whole army formation.

The Hoi Polloi (*Clad in midget costume, entering just as the Arab ceases speaking*): But an army must have something more than King Pins and Sentinels! After all, the hordes of Common Fractions are the ones who constitute the real battle front and we are the only ones qualified for this service.

The Cardinal: Well, if we *must* divide the honors with these Barbarians and the Arab, then our humiliation is complete. We are aristocrats no longer.

The General: But World Progress demands it. You have blocked the world's business for 3,000 years by persistently black-balling these applicants for admission to your Society. Take the Arab, for instance. Without him the ancient Greeks with all their boasted culture could do but little in arithmetic. The mighty Romans could conquer the world in war but they could not master Arithmetic without the help of Zero. It remained for the Hindus and the Arabs to show the world a decimal number system in which Zero was indeed the "king pin." The Arabs brought this wonderful system to Spain and Leonardo of Pisa introduced it to the rest of Europe; but it took Europe three centuries to fully appreciate this great contribution of the East to the West. Now the Arab and the Barbarian shall come into their own. They shall occupy positions of honor on the Number Line and you Cardinals shall recognize them as your equals. You think it will detract from your glory, but, quite the contrary, it will be your making. You have been sitting around for 3,000 years doing practically nothing but teaching people how to count, while the world's great problems have gone unsolved.

The Cardinal: We recognize our great short-sightedness in this matter and we will now follow your directions.

The General: And another thing! This secrecy business must stop. It began way back in Egypt when Ahmes, the Priest, wrote in the minutes of his famous Papyrus "directions for knowing all dark things." This sort of thing was continued down through the Centuries. Some one would solve a problem and carefully conceal the method and then challenge some opponent to find the solution. Even Cardan is said to have gotten his solution of the Cubic from Tartaglia under false pretenses with a promise never to give it away—which promise he deliberately broke.

The Cardinal: Yes, the vow of secrecy was a mistake. We will hereafter do our work in the open.

The General: Finally, you must adopt a Constitution and By-Laws and then examine each applicant for membership on his merits and not act on mere sentiment and prejudice, as you have done all these centuries with the Barbarian and the Arab. The only criterion for membership must be obedience to the By-Laws and allegiance to the Constitution.

The Cardinal: We will do these things, but what about the Arab, the Barbarians, and the Hoi Polloi? Will they also conform?

All in chorus: Yes, we will conform—so say we all of us.

(*Two attendants now enter bearing an exhibition of a well organized Number Line, consisting of a strip of wrapping paper ten or twelve feet long, on which a heavy line is marked off with units and labelled with large figures, positive, negative and zero. The Cardinal, the Barbarian, Zero, and the Hoi Polloi stand back of this Number Line, while General Descartes stands in front with a pointer indicating the respective subdivisions.*)

The General: Henceforth you Cardinals will continue to stand as sentinels on the right wing, and you Barbarians will do duty likewise on the left wing; while the Arab will stand guard at the key position between the two wings, and the Hoi Polloi will fill in the open spaces between the sentinel posts. Thus is the Number Line ready to attack the world's great problems.

(*The Curtain Falls*)

ACT II: INTERNATIONAL ARMISTICE AND PEACE IN THE NUMBER WORLD.

Time: 1825 A.D. Place: International Open Forum.

Dramatis Personae:

The Irrational, $\sqrt{2},$: (The Third Interloper)
The Complex, $\sqrt{-1},$: (The Fourth and Last Interloper).
The Continuum: (The Solid Front Number Line).
Professor Dedekind: (Representing the Great Divide).
Professor Gauss: (Representing the Complex Plane).

Prologue to Act II. (*Same speaker as in the Prologue to Act I, but now grown old, standing between the parted curtains.*)

My fellow devotees of mathematics, it is now 200 years since General Descartes organized his famous Number Line in which the aristocratic whole numbers were compelled to share the honors of sentinel duty with the negative numbers and zero. During these two centuries a lot of things have happened. When the vow of secrecy was removed from the Number Society as Descartes commanded, there was an immediate renewal of the attempt of the Irrational and the Complex to break into the system. Their struggle for recognition had already been going on for at least 600 years, but encouraged by the success of the Barbarians and the Arab in the seventeenth Century, they then battered harder than ever at the doors of the Number Society till the beginning of the nineteenth century when strong friends like Professors Dedekind and Gauss came to their aid and finally secured their admission. You are now to hear these actors in the Number Drama speaking for themselves.

(*The herald withdraws and the curtain is opened showing Professors Dedekind and Gauss, in costumes befitting the time* (*See the Open Court Collection of Portraits of Mathematicians*), *walking and talking together.*)

Professor Dedekind: You recall a bold program proposed by General Descartes two hundred years ago when he commanded the Society of Numbers to abolish secrecy and to adopt a Constitution and By-Laws.

Professor Gauss: Yes, I recall the story. It was, indeed, a bold program and it has taken just about two Centuries for its accomplishment. Oh, well, that is fast work compared to the slow reforms which General Descartes brought to a climax when he compelled the Cardinals to admit the Barbarians and the Arab to full fellowship in the Society.

Dedekind: It is less than ten years ago since those by-laws were fully agreed upon and published, namely, the Commutative, Associative and Distributive Laws which really govern our Number System.

Gauss: It took a long time for people to realize that these Laws do actually control the operations of our Algebra and that they are the only criteria for admission to the Society of Numbers.

Dedekind: Of course the Cardinals were the first to adopt these By-Laws, but it took a long time for people to see that the Barbarians, the Hoi Polloi, and Zero all conform to these same Laws, and hence actually belong to the Society just as much as the Cardinals.

Gauss: It does seem awfully slow progress as we look back upon it now, but when we realize how far we have come since Ahmes, the Egyptian Priest, started the organization 3500 years ago, it seems that we have done wonders in these last two hundred years.

Dedekind: Well, it is up to us to see to it that things move faster in this 19th Century than they did in the preceding centuries. It was a shame and a disgrace to Europe that the Hindu-Arabic symbols and the decimal notation which were brought to us from Arabia by way of Spain 600 years ago should have gone almost unnoticed for three centuries before we began to awaken from our long sleep of the Dark Ages to see that Asia had put into our hands what was destined to become the most powerful instrument ever devised by the mind of man.

Gauss: Yes, a great responsibility rests upon us of this generation. Already the members of the Society are getting busy. The Constitution is clear as to our duty. It states that all hands shall add, subtract, multiply, divide, raise to powers and extract roots unceasingly, in season and out of season. That is our job. Thus shall the world's business be transacted.

Dedekind: It is interesting to note that the first and only ammendment to the Constitution had to be enacted almost at the very start, namely, that division by zero shall be forever prohibited.

Gauss: Yes, I recall how some well-meaning people have tried to divide by zero and have thought that the quotient was a member of our Society, but have finally been convinced that none of our members can act in that capacity.

Dedekind: There is another curious phenomenon which has caused a lot of trouble in the last two centuries, namely, when our members began to extract roots promiscuously they soon ran across

such an anomalous case as the square root of 2 and found that it was neither a Cardinal nor one of the Hoi Polloi. Hence they naturally concluded that such a thing could not belong to our Number Society and so they refused its admission. This case has caused me a lot of trouble and anxiety, but I believe that I now have it satisfactorily explained.

Gauss: That is no more curious than the case that I have been worrying about, namely, the square root of negative one. These two interlopers have caused the Society more trouble than all the orthodox members put together. They have repeatedly applied for admission, and have been consistently rejected for many centuries. Here they come now. Let us hear what they have to say for themselves.

(*The Irrational and the Complex now enter, in costumes appropriate to the age, walking and conversing*).

The Irrational: I surely have had a rocky time of it. I have tried repeatedly to break into the Number Society but have always failed. They have even told me that I am nobody and have no respectable standing at all. But even Pythagoras knew better than that 2300 years ago. He saw that I represent the diagonal of the unit square and that I am just as respectable as the side of the square. Bhaskara in India 600 years ago recognized me as a thoroughly respectable citizen, and the Englishman, Robert Recorde, 300 years ago even wrote a book in which he paid sincere respect to me. But it remained for Professor Dedekind and others of the present time to really vindicate my rights and to atone for the injustice which has been done to me for three thousand years. I want to see him and thank him in person.

The Complex: If you think you have had a rocky time, what do you think of me? I did not have *even one* friend and defendant to do for me what Pythagoras did for you. Not one of the Indian members stood up for me as Bhaskara did for you. I was absolutely an outcast till just 200 years ago when Albert Girard recognized me as a legitimate citizen, while he was solving equations. It is true that Captain Cardan, 100 years earlier, had met me when solving his cubic but he said I was "curious" and "useless." So things went on until a Frenchman by the name of Argand wrote an essay a few years ago in which he defended me and showed why I should be admitted to the Number Society. But nobody paid any serious attention to this essay or to me till Professor Gauss put me on the map and I am hoping to see him and thank him personally for thus atoning for the three thousand years of injustice which *I* have suffered.

The Irrational (approaching Professor Dedekind): Here is Professor Dedekind now. Allow me to thank you kind sir, for what you have done for me. You cannot know what a relief it is to be admitted to the Number Society after thirty centuries of exile. I know that other friends have been working in my behalf, such as Professors Cantor and Weierstrass, but I am sure your name will go down in history as my emancipator and that the "Dedekind Cut" by which my shackles were broken will forever stand as one of the greatest achievements of mankind. Incidentally you will be pleased to know how gracefully and deferentially the Hoi Polloi have separated themselves into two groups at the "Cut" in order to allow me to take my place on the Number Line. Again, kind sir, let me thank you from the bottom of my heart.

Professor Dedekind: You overwhelm me with your words of praise and gratitude. I assure you that it has been one of the greatest joys of my life to be able to render you this service, especially when I realize that you represent an untold multitude of Irrationals, including the famous π and e, all of whom are now given places on Descartes' Number Line. You have my blessing. Go and take your part in the World's Work.

The Complex (approaching Professor Gauss): Ah! Here is Professor Gauss. I am so glad of this opportunity to meet you and to thank you in person for the wonderful way in which you have opened up life's opportunities for me. It was bad enough to be an outcast for three thousand years, but it was the last straw when my friend and companion, the Irrational, was finally given a "place in the sun" on Descartes' Number Line while I was told that there was absolutely no room left there for me.

But just as I was about to die of despair, your brilliant invention, the Gauss Complex Plane, rescued me from oblivion, and now I not only have my place in the sun but I am actually Lord and Master of the whole Number Field, of which the Descartes' Number Line is only a small and very special part. I want you to know, kind sir, that I appreciate this great honor and that I am fully

conscious of the tremendous responsibility which this wide-open Number Field places upon me. Again, kind sir, let me thank you from the bottom of my heart.

Professor Gauss: You have, indeed, been most patient and long-suffering but your long-delayed recognition has been due not so much to the prejudice of the older members of the Society as to their ignorance. They did not know, and I suppose they could not know, the great possibilities for service that your admission to the Number System will open up. I firmly believe that the 19th Century will justify our highest hopes and will, indeed, see you enthroned as Lord and Master of the whole Number Field.

The Continuum (A sturdy fellow bearing a placard on which a section of the Number Line is very closely divided to represent the myriads of real numbers): At last the struggle of mankind to build a Number System adequate to do the world's business is crowned with success. There have been many scenes of conflict in building this System. There have been jealousies, secret plottings, black-ballings, banishments, but now the mathematical world is at peace. There is complete international understanding and only healthful rivalry. As for myself, I am the symbol of complete union and harmony on the real Number Line. I represent the Solid Front that the Society of Numbers presents to the world. We are now fully prepared to underwrite the world's great problems.

(The drama closes with all the members of the Company on the stage beneath a banner on which is a graphic representation of the Real and Complex numbers. General Descartes and Professors Dedekind and Gauss are standing in front of the number Group, and all join in singing a stanza of "Hail, hail, the gang's all here.")

CLUB ACTIVITIES

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OKLAHOMA, Norman, Oklahoma.

During the year 1926–1927, the officers of the club were: First semester—Rothwell Stephens, president; Helen Meister, vice president; L. D. Montgomery, secretary-treasurer. Second semester—Max Heaslet, president; Narcissa Bond, vice president; Ruth Jenkins, secretary-treasurer. The meetings were held fortnightly on Thursday afternoons and refreshments were served before the beginning of the program.

The following topics were discussed:

October 26, 1926. "Welcome" by Dr. Reaves, Dean. "Practical value of mathematics" by Dr. Hassler. "Mathematical research" by Dr. Court. "Possible work of mathematics club" by Dr. Meacham.

November 18. "History of analytic geometry" by Miss Rivers. "Axioms of metrical and projective geometry" by Helen Meister.

December 9. "Excerpts of winnowed wisdom" by Wilma Gorton. "Ancient systems of numerical writing" by Eddie Eaves. "A mathematical story" by Narcissa Bond.

January 13, 1927. "Certain numerical curiosities" by Norma Pate. "Prime numbers" by Professor Townes.

February 10. "The life of Euclid" by Eunice Lewis. "How to find the day of the week having given the month, day of month, and year" by Charles Edmondson. "The development of pi" by L. D. Montgomery. "Multiplication on the fingers" by Dr. N. A. Court.

March 3. "Geometric constructions and paper folding" by Cleo Shaul. "Axioms applied to equations" by Mr. Paine.

March 24. "Non-Euclidean geometry" by Dr. Elsie McFarland. "Permutation and combination" by Houston Roane.

(Report by Mr. L. D. Montgomery)

MATHEMATICS CLUB OF ROCKFORD COLLEGE, Rockford, Illinois.

Organized in October, 1921.

Purpose: "To further interest in the field of mathematics and to provide an opportunity for the presentation of other interesting phases of mathematics and allied subjects not taken up in the class room."

Meetings are held once a month. The officers for 1926–1927 were: Mary Larmore ('27), president; Mary Burchfield ('27), vice-president; Hazel Kelly ('28), secretary-treasurer.

Program for the year 1926–1927:

October 1, 1926. Special meeting at which time it was decided to take in new members at the first regular session.

October 27. Sixteen new members were initiated. The purpose of the club and the plans for the year were discussed. Mathematical puzzles furnished entertainment. Refreshments were served.

November 18. Professor McGavock gave an interesting talk entitled: "If we were on the moon, what we would see." The program illustrated the relation of mathematics and astronomy.

December 15. A program illustrating the relation between mathematics and religion. Talks on the subject were given by Margaret Lauritzen ('27) and Ruby Tjaden ('27).

January 9, 1927. A paper written by True Kimbal ('30) on "Mathematics and architecture" was read.

February 16. The program showed the relation of mathematics and music. Katherine Bogardus ('27) gave a talk on this subject. Ruth Goller ('27) played piano selections and Dorothy May Anderson ('30) gave a cornet solo.

March 16. A talk on mathematics and physics was given by Professor Herrick. She said that mathematics was the language of physics.

April 18. Open meeting held in the Chapel. Dr. Miller gave a very enjoyable lecture on "Mathematics; a bridge between two generations." Refreshments were served.

May 2. Annual mathematics club banquet. The program consisted of a geometry book supposedly written by Dr. Miller. It contained many axioms and theorems concerning the future of the members of the club.

May 19. The relation of mathematics and English. Carolyn Keyt ('27) reviewed an article on the subject and Frances Hills ('29) read an interesting paper she had written. Mildred Greenlee ('27) gave statistics on the number of times mathematical terms were used in literature. The program was concluded by solving mathematical puzzles.

Officers for 1927–1928, elected at this meeting, are: Hazel Kelly ('28), president; True Kimbal ('30), vice-president; Dorothy May Anderson ('30), secretary-treasurer.

(Report by Miss Dorothy May Anderson)

The Mathematics Club of the University of North Carolina, Chapel Hill, N. C.

The Mathematics Club of the University of North Carolina for the session 1926–1927 opened its first meeting October 20, 1926, with membership limited to undergraduates who had continued in mathematics beyond their required work. The following officers were elected for the ensuing year: President, E. A. Cameron; vice-president, J. L. Ring; secretary-treasurer, Dan Hall; faculty adviser, Dr. J. W. Lasley, Jr.

The Club met at 7:30 P.M. as follows:

October 20, 1926. Election of officers and a talk by Mr. M. A. Hill.

November 17. "The maximum triangle in an ellipse." First method, E. A. Cameron; second method, J. L. Ring.

December 15. A social and a lecture on mathematical fallacies and puzzles by Dr. J. W. Lasley, Jr.

January 19, 1927. "Extended synthetic division" by W. S. Barney. "The apothem of a regular pentagon from a geometrical standpoint" by C. W. Hook.

February 16. "Duplication of the cube." First method, E. A. Cameron; second method, C. D. Whisnant. "History of the duplication of the cube" by Alton P. Hall.

March 16. A social and a talk by Mr. L. L. Garner.

April 18. "The trisection of an angle" by Dan Hall. "History of the trisection of an angle" by J. E. Johnson.

May 23. A social and a talk by Prof. E. T. Browne.

(Report by Mr. A. P. Hall)

THE EUCLIDEAN CIRCLE OF THE UNIVERSITY OF INDIANA, Bloomington, Indiana.

The officers for year 1926–1927 were: President, Lehman Shugart; vice-president, treasurer, Mary Francis Atlerton; secretary, Arbys Elizabeth Roberts; faculty advisors, Professors Rothrock, Davis, and Hennel.

Program for the year:

November 15. "Magic cubes and squares" by Richard C. Schneider.
December 6. "Three martyrs of science" by Harold T. Bentley.
December 20. Social Meeting—Memorial Hall.
January 10. "Recent astronomical discoveries" by Prof. K. P. Williams.
March 21. In commemoration of Sir Isaac Newton. 1. Addresses by Professor S. C. Davisson and K. P. Williams of the department of mathematics. 2. Address by Professor W. A. Cogshall of the department of astronomy. 3. Address by Professor Mason E. Hufford of the department of physics.
March 7. Social and Initiation.
April 18. "Conjugate functions" by William J. Kirkham.
May 2. "Generatrix functions" by Irene Price.
May. Picnic.

(Report by Professor H. T. Davis)

THE MATHEMATICS CLUB OF COLUMBIA UNIVERSITY, New York City.

The following meetings were held by the undergraduate mathematical club of Columbia University during the year 1926–1927:

1. "Geometrical fallacies" by Leonard Price.
2. "Number pegs" by A. R. Mosler.
3. "Mascheronian constructions" by P. Roseman.

(Report by Professor W. B. Fite)

THE MATHEMATICAL CLUB OF HARVARD UNIVERSITY, Cambridge, Mass.

During the year 1926–27, the officers of the club were: Mr. Robin Robinson, president; Mr. Morris Marden, secretary-treasurer.

The programs of the Club for 1926–27 were as follows:

October 20, 1926. "Riemann's theory of algebraic functions and functions related to them" by Professor W. F. Osgood.
November 3. "The skew square" by Mr. S. B. Sommerville.
November 17. "Topics in the calculus of variations" by Mr. T. L. Smith.
December 1. "European mathematics of to-day" by Professor G. D. Birkhoff.
December 15. "Legendre's normal forms and some applications" by Mr. T. R. Long.
January 5, 1927. "Motor calculus" by Mr. C. I. Lubin.
January 19. "Multiple integrals with singular integrands dependent on a parameter" by Mr. A. B. Brown.
February 16. A joint meeting was held with the Harvard Engineering Society. Dr. Joseph Slepian of the Westinghouse Electric Company spoke on "Some mathematical ideas in electrical engineering."
March 2. "An application of theta functions in the theory of numbers" by Mr. C. N. Liu.
March 16. "A recent treatment of maxima and minima" by Mr. S. S. Cairns.
March 30. "The determination of analytic functions" by Professor Philip Franklin of the Massachusetts Institute of Technology.
April 13. "The parallelism of Levi-Civita" by Mr. H. B. Hammat.
April 27. "Some circles related to the triangle" by Mr. G. B. Price.
May 11. "Entire functions" by Mr. F. C. Jonah.
May 25. "Parallelism in Finsler space" by Dr. Harry Levy. The winners of the Robert Fletcher Roger's prizes were announced as follows: Mr. C. N. Liu, first prize; Mr. G. B. Price, second prize.

The following officers were elected for the year 1927–28: Mr. Clarence I. Lubin, president; Mr. G. Baley Price, secretary-treasurer.

(Report by Mr. G. B. Price)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3310. Proposed by B. F. Finkel, Drury College.

Find the envelope of a system of circles having for diameters the secants of constant length, $2r$, of a conic.

3311. Proposed by A. Pelletier, Montreal, Canada.

Eliminate $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$ from the following equations:

$$a^{-2}x_1^2 + b^{-2}y_1^2 + c^{-2}z_1^2 = 1, \quad a^{-2}x_2^2 + b^{-2}y_2^2 + c^{-2}z_2^2 = 1, \quad a^{-2}x_3^2 + b^{-2}y_3^2 + c^{-2}z_3^2 = 1,$$

$$a^{-4}x_1x_2 + b^{-4}y_1y_2 + c^{-4}z_1z_2 = 0, \quad a^{-4}x_2x_3 + b^{-4}y_2y_3 + c^{-4}z_2z_3 = 0, \quad a^{-4}x_1x_3 + b^{-4}y_1y_3 + c^{-4}z_1z_3 = 0,$$

$$a^{-2}x_1x + b^{-2}y_1y + c^{-2}z_1z = 1, \quad a^{-2}x_2x + b^{-2}y_2y + c^{-2}z_2z = 1, \quad a^{-2}x_3x + b^{-2}y_3y + c^{-2}z_3z = 1.$$

3312. Proposed by J. V. Uspensky.

For every prime $p=8l+3$, a root of the congruence $\xi^2 \equiv -2 \pmod{p}$ is given by

$$\xi = l!\,(5l+1)!/2l! \pmod{p}.$$

For every prime $p=12l+7$, a root of the congruence $\xi^2 \equiv -3 \pmod{p}$ is given by

$$\xi = C_{6l+3}^{l}/C_{6l+3}^{3l+1} \pmod{p}.$$

3313. Proposed by L. S. Johnston, Pennsylvania State College.

Consider the infinite sequence $[a_n]$ of real positive numbers with the recurrent relation $a_{k+1}^2 = 2a_k/(1+a_k)$ $(k=1, 2, 3, \cdots)$.

(1) Prove $L_{k\to\infty} a_k = 1$ for every a_1.

(2) Prove $L_{n\to\infty} \Pi_{k=1}^{n} a_k$ exists and is different from zero for every a_1.

(3) Express the limit in (2) as a function of a_1.

The problem is of course trivial for $a_1=1$.

3314. Proposed by J. B. Reynolds, Lehigh University.

A uniform flexible chain of length a, weight w, slides down an arch of a smooth inverted cycloid of total length $8a$ in a vertical plane with its vertex lowest. If the chain is released from rest in a position on the arch in which one end is at one end of the arch, find the time until the middle of the chain is at the lowest point of the cycloid, the speed of the chain at that instant, and the tension at its middle point then.

UNSOLVED PROBLEMS

Readers are requested to send in solutions for the following unsolved problems which were proposed in the years 1921, 1922, 1923. The number of each problem is printed in bold face type, with the page number following.

1921

2875, 37; **2882,** 90; **2883,** 139; **2886,** 139; **2888,** 139; **2889,** 139; **2891,** 184; **2905,** 277; **2909,** 326; **2911,** 326; **2915,** 327; **2918,** 327; **2922,** 392; **2926,** 393; **2928,** 466; **2935,** 467; **2937,** 467; **2939,** 468; **2940,** 468.

1922

2946, 29; **2952,** 81; **2953,** 81; **2954,** 81; **2956,** 81; **2960,** 129; **2961,** 129; **2962,** 129; **2963,** 129; **2964,** 129; **2968,** 179; **2969,** 179; **2970,** 179; **2975,** 225; **2977,** 270; **2984,** 314; **2985,** 314.

1923

3004, 76; **3006,** 76; **3007,** 76; **3009,** 76; **3012,** 146; **3015,** 146; **3016,** 146; **3018,** 205; **3020,** 206; **3026,** 275; **3028,** 275; **3031,** 275; **3033,** 276; **3034,** 276; **3038,** 337; **3045,** 449; **3048,** 450.

SOLUTIONS

3250[1927, 156]. Proposed by Norman Anning, University of Michigan.

Give, in a form suitable for high school students, a proof of the theorem that when the perimeter of a triangle is given the area is greatest when the triangle is equilateral.

I. Solution by A. Pelletier, Montreal, Canada

The area A of a triangle may be expressed by the formula

$$A^2 = s(s-a)(s-b)(s-c).$$

Now, if the perimeter $2s$ is constant, the maximum of A^2, and also of A, will take place when the product $(s-a)(s-b)(s-c)$ is the greatest. Since the sum of the three factors is $3s-(a+b+c)=s$, a constant, their product is maximum when they are equal. This principle can be proved in a very simple way. Hence, $s-a=s-b=s-c$, or $a=b=c$.

II. Solution and Remarks by Otto Dunkel, Washington University

The proof below will be based on the following proposition which is proved in many texts on plane geometry.

Proposition. *Of all triangles having a given perimeter and a given base, the isosceles triangle with the other two sides equal has the greatest area.*

Let S be the area of any triangle with the sides a, b, c which are not all equal, but such that $a+b+c=3e$, the given perimeter. Let E be the area of the equilateral triangle of side e; then it will be proved that $S<E$. Let c be the longest side, so that $c>e$, and consider the isosceles triangle with base c and with each of the other two sides equal to $a'=(a+b)/2$. Then its area $S'\geqq S$ by the above proposition, where the upper sign is used if $a\neq b$. We now compare S' and E by constructing a half of each triangle. Draw the right triangle AMC so that $\angle M=90°$, $AC=e$, $AM=e/2$. Then $\angle A=60°$. Produce MA to A_1 so that $MA_1=c/2$. Lay off on CA the length $CK=AA_1$, then $AK=AC-A_1A=a'$. Draw the parallelogram $AKDA_1$, and take G on CA so that CG is bisected at K. Draw GP perpendicular to MC cutting it in P. From the form of triangle CGP it follows that $GP=GK=KC=DK=A_1A$. Also $KP=KG$. Hence DP is perpendicular to AC and to A_1D, since DKG and PKG are equal equilateral triangles. With A_1 as center and A_1D as radius describe a circle cutting the segment MC in C'. Since DP is tangent to this circle at D, $CP<CC'$. Draw A_1C' cutting AC in F, and $C'G'$ parallel to MA cutting AC in G'. Then $C'G'>PG=A_1A$, and A_1MC' is one half of the triangle of area S'. Since $G'C'>A_1A$, the area of $FG'C'$ is greater than that of the similar triangle FA_1A. Hence

$$E-S'=2[\text{area } CG'C'+\text{area } FG'C'-\text{area } FA_1A]>2 \text{ area } CG'C'.$$

We have then $E>S'\geqq S$, and the theorem is proved.

REMARKS. The treatment of this matter as given in our plane geometry texts cannot be regarded as a proof of the proposition as stated in this problem. In such texts the proposition above or a similar one regarding isosceles triangles is first proved, and then there follows a corollary sometimes worded about as follows: Of all triangles having the same length of perimeter the one of maximum area is equilateral. In this form of wording the facts stated have really been proved, but this falls far short of proving that the equilateral triangle with the given perimeter has the maximum area. In order to prove this last fact with the aid of the corollary it must be shown that there exist one or several triangles of some sort of shapes with the given perimeter such that they have the maximum area. This can be done by continuity considerations, but in this case a satisfactory treatment is likely to be unsuitable for an elementary text. It seems to be better and simpler to prove directly that the area of the equilateral triangle is greater than that of any other triangle with the same perimeter. The following process has been used.

Starting with any triangle which is not equilateral, we may consider an unending sequence of isosceles triangles formed by replacing two unequal sides of one triangle by two equal sides having the same total length as before and thus obtain the next triangle of the sequence. In such a sequence the areas increase and it may be easily shown that the triangles approach an equilateral triangle with the given perimeter, also that the areas approach a limit. But here we meet with a difficulty, it must be shown that the limit approached by the areas is the area of the equilateral triangle approached. Here again we may show this by continuity considerations but in this case also such considerations would be unsuitable for elementary treatment. Such a process was used by Lhuilier, as stated by Steiner in some preliminary remarks to his own proof of this theorem (Crelle's Journal, vol. 24, 1842, pp. 96–99). This proof of Steiner is not only elegant and elementary but is has the additional merit of applying also to spherical triangles. However, from an elementary geometry standpoint it is not complete. For in the course of the proof it is required to prolong the side BC of a triangle DBC beyond C and to take a point E on this prolongation so that $DE+EC$ will have a given total length. No construction is given for E as is required in a geometry proof. This may be supplied without great difficulty in the case of a plane triangle, but this addition would complicate and lengthen the proof considerably. Another form of proof has been employed which depends upon the expression for the area of the general triangle in terms of the sides and the algebraic theorem that the arithmetic mean of any number of positive quantities is greater than their geometric mean. This furnishes a very brief and satisfactory proof, but it requires a proof of this algebraic theorem since it is not usually given in elementary texts.

Since it is quite likely that all students of plane geometry and many of their teachers have been led to believe that the theorem of this problem has been proved in their geometry texts, it would be desirable in such texts either to prove the theorem completely or to make clear just how much has been proved in the treatment given.

Also solved by H. C. BRADLEY, ALICE A. GRANT, MICHAEL GOLDBERG, S. B. LITTAUER, J. ROSENBAUM, and PAUL WERNICKE.

3253 [1927, 217]. Proposed by N. A. Court, University of Oklahoma.

Through the vertices of a given tetrahedron, planes are drawn parallel to the opposite faces. Prove that the faces of the new tetrahedron thus obtained have for their centroids the corresponding vertices of the given tetrahedron.

Note: The new tetrahedron may, by analogy with the case in the plane, be called the "anticomplementary tetrahedron" of the given tetrahedron.

SOLUTION BY THEODORE BENNETT

Using tetrahedral coordinates, let the given tetrahedron have vertices P_i and altitudes h_i; let the areas of the faces be A_i, and let the volume be V. If we use this as a coordinate tetrahedron, any point in space has coordinates x_i connected by the identity

$$A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 \doteq 3V, \tag{1}$$

the x's being the perpendicular distances from the faces of the tetrahedron to the point in question. We note that

$$A_1h_1=A_2h_2=A_3h_3=A_4h_4=3V. \tag{2}$$

The four planes through the vertices parallel to the opposite faces have the equations

$$x_1 = h_1,\ x_2= h_2,\ x_3 = h_3,\ x_4 = h_4. \tag{3}$$

From (1), (2), and (3) we find that the last three of these planes meet at the point

$$Q_1 = \left(\frac{3V - A_2h_2 - A_3h_3 - A_4h_4}{A_1}, h_2, h_3, h_4\right) = (-2h_1, h_2, h_3, h_4).$$

Similarly, the remaining vertices of the new tetrahedron are

$$Q_2, Q_3, Q_4 = (h_1, -2h_2, h_3, h_4),\ (h_1, h_2, -2h_3, h_4),\ (h_1, h_2, h_3, -2h_4).$$

As in Cartesian coordinates, the centroid of any triangle has for coordinates one third the sum of the corresponding coordinates of the vertices. Thus the centroid of $Q_1Q_2Q_3$ has coordinates $(0, 0, 0, h_4)$ and is therefore the point P_4. Similarly for the other faces.

Also solved by MICHAEL GOLDBERG, A. PELLETIER, J. ROSENBAUM, and PAUL WERNICKE.

A second solution by THEODORE BENNETT was received.

3256[1927, 217]. Proposed by C. N. Schmall, New York City.

Two spheres, with radii R and r, are inscribed in a right circular cone so that the greater touches the smaller and the base of the cone. If v is the volume of the cone, show that $v=2\pi R^5/3r\,(R-r)$.

SOLUTION BY D. LAWRENCE BARRICK, Montezuma College, Montezuma, N. M.

Let A be the vertex of the cone and AD its altitude. Then AD contains the centers O and H of the two inscribed spheres of radii r and R, respectively. A plane through AD cuts out the element AB and the radius DB of the base. Let OX and HW be the radii to the points of contact of the spheres with AB and let OK be perpendicular to HW at K. Then $HD=R$, $OH=R+r$, $HK=R-r$, and hence $OK=2(Rr)^{1/2}$ From similar triangles, we have $AH/R=(R+r)/(R-r)$; and hence $AD=2R^2/(R-r)$. Then $DB/AD=(R-r)/2(Rr)^{1/2}$; and hence $DB=R^2/(Rr)^{1/2}$. It now follows that $v=2\pi R^5/3r(R-r)$.

Also solved by R. P. AGNEW, E. F. ALLEN, THEODORE BENNETT, H. C. BRADLEY, W. B. CURRIE, L. A. DYE, PHILIP FITCH, J. S. GEORGES, MICHAEL GOLDBERG, ALICE A. GRANT, H. M. LUFKIN, R. E. MORRIS, R. M. MCFARLAND, NINA M. ALDERTON, and A. PELLETIER.

3260[1927, 272]. Proposed by B. C. Keeler, New York City.

Prove that the following series is convergent.

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots+\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}+\cdots$$

SOLUTION BY LAURENCE HAMPTON, University of Oklahoma.

From the binomial expansion of $(1+n^{-1})^{n+1}$ we have

$$(1+n^{-1})^{n+1}-(1+n^{-1}) \geqq 1+(n+1)/2n > 3/2.$$

Hence the nth term of the series is less than $(2/3)^n$, and the sum of any number of terms is less than 2.

Also solved by THEODORE BENNETT, MICHAEL GOLDBERG, R. E. MORITZ, A. PELLETIER, and the PROPOSER.

3261[1927, 272]. Proposed by J. Rosenbaum, Milford, Connecticut.

On the sides AB, BC, CD, and DA of a quadrilateral $ABCD$, the points P, Q, R, and S are taken so that $AP/AB=BQ/BC=CR/CD=DS/DA$, thus forming the quadrilateral $PQRS$.

Prove that if the two quadrilaterals are similar then they are parallelograms.

SOLUTION BY PAUL WERNICKE, Washington, D. C.

The quadrilaterals $ABCD$ and $PQRS$ are assumed to be similar as written. Let $m=AP/AB$, and the vectors $AB=\beta$, $AD=\delta$, $AC=c\beta+k\delta$. Let the diagonals of $ABCD$ meet in F, those of $PQRS$, in V.

$$\text{Then } AF = (c\beta + k\delta)/(c + 1), \quad BF = k(\delta - \beta)/(c + k) = k \cdot BD/(c + k).$$

On account of the similarity, V divides the diagonals

$$PR = AC + CR - AP = c\beta + k\delta + m(-c\beta - k\delta + \delta) - m\beta = (c - mc - m)\beta + (k - mk + m)\delta$$

$$\text{and } QS = AS - (AB + BQ) = (1 - m)\delta - [\beta + m(c\beta + k\delta - \beta)]$$

$$= (-1 + m - mc)\beta + (1 - m - mk)\delta$$

as AC and BD are divided in F. Therefore

$$PV = [(c - mc - m)\beta + (k - mk + m)\delta]/(c + k);$$

$$QV = [k(-1 + m - mc)\beta + k(1 - m - mk)\delta]/(c + k).$$

The relation $PB+BQ=PV+VQ$ can be put into the form

$$(1 - 2m + cm)\beta + km\delta = [(c + k - cm - km - m + ckm)\beta + m(1 + k^2)\delta]/(c + k)$$

whence

$$1 - 2m + cm = [c + k - cm - km - m + ckm]/(c + k), \quad km = m(1 + k^2)/(c + k).$$

Discarding the trivial case $m=0$, we find $c=k=1$, or $c=-k=\pm i$. Thus, if the quadrilaterals are both real $c=k=1$, and they are both parallelograms.

Also solved by A. PELLETIER.

NOTE BY THE EDITORS: It can be shown further that a necessary and sufficient condition for the similarity of the quadrilaterals $ABCD$ and $PQRS$ (as written) is that they be squares.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Additional circulars of information concerning the Congress of Mathematicians to be held at Bologna next September, and blanks for abstracts of papers to be presented have been sent to all those whose names are on the list of prospective participants, as received at the office. Additional copies may be obtained upon request from the central office of the American Mathematical Society, 501 West 116th Street, New York City.

Dr. RALPH D. DONER and Dr. DUNCAN C. HARKIN have been made associate professors of mathematics at the Alabama Polytechnic Institute.

Professor GEORGE W. GORRELL has been made head of the department of mathematics at the University of Denver.

Miss MARY GERTRUDE HASEMAN has been made head of the department of mathematics and advisor of women at the Junior College of Connecticut, which opened its doors for the first time on February 1, 1928.

Professor GEORGE H. HENDERSON of Dalhousie University is on leave of absence. Mr. W. J. JACKSON and Mr. J. G. ADSHEAD have been made lecturers at this university.

Dr. M. R. RICHARDSON has been made assistant professor of mathematics at the Georgia School of Technology.

Professor J. E. ROWE of the College of William and Mary has been allowed two sets of letters patent on his trinometer and associated instruments by the United States Commissioner of Patents.

Mr. H. E. WAHLERT of the Georgia School of Technology resigned inJanuary 1928. He has been succeeded by Mr. R. F. WATKINS.

Assistant Professor HERBERT H. WHEATON has been promoted to an associate professorship at Fresno State College.

The following appointments to instructorships in mathematics are announced:

Alabama Polytechnic Institute, Mr. L. L. GARNER, Mr. Z. M. PIRENIAN;

Colorado College, Miss MARTHA C. BELSCHNER, Miss HELEN E. VON BOSTON;

University of Denver, Mr. T. R. CUYKENDALL;

Fresno State College, Mrs. MABEL KELLY STURM;

Georgia School of Technology, Mr. ROY A. SMITH, Mr. CARL A. BENANDER.

The following courses in mathematics are announced for the summer of 1928.

University of Indiana. In addition to the usual courses in general mathematics, calculus, and mathematical theory of investment, the following advanced courses are offered. By Professor S. C. DAVISSON: Advanced differential calculus; Advanced algebra; Differential geometry. By Professor D. A. ROTHROCK: History of mathematics; Fourier series. By Professor K. P. WILLIAMS: Advanced analytic geometry; Differential equations.

University of Michigan, June 25 to August 17. In addition to courses in algebra, plane and solid geometry, trigonometry, analytic geometry, elementary calculus, mathematical statistics, and the mathematical theory of

interest and insurance, the following advanced courses are offered. By Professor W. B. FORD; Advanced calculus; Infinite series with special reference to Fourier series. By Professor L. C. KARPINSKI: Teaching of algebra; History of mathematics. By Professor PETER FIELD: Vector analysis. By Professor T. R. RUNNING: Graphical methods. By Professor J. W. BRADSHAW: Figures of solid geometry; Selected topics in projective geometry. By Professor T. H. HILDEBRANDT: Theory of functions of a complex variable; Theory of the potential. By Professor H. C. CARVER: Advanced mathematical theory of statistics. By Professor L. A. HOPKINS: Elements of mechanics. By Professor C. J. COE: Integral equations. By Professor NORMAN ANNING: Differential equations; Determinants and theory of equations. By Professor J. A. NYSWANDER: Theory of probability; Finite differences. By Mr. O. J. PETERSON: Solid analytic geometry.

University of Minnesota, first term, June 16 to July 28; second term, July 30 to September 1. In addition to the usual elementary work, the following courses will be offered, First term: By Assistant Professor GLADYS GIBBENS: Differential equations; Synthetic metric geometry. By Professor W. L. HART: Advanced calculus; Reading in advanced mathematics. Second term: By Professor A. L. UNDERHILL: Reading in advanced mathematics.

University of Missouri, June 7 to August 3. In addition to the usual elementary courses, the following advanced courses are offered: By Professor INGOLD: Modern geometry; Selected topics from geometry. By Professor WAHLIN: Advanced algebra; Selected topics from algebra. By Professor L. R. FORD (Rice Institute): Calculus; Selected topics from analysis.

University of Texas, first term, June 5 to July 16. By Professor R. L. MOORE: Functions of real variables; Foundations of geometry. By Professor E. L. DODD: Infinite processes; Mathematical statistics. By Professor H. J. ETTLINGER: Boundary value problems; Ruler and compass constructions. By Professor C. D. RICE: Differential geometry; Advanced calculus. By Professor P. M. BATCHELDER: Teaching problems in mathematics. By Professor C. M. CLEVELAND: Advanced calculus. By Professor MARY DECHERD: Calculus. By Professor GOLDIE HORTON: Theory of equations. All freshman courses, both terms. Second term, July 16 to August 27. By Professor H. J. ETTLINGER: Boundary value problems; Definite integrals. By Professor P. M. BATCHELDER: Teaching problems in mathematics. By Professor C. M. CLEVELAND: Advanced calculus. By Professor R. G. LUBBEN: Non-Euclidean geometry; Calculus. By Professor G. T. WHYBURN: Functions of real variables; Calculus.

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Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

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By GEORGE W. SNEDECOR, Iowa State College

Two papers[1] recently appearing in the Monthly have served to acquaint its readers with some of the varied adaptations of the punched card sorting and tabulating machines, especially to accounting and to the compilation of statistics of population, insurance, and economics. In some institutions the equipment, being already installed, may be available for use by members of the mathematics departments. In other institutions the outfit might be secured, as it has been at Iowa State College, by the cooperation of several interested departments. It is the purpose of this article to explain in some detail how such an installation may be used by mathematicians, especially those interested in statistics.

The operation of the various units of the Hollerith equipment was fully described by Mr. Johns.[2] One feature only will be amplified, viz., the manner in which facts may be transferred to cards by punching.[3] For this purpose as many of the columns of a standard card as may be needed are grouped into "fields" of one or more columns each, one field for each variable. In the case of a non-numerical variable, such as "nationality" or "department of instruction," a "code" is adopted in which each individual is designated by a suitable number. Numerical data may also be coded, a process which will be discussed below. The number of columns required for a field is the same as the number

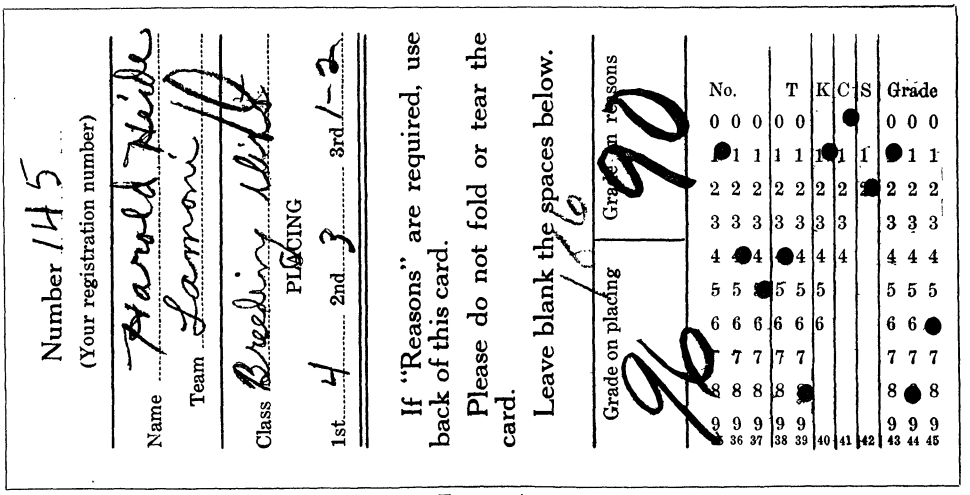


FIGURE 1

[1] Victor Johns, *On the Mechanical Handling of Statistics*, vol. 33, (1926), p. 494.
F. A. Harper and C. W. Vaughn, *Machines for Handling Statistics*, vol. 34 (1927), p. 449.

[2] Loc. cit.

[3] See also Edmund E. Day, *Statistical Analysis*, Macmillan, 1925, p. 394; and B. F. Young, *Statistics as Applied in Business*, Ronald Press, 1925, p. 493.

of digits in the largest value of the corresponding variable. The fields with their designations may be printed on the cards as in figure 1. Often, however, the standard card is used without further printing as in figure 2. In this case some record of the fields must be kept for reference, a convenient form being il-

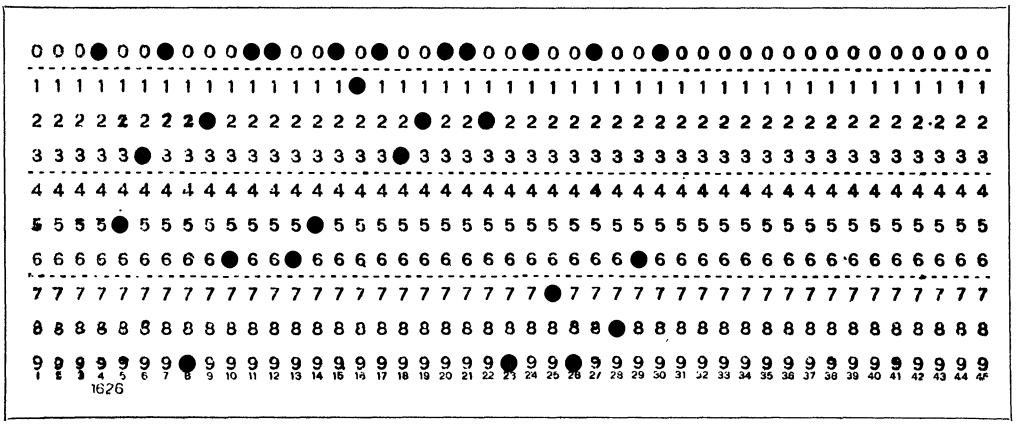


FIGURE 2

lustrated in figure 3. Finally, using one card for each individual, the values of the variables are punched each in its designated field.

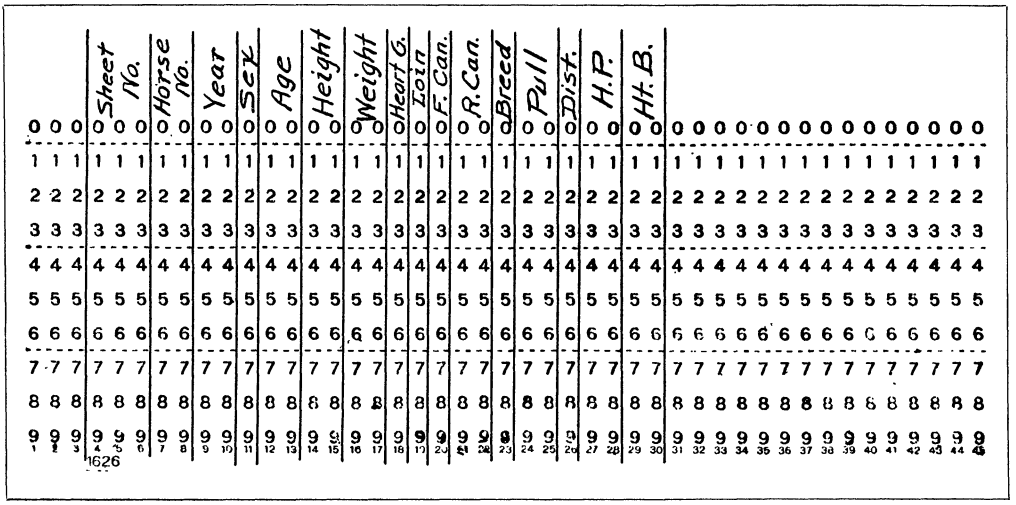


FIGURE 3

The tabulating machine with its listing and total-printing attachment is being used in rather a novel manner by Professor J. S. Turner of our department. In some work he is doing in collaboration with Professor Dickson of the University of Chicago, it is required to evaluate $a^3+b^3+2c^3+2d^3$ for various values of a^3+b^3 each combined with a set of values of $2c^3+2d^3$. Each sum must be recorded—approximately 100,000 of them. This is accomplished by

using two sets of cards, each of the first set punched with a value of a^3+b^3 and of the second set with the difference between two successive values of $2c^3+2d^3$. The same field is used for the two variables. Figure 4 shows part of one record made by placing a single card of the first set at the beginning of the second set, and accumulating in column III the totals after the successive cards. For convenience in verifying, columns I and II are included in the record, the former being merely a serial number punched in the cards. The latter column is headed by $(22)^3+(9)^3$, and contains thereafter the successive differences referred to above. The numbers in column III, therefore, are the sums,

$$(22)^3 + (9)^3 + 2c^3 + 2d^3,$$
$$c = 0,1,2, \cdots ; d = 0,1,2, \cdots ; c \geqq d.$$

The sorting machine separates punched cards into groups ordered according to the position of the holes in any column. The sorting is on only one column at a time. If the field covers two or more columns, successive sortings are made on them from right to left. If one column constitutes the entire field of a variable, a single sorting yields the frequency distribution of that variable. Figure 5 shows cards sorted into two such distributions. In the lower one, the independent variable is the "number of pigs in a litter," the sample comprising 2789 litters of Poland China pigs. This set of cards was selected for the illustration because two extra positions were punched in a single-column field, making twelve classes instead of the usual ten. The cards in the thirteenth class on the extreme right were unpunched. A column utilized in this manner can be used only for sorting because, although the sorter is

I	II	III
541	11377	11377
1	2	11379
2	2	11381
3	12	11393
4	2	11395
5	14	11409
6	22	11431
7	2	11433
8	14	11447
9	38	11485
10	20	11505
11	2	11507
12	14	11521
13	38	11559
14	68	11627
15	2	11629
16	4	11633
17	10	11643
18	38	11681
19	74	11755
20	54	11809
21	2	11811
22	14	11825
23	38	11863
24	14	11877
25	60	11937
26	122	12059
27	4	12063
28	2	12065
29	14	12079
30	38	12117
31	74	12191
32	50	12241
33	72	12313
34	88	12401
35	2	12403
36	14	12417
37	38	12455
38	40	12495
39	34	12529
40	122	12651
41	98	12749
42	84	12833

FIGURE 4

capable of handling this maximum of thirteen groups in any one column, the tabulating machine adds only such numbers as are punched in the ten positions, 0, 1, 2, · · · , 9.

After the illustrated sorting the cards are transferred to the tabulator for the determination of the class frequencies. This is accomplished by a "card count" connection causing the addition of one unit for each card passing, independent of the position of the hole punched in any field. The result is the following frequency distribution:-

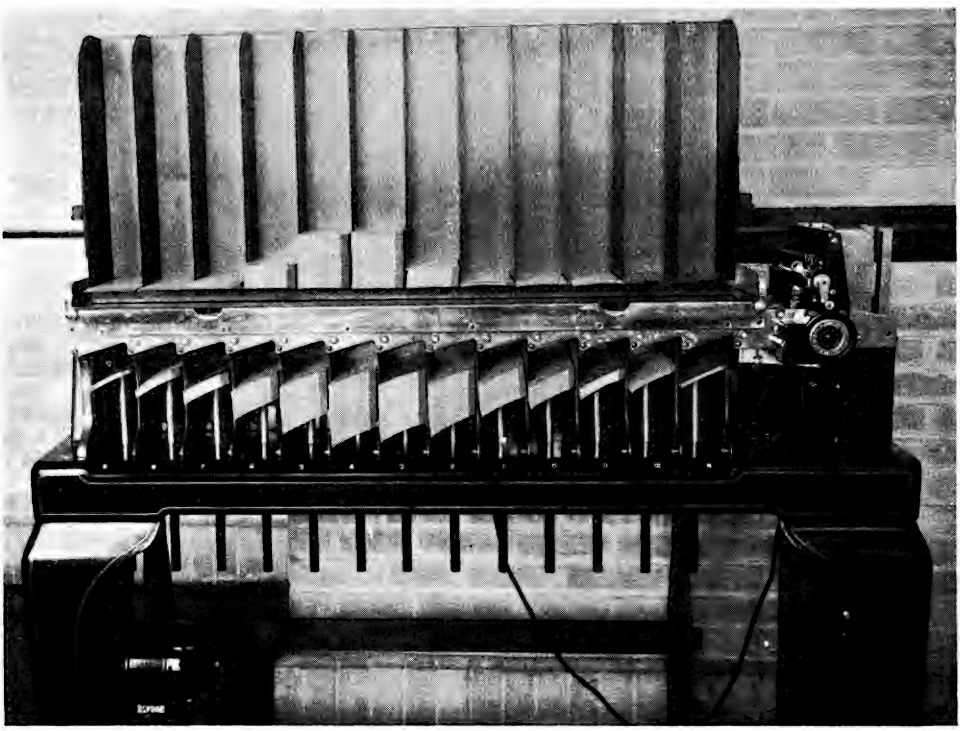


FIGURE 5

Number pigs in litter	2–3	4	5	6	7	8	9	10	11	12	13	14	15–17
Frequency	29	59	100	232	392	530	469	419	263	164	76	35	21

The constants of the distribution are now computed in the usual manner. The distribution illustrated in the upper part of Fig. 5 is that of 1352 horses sorted on the coded values of the variable, "heart girth," appearing in column 18 of the cards illustrated in Figs. 2 and 3. The records were furnished by the

Horse Association of America as collected from the horse-pulling contests conducted during 1926, and have been studied by Mr. A. E. Brandt[1] of our department.

Most of the statistical sets coming into our laboratory contain two or more variables, usually continuous. In the preliminary treatment such of the variables as may require it are coded before they are punched in the cards. This process differs only in minor details from that usually employed in grouping data into frequency distributions and preparing the tables for computation, but some of the likenesses and differences are important enough to warrant explanation. The first step in the two processes is identical,—for each variable, A', B', C', $\cdots$, a class interval is selected, and such interval then serves as a unit of measurement. The second step in the usual procedure is the tallying of each observed value in the corresponding interval, or of each pair of correlated values in the appropriate cell; and this is followed by the third step, the choice of an origin for each variable, usually at a class mark near the mode. (The variables now having each a new origin and a new unit of measurement may be represented by A, B, C, $\cdots$). In the punched card system, the latter of these steps is the second, the origin being customarily fixed at the lowest class mark of the sample, and the coded values, A, B, C, $\cdots$ being punched in the cards. Finally, the tallying process, designated as step two above, is replaced by the sorting of the cards on the several variables in succession.

The total-printing and listing tabulating machine, receiving the cards after they are sorted on one of the variables, say A, records the following information as illustrated in block A of Fig. 6; first, the successive coded values of A, column I; second, the frequency, f_A, of each A-class, column II; third, the sum of the values, corresponding to the several A-classes, of each of the variables, i.e., $(\sum A)_A$, $(\sum B)_A$, etc., columns III, IV and V. This is all done simultaneously during a single run of the cards through the machine. After successive sorts are made on the other variables in turn, similar records are made as shown in blocks B and C.[2] The identity of the totals in any column constitutes one of the checks on the accuracy of the machine work.

In Fig. 6 are found all the data required for the calculation of the means, standard deviations, correlation coefficients and correlation ratios. The formulas[3] are as follows:

[1] *A preliminary study of the relation between form and power in the horse*, a paper read before the Am. Soc. Ag. Eng., at Saint Paul, 1927.

[2] These data are from A. E. Brandt, *Effect of breed and body measurements on dressing percent in swine*, Master's thesis, Iowa State College, 1926. Variable A is "length," B is "heart girth," and C is "live weight."

[3] Formulas adapted from J. A. Harris, American Naturalist, vol. 44 (1910), pp. 693-699; and from J. Blakeman, Biometrika, vol. 4 (1906), pp. 332-350.

	I	II	III	IV	V
	A	f_A	$(\sum A)_A$	$(\sum B)_A$	$(\sum C)_A$
		6		3	6
	1	30	30	47	62
	2	57	114	119	146
	3	69	207	238	225
	4	78	312	278	269
A	5	33	165	147	137
	6	12	72	67	70
	7	4	28	22	23
	9	2	18	16	18
		291	946	937	956
	B	f_B	$(\sum A)_B$	$(\sum B)_B$	$(\sum C)_B$
		16	18		17
	1	34	76	34	71
	2	69	188	138	172
	3	59	190	177	177
	4	33	129	132	122
B	5	44	173	220	190
	6	22	101	132	113
	7	11	51	77	72
	9	3	20	27	22
		291	946	937	956
	C	f_C	$(\sum A)_C$	$(\sum B)_C$	$(\sum C)_C$
		1	2		
	1	21	22	11	21
	2	76	195	147	152
	3	87	283	250	261
	4	53	197	231	212
C	5	29	127	150	145
	6	12	52	67	72
	7	6	31	38	42
	8	3	13	20	24
	9	3	24	23	27
		291	946	937	956

FIGURE 6

$$N = \sum f_A, \quad M_A = (\sum A)/N, \quad \sigma_A^2 = [\sum A^2 - (\sum A)M_A]/N,$$

$$r_{AB} = [\sum AB - (\sum A)M_B]/[\sum A^2 - (\sum A)M_A]^{1/2}[\sum B^2 - (\sum B)M_B]^{1/2},$$

$$\eta_{BA}^2 = \left[\sum \frac{(\sum B)_A^2}{f_A} - (\sum B)M_B\right]/[\sum B^2 - (\sum B)M_B].$$

A few particulars will serve to make the whole process obvious. First, each of the totals in column III is $\sum A$, each of those in column IV is $\sum B$, and so on. Secondly, in columns I and IV, block A, the pairs of numbers in each line, upon multiplication and addition, yield the product-sum, $\sum AB$, because $\sum AB = \sum A(\sum B)_A$. The correctness of the calculation is verified by using columns I and III, block B. Thirdly, by the same multiplication-addition process, $\sum A^2$ is obtained from columns I and III, block A, and verified by calculating $\sum f_A A^2$ directly from columns I and II. Fourthly, $\sum(\sum B)_A^2/f_A$, required for the computation of η_{BA} is taken from columns II and IV block A, and verified by computing this sum as $\sum(\sum B)_A[(\sum B)_A/f_A]$. For economy of effort and time the computations should be completed in some such systematic form as that indicated by B. B. Smith[1] or by Wallace and Snedecor.[2]

If desired, the computations may be verified by using another variable, $S = A + B + C + \cdots$, whose values are computed from the coded values of the observed variables and punched in each card. A sixth column would be added to the form illustrated in Fig. 6, but complete verification would be obtained with the following parts of the record deleted:—column III, block A; columns III and IV, block B; and columns III, IV, V, block C. The corresponding correlation ratios are sacrificed.

The multiple correlation constants are best obtained by direct solution of the normal equations, using either the method of Gauss[3] or one of the adaptations.[4] An idea of the rapidity with which such constants may be computed is conveyed in this statement; one of our operators, starting with the simple correlation coefficients among eight variables, including the sum, completes in forty minutes the regression equation and the multiple correlation coefficient.[5]

[1] *The use of punched card tabulating equipment in multiple correlation problems*, a mimeographed pamphlet issued by Bureau of Ag. Ec., U.S.D.A., 1923.

[2] *Correlation and machine calculation*, Iowa State College Bulletin, vol. 23, No. 35, 1925.

[3] *Disquisitio de Elementis Ellipticio Palladis*, Societati Regiae Tradita, vol. 25, 1810. See Carl Friedrich Gauss, Werke, vol. 6, p. 21.

[4] M. H. Doolittle, U.S. Coast and Geodetic Survey Report (1878), p. 115.

Tolley and Ezekiel, Journal of the American Statistical Association, vol. 18 (1923), p. 993.

Wallace and Snedecor, loc. cit.

[5] Since this was written there has come to hand an article advocating the direct solution of the normal equations. See Tolley and Ezekiel, *The Doolittle vs. the Kelley-Salisbury method*, Journal of the American Statistical Association, vol 22 (1927), p. 497.

The punched card equipment is well adapted to the study of multiple curvilinear regression using the method of Ezekiel.[1] The errors of estimate, e, are punched in the corresponding cards. The cards are then sorted successively on the several independent variables. After the sort on A, for example, the errors of estimate are tabulated for each coded value of A, yielding $(\sum e)_A$. The means, $(\sum e)_A/f_A$, are then plotted as deviations from the line,

$$\overline{X} = M_X + b_{XA \cdot BC} \ldots (A - M_A),$$

where X is the dependent variable, and $\overline{X}$ its value estimated from the complete linear regression equation. This is done for each independent variable. A curve is then fitted to each of the sets of points so plotted, and these curves are made the basis of the first approximations to the assumed curvelinear values of X. The new errors of estimate, e', are computed, and the stage is now set for the second approximation.

To convey an idea of the uses to which the punched card equipment has been put during the first year of its installation at Iowa State College, there is appended a partial list of the problems studied, not including those already cited in this article;

1. Relation of college grades to high school averages, mental tests, physical condition and other characters.
2. Correlation among mental test grades.
3. Butter fat production as influenced by temperature, advancement of the gestation and lactation periods, etc.
4. Cost per hundred pounds gain in swine as related to season, feeding system, average daily gain, and initial weight.
5. Differences in cattle prices at the Omaha, Kansas City, and Chicago markets.
6. Analysis of demand meter records obtained in farm electric installations.
7. Results of the Iowa corn contests.
8. Inheritance of characters in corn.
9. Inheritance in tomatoes.
10. Relations among number of stalks per hill, number of ears per stalk, number of ears per hill, size of ears, fodder weight, and yield in sweet corn.
11. Correlations among number of plants and fruits per hill and size of fruits in strawberries.
12. Results of high school contest in judging grains and animals.
13. Biological significance of partial regression coefficients from the standpoint of probable errors.

[1] Journal of the American Statistical Association, vol. 19 (1924), p. 431.

14. Magnitude of arithmetical changes in statistical constants due to grouping.
15. Relations among factors affecting farm profits.
16. Man labor and horse labor used in various farm enterprises and operations.
17. Farm mortgage study.
18. Livestock shipping associations survey.
19. Bank and credit conditions in Iowa.
20. Land and crop utilization as related to type of farming.

THE LAW OF SMALL NUMBERS[1]

By A. R. CRATHORNE, University of Illinois

In his *Sur la probabilité des jugements*, published in 1837, Poisson discussed in his chapters on the law of large numbers the probability that the number of happenings of an event E in n trials will fall within certain assigned limits. Letting p be the probability of success, he starts with the binomial expansion

$$(p+q)^n = p^n + np^{n-1}q + \frac{n\cdot(n-1)}{2!}p^{n-2}q^2 + \cdots,$$

the terms of which give the probability of the success of E, n times, $(n-1)$ times, and so on, out of n trials. In general the probability P_x of success x times out of n trials is given by the $(n-x+1)$th term of the series

$$\frac{n(n-1)\cdots(n-x+1)}{x!}p^xq^{n-x}.$$

To avoid the large factorials in this expression he reduced the problem to the consideration of the integral $\int e^{-t^2}dt$. In doing this it was necessary to assume not only that n was large but also to assume that p was not too small. The general idea underlying this treatment of the problem was of course not original with Poisson. What Poisson offered that was new at this point was a special consideration of the case where p was small, n still remaining large. He showed that if n was allowed to increase indefinitely and p to decrease towards zero in such a way that np remained constant, the expression

$$P_x = \frac{n(n-1)\cdots(n-x+1)}{x!}p^xq^{n-x} \text{ approaches } P_x = \frac{m^xe^{-m}}{x!},$$

[1] Read at the twelfth annual meeting of the Association, Nashville, December 30, 1927.

where $m=np$ and x is not a large number. This gives the probability of x successes in n trials when p is too small for the more general formula.

This result was more or less a by-product of Poisson's discussion of the law of large numbers and seems to have received little attention until 1898 when Bortkewitsch published his "Gesetz der kleinen Zahlen," in which he called attention to Poisson's formula and emphasized the very decided difference between the behavior of small and of large frequencies. He pointed out the facts that small frequencies from large fields tend to fit the norm given by Poisson's formula and that they have a certain stability of their own. Bortkewitsch called this realization in statistical experience the law of small numbers.

If the frequencies for various values of x in a series of observations are given by the expression $(m^x e^{-m})/x!$, we have a maximum for $x=m$ where m is the mean. There is a rather rapid decrease on either side as x takes on integral values. Hence the observations never vary very much from their mean value and may be considered as quite stable. It should be emphasized that the above expression is defined only for integral values of x.

The discussion by Bortkewitsch is somewhat as follows: If we are investigating a series of observations of some phenomena, it is important to know whether the variations in the observations can be explained by the theory of probability and likened to some game of chance. Consider an event E with a probability of happening p. Out of n trials, E happens s_1 times, in another series of n trials, E happens s_2 times, and so on. In this way we get a series of numbers $s_1, s_2, s_3 \cdots$. In a large number of s's we shall find that the number $s=np$ occurs more than any other. If we mark off an interval on each side of np so that the double interval contains one half the s's, then it can be shown that this interval is $0.6745\ [np(1-p)]^{1/2}$, the so called probable deviation. If now we consider a series of relative numbers coming from actual observation and mark off on each side of the mean an interval in which half of the observations lie, and if we find that this interval is the same as the one given by some urn problem, then we are justified in investigating further as to whether a constant probability underlies the observed phenomena. In other words a constant complex of causes probably underlies the phenomena. It is, of course, not necessary to use the interval about the mean containing one half of the observations. Some other fractional part would serve. For most purposes the so-called standard deviation is preferable.

In his special case, Bortkewitsch considers the numbers $x_1, x_2, \cdots, x_n$ which come from observation and represent the number of times a rare event has happened, say during a series of years. For example, it may be the number of triplets born in a country each year for as many years as we have information; or it may be the number of alpha particles colliding with a small screen as in the Rutherford and Geiger experiment; or again it may be the number of

triple plays per year in one of our national leagues. If the frequency distribution of the x's fit Poisson's formula, then it is easy to show that the standard deviation of the x's is $\sigma_x = (np)^{1/2} = m^{1/2}$. Bortkewitsch calls this the calculation of the standard deviation by the indirect method. Since the theoretical m is not known, an approximation m', the mean of the x's, is used. Then computing the standard deviation directly from the observations, we have

$$\sigma_x' = \left[\sum \frac{(x_i - m')^2}{n}\right]^{1/2}.$$

If σ_x and σ_x' are statistically equal, then according to Bortkewitsch we are justified in saying that the theoretical formula fits the actual observations. Of course σ_x and σ_x' are accompanied by errors and do not necessarily have to be exactly equal. Bortkewitsch showed that the standard deviation of σ_x^2 is is $(m'/n)^{1/2}$ and of $\sigma_x'^2$ is $[(2m'^2+m')/n]^{1/2}$.

To give one example, the number of quadruplets born in Prussia each year during a period of 69 years of observation is given by Schmidt. In 14 of the years no quadruplets were born; in 24 years, 1 each year; in 17 years, 2 each year; and so on according to the following table:

x	Observed	Theoretical
0	14	14.2
1	24	22.5
2	17	17.7
3	9	9.3
4	2	3.7
5	2	1.2
6	1	0.3
7	0	0.1
	69	69

In the table, the first column gives, the number, x, of quadruplets, the second column gives the number of years in which x quadruplets were actually born, the third column gives the theoretical results from Poisson's formula where $m = 1.58$.

From these figures we find $\sigma_x^2 = 1.58$, $\sigma_x'^2 = 1.72$. The standard deviation of $\sigma_x'^2$ is 0.31. The difference $1.72 - 1.58 = .14 < .31$ is then not so much as might be expected.

In most statistical problems of the above type dealing with events that are more common, the difference between the σ_x and the σ_x' is more marked. The σ_x' may be quite a large multiple of σ_x. Bortkewitsch states this fact by saying that rare events are more stable than other events.

This apparent stability of series of small numbers has an explanation in the so-called Lexian theory. Lexis pointed out that the variations of certain series of observations which he called typical may have at their foundation a variable probability. Consider a large pile of thoroughly mixed black and white balls. From the pile m urns are filled say by dipping the urns into the pile. Let p_0 represent the proportion of white balls in the pile. Due to accidental variations

the proportion of white balls in the urns will vary from urn to urn. We may say then that the probabilities of drawing a white ball from the urns are $p_1, p_2, \cdots, p_m$ and that the mean of the p's is an approximation to p_0. Take n drawings from each urn, replacing the ball after each drawing.

Let s represent the proportion of white balls drawn. We shall have a series of numbers $s_1, s_2, \cdots, s_n$. According to the theory of probabilities, the standard deviation of the s's is $R=(r^2+a^2)^{1/2}$, where $r=(p_0q_0/n)^{1/2}$ is the standard deviation if each urn contains the same proportion, p_0, of white balls. This Lexis called the unessential component of the fluctuation. The a is the standard deviation of the variable probabilities p and is equal to $[(1/m)\sum(p_i-p_0)^2]^{1/2}$ and is called the physical component. Lexis then used for his criterium of dispersion the quotient $Q=R/r$. When $Q=1$, he called the dispersion normal; and when $Q>1$, supernormal. For $Q=1$ we have a maximum of stability and for any series of observations for which this condition is filled we may say that there is a constant complex of causes which may be analogous to some game of chance.

Substituting the above values of r and a we find

$$Q=\left[1+\frac{n\sum(p_i-p_0)^2}{mp_0q_0}\right]^{\frac{1}{2}}.$$

Q may be made to approach unity by decreasing n, the number of drawings. In statistical phraseology this is the same as restricting the field of observations more and more. Q may also approach 1 if (p_i-p_0) is small. For $p_i=p_0$ we have normal dispersion. It is usually not possible to know from just the observations whether (p_i-p_0) is small or not; but if p_i and p_0 are both small, their difference is small. Other things being equal we may then say that the smaller the p's the less the value of Q and, accordingly, the less the dispersion and the greater the stability of the series. To the two methods of increasing stability we may add combinations of the two.

In a statistical problem we do not have the convenient urn but have a vaguer term, field of observation. For example, in statistics concerned with male births from year to year in a district, n will be the total number of births and m the number of years under observation. n is easy to find here, but it is not usually so easy, and an essential difficulty arises. For example, our statistics may be a list of the number of cases of death from Addison's disease in the country for a series of years. Shall we consider our field of observation to include the whole population or the whole population of adults? Or shall we limit it to the total number of deaths during the year, or perhaps to those dying of disease? Or we may be considering the number of bachelors who marry when over seventy. Shall we consider our field to be all bachelors, or all bachelors over seventy, or merely all exposed bachelors? What is n in the case of

alpha particles striking a small screen? It is due to Bortkewitsch that we can give a theoretical representation of the frequency of such rare events without knowing anything about the number in the field of observation except that it is a large number. All we know is an empirical estimate of $m' = np$.

The theory of statistics is lucky in having among its workers vigorous and sometimes caustic writers who are not altogether impersonal. This has livened up and invigorated some otherwise rather formal discussions. Bortkewitsch, with his law of small numbers, has done his share in furnishing material for these writers, and his answers have been equally pungent.

The name itself, "Bortkewitsch's law of small numbers," has been thoroughly criticised. It has been said that there is no law; that if there were a law, it was not Bortkewitsch's; that it was not about numbers; and if it were the numbers were not small. In mathematical statistics we have a well known and fundamental group of propositions which make up what is called the law of large numbers. This law is the bed rock upon which the greater part of the theory is built. We are accustomed to think of small numbers as interlopers and we ruthlessly cast them out. So it is somewhat startling to find that they claim to be law abiding even if it is to a code of laws of their own making. But the law of small numbers is not something opposed to the law of large numbers or something to be set up alongside and compared with it, but it is part and parcel of the older fundamental law and was so considered by Poisson. A great deal of confusion seems to have centered about the term "small numbers," as if the reference were to the number n. In fact, a great deal of the mistrust in the theory is due to this misunderstanding. The factor n in the expression np is always assumed to be large, although we do not know enough of the composition of np to be able to separate it into its factors.

It has been suggested that a better name would be the law of small probabilities. It is no doubt a law of small probabilities, because the probabilities considered are always small. If we want a short title, this would no doubt be better and save confusion. But if we want a title to give a complete description, this one does not tell the whole story. Bortkewitsch insists that the actual magnitude of the observed numbers is an important factor and that the main fault with the phrase, law of small numbers, lies in the fact that it emphasizes the actual observed numbers and not the corresponding small probabilities. In the limiting step leading to the derivation of the fundamental formula of small numbers, Bortkewitsch stresses the fact that p approaches zero and at the same time np and x should not be large. The number n is the only one in the formula which is large. Other names that have been suggested are "law of probability of rare events," "law of the permanence of small numbers," and one suggested recently by von Mises, "law of large numbers in the case of small frequencies." This latter name seems to be truly descriptive.

The attempt to cloud Bortkewitsch's title as originator of the law is rather futile. No doubt Poisson was the first to derive the underlying formula, but Bortkewitsch was the first to use the name and the first to point out the statistical importance of Poisson's formula. Poisson was the inventor of the phrase "law of large numbers," but he did not understand by it this or that formula from the theory of probabilities, he meant the characteristic behavior of the frequencies and the realization in experience.

The objection has been made that the use of the law of small numbers should be limited to cases where a test can be made to find out whether p is actually small and n large. The test is to find the fitted binomial. Even if p and n do satisfy these conditions the binomial will give just as good or a better fit. When the data in the problems in the literature are represented by binomials, all sorts of values of p and n are found. From this the objectors conclude that the application of the law of small numbers is not justified.

In regard to this objection Bortkewitsch points out that the *a priori* assumption of representation by a binomial with small p and large positive n carries with it the assumption that q is less than 1. It can easily be shown that the Lexis criterium of dispersion Q is equal to $\sqrt{q}$ for the binomial. Hence the assumption is the same as assuming a Q less than unity. This leads to the paradox that in an investigation of a series of small frequencies with regard to the question of dispersion, the assumption is made that subnormal dispersion is present. In other words, Bortkewitsch accepts the Lexis theory of dispersion as a measure of stability and his opponents reject it.

The binomial no doubt makes a better fit in many cases. In using the exponential form one uses only the mean of the observations and postulates absolute ignorance as to the deviations from the mean. On the other hand, in the binomial with its two constants at our disposal we can use the standard deviation of the observations about the mean. But part of the problem is the representation of the frequencies of a series of observations by the simplest possible mathematical expression. The dubious case of the binomial with negative n and p can be naturally explained from the standpoint of the Lexis theory. They simply express the fact that in the case in question the divergence coefficient is greater than unity, which may be considered as accidental or a mark of supernormal dispersion. It is not necessary that we consider the Poisson formula as coming out of the binomial expansion. In the semi-invariant theory of frequency functions the Poisson exponential follows from the special case in which the semi-invariants are all equal.

The law of small numbers is so closely tied up with the Lexis theory that they stand or fall together. Keynes suspects that the apparent paradox in the statement that the regularity of the occurrence of rare events is more stable than that of commoner events, may be explained by the peculiar measure of

stability which has been selected. There is considerable truth in this statement, nevertheless there is much common sense in the Lexis idea. The man of the street is not much concerned with formal definitions of stability, but he does think it a rather interesting fact that the average number of triple plays in one of our national base ball leagues is 4 and the chances are more than even that the number of triple plays in a season will not differ from 4 by more than 1. There may be more general and more accurate ways of describing the frequency of rare events, but experience has shown that the law of small numbers gives a simple and useful mathematical description.

A NEW FORMULA FOR VOLUME

By J. B. REYNOLDS, Lehigh University

One of the most remarkable theorems known to the Greeks is that for th volume generated when a plane area is revolved about a line. The theorem as stated by Pappus, who lived about the end of the third century A. D., is as follows:

Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded of (1) the ratio of the areas of the figures and (2) of the ratio of the straight lines similarily drawn to (i.e. drawn to meet at the same angles) the axes of rotation of the respective centres of gravity. Figures generated by incomplete revolutions are in the ratio compounded (1) of the ratio of the areas of the figures and (2) of the ratio of the arcs described by the centres of gravity of the respective figures, the latter ratio being itself compounded (a) of the ratio of the of the straight lines similarily drawn (from the respective centres of gravity to the axes of revolution) and (b) of the ratio of the angles contained (i.e. described) about the axes of revolution by the extremities of said lines (i.e. the centres of gravity).[1]

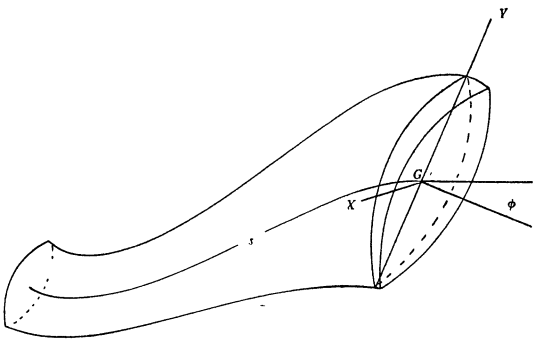
The theorem was stated without proof by Pappus in Book VII of his Mathematical Collections with the suggestion that proof would be given in Book XII which, if it ever existed, is not now extant. The theorem was apparently lost to be rediscovered, without satisfactory proof, by Guldin (1577–1643), a Swiss mathematician, whose name it often bears. Edwards in his recent work on integral calculus[2] shows that the theorem as stated by Pappus holds when the line of revolution is in the plane of the area revolved and does not cut the area (the usual interpretation) and also when the line of revolution is parallel to the plane of the area, or oblique to it, provided the projection of the line upon

[1] See Sir Thomas Heath, *A History of Greek Mathematics*, vol. 2, p. 403.

[2] See Joseph Edwards, *Integral Calculus*, vol. 1, p. 783.

the plane of the area does not cut the area. Routh[1] drew attention to the fact that the line of revolution need not be fixed but may be the instantaneous axis of the moving area. From this point of view the volume generated equals the area times the length of path of its centroid.

It is the purpose of this article to develop a formula for volume for which the above are special cases. When a body may be considered as composed of a succession of laminas such that no lamina has faces that intersect as they approach coincidence and such that the face of lamina never becomes reentrant upon the volume, the volume may be considered as follows.



Let s be the space curve which is the locus of the centers of gravity of the laminas composing the body. Consider the volume of a differential lamina. Let the plane of one face be obtained from that of the other by rotation through an angle $d\theta$ about an instantaneous axis outside the face. Through the center of gravity G of one face draw an x-axis GX, parallel to the instantaneous axis. Let GY be a y-axis perpendicular to GX in the plane of the face, in which G lies, drawn away from the instantaneous axis. If the thickness of the lamina at G is dt, at the point (x, y) its thickness is $dt+yd\theta$. Hence, V being the volume, we have, to first order differentials in x, y, t and θ.

$$V = \iiint dx\,dy\,dt + \iiint y\,dx\,dy\,d\theta.$$

Since θ is independent of x and y, and since $\iint y\,dx\,dy = 0$ because G is the center of gravity of the face,

$$V = \iiint dx\,dy\,dt. \tag{1}$$

If s makes an angle ϕ with the normal to the plane, $dt = ds \cos \phi$ and[2]

[1] See E. J. Routh, *Analytical Statics*, vol. 1, p. 293.

[2] In the case where $\phi = 0$, GX lies in the binormal to the curve s and GY in the principal normal and $Rd\phi = ds$, where R is the radius of curvature of the curve s.

$$V = \iiint dx\,dy\,\cos\theta\,ds. \tag{2}$$

Supposing the integration in x and y to have been made for a general position of the lamina of face area A, we have

$$V = \int A\,\cos\vartheta\,ds, \tag{3}$$

in which A and ϕ are, in general, functions of s.

There are many special cases, some of which will be mentioned. If A is constant and $\phi = 0$, $V = As$, the case mentioned by Routh.

If A is constant; ϕ, constant, $= B$, and s is the circumference of a circle of radius r, $V = 2\pi rA\cos B$, the equivalent of special cases discussed by Edwards. If $B = 0$, $V = 2\pi r\,A$, the usual theorem of Pappus for volume. If the normal to A retains a fixed direction OX, then $\cos\phi ds = dx$ and we have the usual formula for volume by parallel sections. If the normal to A retains a fixed direction, s is a straight line, and A varies as s^2, we have a pyramid or cone; and if A is a constant, a cylinder.

A formula similar to (3),

$$S = \int L\,\cos\phi\,ds, \tag{4}$$

can easily be developed for the surface of the volume generated. L, being the perimeter of the face of a general lamina, is, in general, a function of s, the locus of the center of gravity of the perimeter.

A few applications may make clear some uses of the formulas. To find the volume of a truncated cylinder standing on a base of area A_0, we may consider it to be generated by a general lamina making a variable angle ϕ with the base which coincides with the base in its first position and with the truncated base in its final position. Then $A = A_0 \sec\phi$, and $V = \int A\,\cos\phi ds = A_0 s$, the area of the base times the height at the center of gravity.

Suppose a cornucopia is generated by a variable circle whose plane moves normal to the helix $x = 4a\,\cos\theta$, $y = 4a\,\sin\theta$, $z = 3a\theta$, its center being on the helix and its radius equal to $s/8$ where s is the length of the arc of the helix measured from $\theta = 0$. If θ varies from 0 to $3\pi/2$, find the volume and surface of the cornucopia.

Here $A = \pi s^2/64$; $\phi = 0$, and $V = (\pi/64)\int s^2 ds$, from $s = 0$ to $s = 15\pi a/2$. Hence $V = 1125\pi^4 a^3/512$, and $S = \int L\cos\phi ds = (\pi/4)\int s ds$, from $s = 0$ to $s = 15\pi a/2$; whence $S = 225\pi^3 a^2/32$.

Again, consider the figure generated by a circle of constant radius r $(r < 2a/3)$ moving with its plane perpendicular to the plane of a fixed circle of radius a,

the center of the moving circle being always on the circumference of the fixed circle and the normal to the plane of the moving circle making an angle $\phi=\theta/2$ with the tangent to the fixed circle where it pierces the plane of the moving circle; to find the surface and volume generated for θ varying from 0 to π where $a\theta$ is the distance moved by the center of the generating circle.

In this example, $\phi=\theta/2$, $s=a\theta$ and hence $S=\int L\cos\phi ds=2\pi ra\int_0^\pi\cos(\theta/2)d\theta=4\pi ra$; and $V=\int A\cos\phi\, ds=\pi r^2a\int_0^\pi\cos(\theta/2)d\theta=2\pi r^2a$.

Many possible applications of the formulas will readily occur to anyone interested.

NOTE ON THE GEOMETRIC DESCRIPTION OF LINEAR FAMILIES OF CONICS BY MEANS OF APOLARITY

By ALAN D. CAMPBELL, Syracuse University

It is easy to describe geometrically one-parameter linear families (pencils) of point or line conics. For example the pencil of point conics $\lambda x^2+\mu(y^2+2zx)=0$, with discriminant $\Delta\equiv-\mu^3$, consists of all the point conics that intersect in four coincident points at (0, 0, 1) and hence has just one degenerate conic, a double line (the common tangent at this point). The pencil of line conics $\lambda(u^2-v^2)+\mu(u^2-w^2)=0$ consists of all the line conics with four common tangents and therefore has three degenerate conics, the pairs of opposite vertices of the complete quadrilateral formed by these four tangents.

To describe two-parameter linear families (nets) of conics and to derive as well as describe typical three- and four-parameter families we can use a projectively invariant property that links point and line conics and is called apolarity.[1] A point conic

$$C\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is said to be apolar to line conic

$$C'\equiv A'u^2+B'v^2+C'w^2+2F'vw+2G'wu+2H'uv=0$$

if

$$K=aA'+bB'+cC'+2fF'+2gG'+2hH'=0.$$

The following facts about apolarity are well known. If a line conic C' is a double point, then C' lies on any apolar point conic C; if C' is a pair of points, then it is a pair of conjugate points with respect to any apolar point conic C; if C' is a nondegenerate line conic, then it is inscribed in complete quadrilaterals

[1] See G. Salmon, *Conic Sections*, 2nd edition, §375; H. J. S. Smith, *Collected Works*, vol. 2, pp. 524–540; Reye-Holgate, *Geometry of Position* pp. 222–240; and, for the space analogue, Snyder and Sisam, *Analytic Geometry of Space*, pp. 180–187. We note that apolarity is an invariant property in the complex domain, in the real domain, and in the Galois Fields of order p^n where $p>2$.

that are self-polar with respect to any apolar point conic C (i.e., the pairs of opposite vertices of such a quadrilateral are pairs of conjugate points with respect to C).

By an r-parameter linear family ($r=0, 1, 2, 3, 4, 5$) of point (or line) conics, we mean all conics given by the equation

$$\lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_{r+1} C_{r+1} = 0, \text{ where } C_i = 0 (i = 1, 2, \cdots, r+1)$$

are linearly independent point (line) conics that are called fundamental conics of the family. If a point (line) conic is apolar to each of a set of fundamental line (point) conics of an r-parameter family, it is apolar to every conic of this family. Also if a set of fundamental conics of a family of point (line) conics are apolar to the conics of another family of line (point) conics, then every conic of the first family is apolar to every conic of the second. Two such families are said to be apolar to one another. To every r-parameter family $M_r (r=0, 1, 2, 3, 4)$ of line conics there corresponds a unique apolar $(4-r)$-parameter family N_{4-r} of point conics, and conversely. Hence if we project any family M_r into another family M_r' of the same class of families (for which M_r may be taken as a typical family), then (since apolarity is an invariant property) the corresponding unique apolar family N_{4-r} must go into the unique family N_{4-r}' apolar to M_r'; and conversely. So to the class of families of line conics to which M_r belongs there corresponds a unique class of apolar families of point conics to which N_{4-r} belongs and conversely. We call two such classes apolar classes of families.

Thus apolarity gives us a very useful one-to-one reciprocal correspondence between the classes of line (point) conics and their apolar classes of four-parameter families of point (line) conics, between the classes of pencils of line (point) conics and their apolar classes of three-parameter families of point (line) conics, and between the classes of nets of line (point) conics and their apolar classes of nets of point (line) conics. To describe geometrically a family of point conics apolar to a family M_r of line conics, we choose $r+1$ linearly independent conics in M_r, preferably double points (if these are present in the family), otherwise pairs of real points or, finally, pairs of conjugate imaginary points, or nondegenerate real or imaginary conics.

We shall now give a few illustrations of this use of apolarity, first of all in the study of three- and four-parameter families of conics. It is easy to see that, apolar to the double point $u^2=0$, we have the four-parameter family

$$\lambda y^2 + \mu z^2 + 2\nu yz + 2\rho zx + 2\sigma xy = 0,$$

which consists therefore of all the conics through a given point $P(1, 0, 0)$. Apolar to the typical pencil $\lambda(u^2-v^2)+\mu(u^2-w^2)=0$, we have (using K) the three-parameter family

$$\lambda(x^2 + y^2 + z^2) + 2\mu yz + 2\nu zx + 2\rho xy = 0,$$

which consists of all the conics having a common self-polar quadrilateral.

The typical nets of point (line) conics are derived by other methods than apolarity,[1] but they are readily described by means of their apolar nets of line conics. Let us consider the net of point conics

$$\lambda(z^2 + 2xy) + 2\mu yz + \nu x^2 = 0.$$

Any line conic apolar to the fundamental conics of this net must satisfy the equations (see K): $C'+2H'=0$, $F'=0$, $A'=0$. So every line conic of the form, $B'v^2+C'w^2+2G'wu-C'uv=0$ is apolar to this net. We put $C'=\lambda$, $G'=\mu$, $B'=\nu$, and we have an apolar net of line conics

$$\lambda(w^2 - uv) + 2\mu uw + \nu v^2 = 0.$$

In this net we find a complete quadrilateral with vertices given by $uw=0$, $(w-v)(w+v+u)=0$ (the conic given by $\lambda=1$, $\mu=\frac{1}{2}$, $\nu=-1$) and $(w+v)(w-v-u)=0$ (the conic given by $\lambda=1$, $\mu=-\frac{1}{2}$, $\nu=-1$) and also the point $v^2=0$ on one of the sides of this quadrilateral. Hence any net of point conics of the class that has the above typical net consists of all the conics having a common real self-polar quadrilateral and also passing through a given point on one of its sides. The net of point conics

$\lambda x^2 + \mu y^2 + 2\nu yz = 0$ is apolar to the net of line conics $\lambda w^2 + 2\mu wu + 2\nu uv = 0$;

hence this net of point conics consists of all the conics that have a given point (1, 0, 0) and a given line $x=0$ as pole and polar and that cut $x=0$ in the given point (0, 1, 0).

[1] For the complex domain, see C. Jordan, *Réduction d'un réseau de formes quadratiques*, Journal de Mathématiques (6), vol. 2 (1906), p. 412. For the Galois fields of order p^n where $p>2$, see A. H. Wilson, *The canonical types of nets of modular conics*, American Journal of Mathematics, vol. 36 (1914), pp. 187–210. For the real domain, see A. D. Campbell, *Nets of conics in the real domain*, American Journal of Mathematics (not yet published).

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. AN ANALYTICAL DISCUSSION OF PROBLEM 3221

BY ROSCOE WOODS, University of Iowa

In the October, 1927 issue of this Monthly there appeared my geometric solution of Problem 3221 set by H. E. Trefethen. The problem read as follows: A variable rectangle has a diagonal of constant length and two sides lying upon two fixed perpendicular straight lines. Determine geometrically the locus of the foot of the perpendicular from the vertex opposite the fixed vertex upon the diagonal which does not pass through that vertex.

Let us consider the problem somewhat extended, using analytical methods. Let the vertices of the variable rectangle be O, A, B, C, where O is the fixed vertex, A and C are the vertices on the two fixed perpendicular straight lines, and B is the remaining vertex whose locus is evidently a circle with O as center and radius r which is the length of the constant diagonal of the variable rectangle. There is no loss of generality if the units are so chosen that $r=1$. Let the vertices O, A, B, C have the coordinates $(0, 0)$, $(X, 0)$, (X, Y) and $(0, Y)$ respectively where $X^2+Y^2=1$. Instead of finding the locus of $P(x, y)$, the foot of the perpendicular from B upon AC, let the locus of P, the foot of the perpendicular from $B'(X', Y')$ upon AC, be discussed, where X and Y are related to X' and Y' as follows:

$$(1)\qquad \begin{aligned} X' &= (a_1X + b_1Y + c_1)/(a_3X + b_3Y + c_3), \\ Y' &= (a_2X + b_2Y + c_2)/(a_3X + b_3Y + c_3), \end{aligned}$$

where a_3, b_3, c_3 are not all zero simultaneously and where a_i, b_i, $c_i (i=1, 2, 3)$ are any real numbers including zero. The coordinates of $P(x, y)$ are

$$(2)\qquad x = X(XX' - YY' + Y^2), \qquad y = Y(YY' - XX' + X^2).$$

For convenience of discussion we introduce a parameter t as follows:

$$X = (1 - t^2)/(1 + t^2), \quad Y = 2t/(1 - t^2).$$

It then follows that

$$(3)\qquad \begin{aligned} X' &= [t^2(c_1 - a_1) + 2b_1t + a_1 + c_1]/[t^2(c_3 - a_3) + 2b_3t + a_3 + c_3], \\ Y' &= [t^2(c_2 - a_2) + 2b_2t + a_2 + c_2]/[t^2(c_3 - a_3) + 2b_3t + a_3 + c_3]. \end{aligned}$$

From (2) and (3) it now follows that the locus of P is in general a rational octavic curve. This locus has cusps at the circular points of the plane and cuts the line at infinity in two other points which are coincident when $b_3^2+a_3^2=c_3^2$. It is also easy to see that this locus cuts the x- and y-axes in the points

$$[c_1 \pm a_1)/(c_3 \pm a_3),\quad 0] \quad \text{and} \quad [0,\ (c_2 \pm b_2)/(c_3 \pm b_3].$$

Let the determinant of the transformation (1) be denoted by D. If D is different from zero, (2) is an eight parameter system of rational curves. For $D\neq 0$, P' lies upon a conic, the transform of the circle $X^2+Y^2=1$ under (1). There are many interesting curves in this system when the arbitrary constants are subjected to various conditions. For instance, when (1) reduces to the identity, the locus of P is the hypocycloid of four cusps. As is known this curve is a sextic with cusps at the circular points.

Also there are septimics, sextics, quartics, ellipses, and a circle. There are no quintics or cubics. In what follows are set forth some of the conditions that give rise to these curves.

(I) Septimics with cusps at the circular points arise when $\pm a_3=\mp c_3=b_3=k/2$, where k is any constant different from zero. In this case $D\neq 0$ and the unit circle is transformed into a hyperbola, the discriminant of the second degree terms being proportional to D^2. Septimics can also arise when $a_i=\pm c_i (i=1, 2, 3)$. In this case $D=0$ and the locus of P' is the line whose equation is

$$(a_3b_2)X' + (a_1b_3)Y' + (a_2b_1) = 0 \qquad \text{where} \qquad (a_ib_k) = a_ib_k - a_kb_i.$$

(II) Sextics with cusps at the circular points arise: (a) when $a_3=b_3=0$, $c_3=k(k\neq 0)$, and $D\neq 0$. If we impose the further conditions $a_2=c_2=b_1=c_1=0$ and $k=1$, the system so obtained is symmetrical to the x- and y-axes and the origin and is of such a nature that a great deal of information is readily obtained. We shall return to this system later. (b) when $c_3=b_3=c_2=b_2=0$, $a_3=a_2=k$ $(k\neq 0)$ and $D=0$. In this case P' lies on the line $Y'=k$. (c) when the rank of D is 1. In this case P' is now a fixed point in the plane and the locus so found is the pedal curve of this point with respect to the hypocycloid of four cusps. If P' lies inside the hypocycloid the pedal has four real branches passing through P', and if on the outside of the hypocycloid there are only two real branches passing through P'. In particular if P' is at the origin the pedal is the four-leaved rose.

(III) When $a_1+b_2=k$, $b_1-a_2=0$, $a_3=b_3=0$, $c_3=k$ $(k\neq 0)$ and $D\neq 0$, the locus becomes bi-circular quartics with double points at the circular points. The locus of P' is now a parabola. The x- and y-intercepts of these curves are easily found but are not set down here.

(IV) In (III) if the further conditions are imposed, that is if $c_2=c_1=0$, $D=0$, the bi-circular quartics degenerate into ellipses. In this case the locus of P' is an ellipse. If we set $2a_1=k$, the locus becomes a circle. It ought to be pointed out that in this case (IV) P' is a fixed point on the line AC.

Returning now to the system of sextics in (IIa), we note several cases of interest:

(1) When a_1, $b_2>1$, $a_1+b_2>1$, the sextics cross the axes at A_1 $[\pm a_1, 0]$, $B_1[0, \pm b_2]$, $A_2[\pm b_2^{1/2}(a_1+b_2-1)^{-1/2}, 0]$, and $B_2[0, \pm a_1^{1/2}(a_1+b_2-1)^{-1/2}]$. The points A_2 and B_2 are double points.

(2) When $0<a_1<1$, $0<b_2<1$, the curves have intercepts at A_1 and B_1 but the points A_2 and B_2 are imaginary.

(3) When a_1, $b_2<0$, the curves cross at A_1 and B_1 and have double points at A_2 and B_2 and there is also a double point in each quadrant.

(4) When $a_1=1$, $b_2>0$, $(b_2\neq 1)$ the points A_1 and A_2 coincide and become cusps, B_1 remain intercepts and B_2 remain double points.

(5) When $a_1=1$, $b_2=0$, $D=0$, the locus is now the diameter of the unit circle that lies along the x-axis.

(6) When $a_1=1$, $b_2<0$, A_1 and A_2 coincide and are cusps, B_1 are intercepts and B_2 are imaginary.

(7) When $a_1>1$ or $a_1<1$, $b_2=0$, $D=0$, the locus has A_1 for intercepts, B_1 and A_2 are at the origin, and B_2 remain double points.

(8) When $0<a_1<1$, $b_2=0$, $D=0$ the locus has intercepts at A_1, B_1 and A_2 fall at the origin and the points B_2 are imaginary.

(9) The interchange of the conditions imposed upon a_1 and b_2 merely has the effect of interchanging the x- and y-axes in the foregoing.

(10) When $a_1=b_2=1$, the locus is the hypocycloid of four cusps, A_1 coinciding with A_2, B_1 with B_2, forming cusps.

II. A Method of Solving A Biquadratic

By C. H. Smiley, University of Illinois

The problem of finding the four roots of the biquadratic

$$f(x) \equiv a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \qquad (a_0 \neq 0)$$

may be considered as equivalent to the problem of factoring $f(x)$ into two quadratic factors; that is, the determination of b_0, b_1, b_2; c_0, c_1, c_2 so that $f(x) \equiv (b_0x^2+b_1x+b_2)\ (c_0x^2+c_1x+c_2)$.

We must then determine these six quantities to satisfy the five equations:

$$a_0 = b_0c_0\ ;\ a_1 = b_0c_1 + b_1c_0\ ;\ a_2 = b_0c_2 + b_2c_0 + b_1c_1\ ;\ a_3 = b_1c_2 + b_2c_1\ ;\ a_4 = b_2c_2.$$

Having one condition at our disposal, let us set $b_1=c_1$. (We shall discuss

the legitimacy of this assumption later.) Then our conditions may be written:

(1) $b_1=c_1$; $b_0c_0=a_0$; $b_0+c_0=a_1b_1^{-1}$; $b_0c_2+b_2c_0+b_1^2-a_2=0$; $b_2+c_2=a_3b_1^{-1}$; $b_2c_2=a_4$.

Then b_0 and c_0 are the roots of the quadratic equation, $y^2-a_1b_1^{-1}y+a_0=0$.

And likewise, b_2 and c_2 are the roots of the quadratic equation, $z^2-a_3b_1^{-1}z+a_4=0$.

Then, for an assumed b_1, we can compute the roots of these two quadratics, which will be the corresponding values of b_0, c_0, b_2, c_2. Using these values in the fourth equation of (1), we may obtain a better value of b_1, and by a method of successive approximations determine a set of constants which will satisfy the fourth equation of (1) to any desired degree of accuracy. Or we may use a method of interpolation to secure this same result.

Having determined a set of constants which will satisfy the conditions (1), we may proceed to solve the two quadratics to obtain the four roots of the biquadratic.

It is interesting to note the nature of b_1, when expressed as a function of the roots of the biquadratic. For the sake of simplicity, let us set $a_0=1$ and note that the fourth equation of (1) may be written in the form:

$$b_1^2 = a_2 - \frac{b_2}{b_0} - \frac{c_2}{c_0} ; \quad b_1^2 = x_1x_3 + x_2x_4 + x_1x_4 + x_2x_3 ;$$

the corresponding resolvent cubic is

$$b_1^6 - 2a_2b_1^4 + (a_2^2 + a_1a_3 - 4a_4)b_1^2 + a_1^2a_4 + a_3^2 - a_1a_2a_3 = 0.$$

Now as to the legitimacy of our assumption, $b_1=c_1$, obviously there are two cases which we will need to avoid;—first, the case where the only possible factorization into quadratic factors with real coefficients leaves one factor with and the other factor without a term in x; and secondly, where the only possible factorizations into quadratic factors with real coefficients leave the two coefficients of x in each pair of factors, of opposite sign.[1] The first case may be handled by the method of this paper if we first transform our equation by means of a simple translation of the origin. And the second case may also be handled if we change the signs of all the a's, making a_0 negative, before starting our approximations. There may be some difficulty in recognizing these cases—and it seems desirable to avoid them entirely. This we may do, if we first transform our given biquadratic in such a way as to make a_1 zero, and then change the signs of all the a's to make a_0 negative. Then the method will work.

[1] We have tacitly assumed that $a_0>0$ in the given biquadratic.

III. On the Approximate Division of a Circumference

By Tobias Dantzig, University of Maryland

In a recent issue of this Monthly,[1] R. A. Johnson discussed two approximate constructions for the division of a circle in n equal parts. While the accuracy of either construction, particularly the second one, was reasonably good, the error did not approach zero with increasing n. The approximation method given below is also rather simple and accurate, but has the advantage that the error very rapidly approaches zero with increasing n.

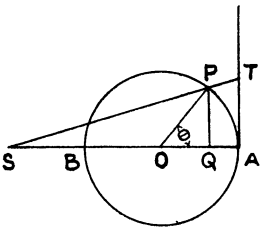


Fig. 1

Let AB (Fig. 1) be the diameter of a unit circle, S a point on the extension of this diameter, and P the representative point of an acute angle θ. Let SP continued meet the tangent at A in T and set $AT=t$, $BS=c$. Then by similar triangles:

$$\frac{AT}{QP}=\frac{AS}{QS} \text{ or } t=\frac{(c+2)\sin\theta}{c+1+\cos\theta},$$

and consequently

$$t-\theta=\frac{(2+c)\sin\theta-(1+c)\theta-\theta\cos\theta}{1+c+\cos\theta}.$$

Expanding the numerator into series, we obtain

$$t-\theta=\frac{(1-c)\theta^3}{6(1+c+\cos\theta)}+F(\theta)\cdot\theta^5,$$

where $F(\theta)$ is a function of θ such that

$$\lim_{\theta\to 0}F(\theta)=(c-3)(c+2)^{-1}/120.$$

Thus it appears that $t-\theta$ is, for any value of c, an infinitesimal of the third order and *for $c=1$ the difference $t-\theta$ becomes an infinitesimal of the fifth order with respect to θ*. This gives the approximation formula

$$\theta=\frac{3\sin\theta}{(2+\cos\theta)}. \tag{1}$$

The accuracy of the formula is shown in the following table:

[1] Vol. 34 (1927), pp. 429–431.

n	Angle	Arc	Approximation	Error in percentage
5	72°	1.2566	1.2360	1.6
6	60°	1.0472	1.0392	0.8
12	30°	.5236	.5234	0.4
30	12°	.2094	.2094	0.005

This leads to the approximate division shown in Fig. 2. On the tangent CD lay off a uniform scale with a unit equal to $R/6$. Let N be the point on this scale representing the number n of parts in which we wish to divide the circumference. Let NO meet AD in T and let ST meet the first quadrant in P_n; then the arc AP_n is approximately $1/n$ of the circumference.

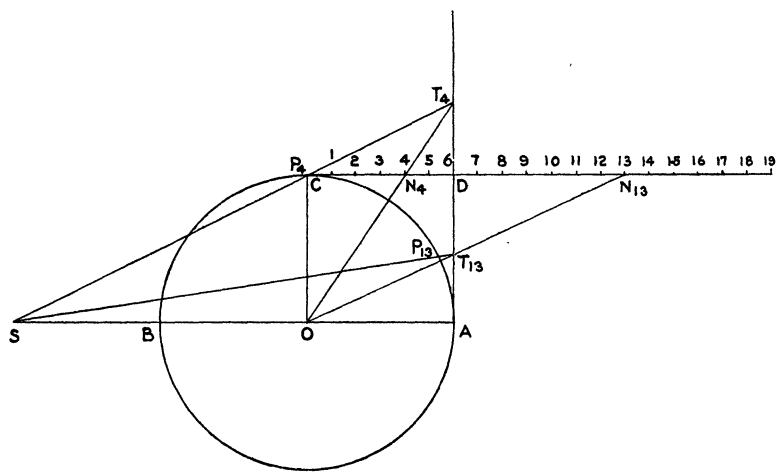


Fig. 2

Incidentally, formula (1) for $\theta = \pi/6$ gives the following approximation for π: $\pi = 18/(4+\sqrt{3})$, which is correct to two decimal places.

Figure 2 shows the construction for $n=4$ and $n=13$.

While the construction here given is believed to be new, formula (1) on which it is based dates back to Cardinal Cusanus (1401). It was later derived independently by Willebrord Snellius (1627), who also gave another approximation formula $\theta = \frac{1}{3}(2 \sin\theta + \tan\theta)$. It is of considerable historical interest to note that while Snellius regarded the formulae as mere approximations, Cusanus actually believed that he was offering an exact construction. As a matter of fact he utilized this method for a determination of the number π.

In this connection we wish to note two other formulae for the approximation of an arc of a circle:

(2) $\theta = (\sin^2\theta \tan\theta)^{1/3}$; (3) $\theta = \dfrac{(14 + \cos\theta)\sin\theta}{(9 + 6\cos\theta)}$

The first one is due to Orontius Finaeus whereas the other was devised by Newton and gives an exceedingly accurate construction. As a matter of fact Newton used the formula for getting the number π correct to 12 decimal places.

For references to the problem of division of the circumference and allied questions we refer the reader to Moritz Cantor, *Geschichte der Mathematischen Wissenschaften*, volume 2 and to Theodor Vahlen, *Konstruktionen und Approximationen*.

IV. A Problem in Maxima and Minima

By Roger A. Johnson, Hunter College of the City of New York

Problem. *A frustum of a right circular cone with a fixed altitude h and a fixed lower base of radius r has a variable upper base of radius x. Consider the area of the lateral surface as a function of x in the interval from $x=0$ (cone) to $x=r$ (cylinder). In particular, for what values of x in this interval is the area a maximum or minimum?*

This problem is noteworthy for several reasons. It is suggested by a standard problem of the calculus of variations, that of the surface of revolution of minimum area.[1] It is a typical problem in maxima and minima, of the class ordinarily solved in a first course in the calculus, but the author has not seen it proposed in any calculus text. Finally, the nature of the solution is by no means intuitionally evident.

The reader is invited at this point to consider the problem, and before turning the page to formulate intuitionally his own conclusions as to its solution.

The straightforward solution of the problem by the usual methods of the calculus should not tax the powers of the average undergraduate. The area in question is

(1) $$S = \pi(r + x)\left[h^2 + (r - x)^2\right]^{1/2},$$

the last factor representing the slant height g. In particular, the areas of the cylinder ($x=r$) and of the cone ($x=0$) are, respectively,

$$S_c = 2\pi rh, \quad S_0 = \pi r\left[h^2 + r^2\right]^{1/2}.$$

[1] For if we consider the surfaces generated by revolving about an axis the various curves joining two fixed points A and B, where AB is parallel to the axis, we naturally think first of the cylinder generated by AB itself, and secondly of the double frustum generated by a broken line ACB, with $AC=CB$.

V. On the Integration of a Certain Type of First Order Differential Equation

By John P. Dalton, University of the Witwatersrand, Johannesburg, S.A.

1. Problem 3272 recently proposed in this Monthly[1] by L. M. Berkeley is readily solved by either of the following methods. These methods have been used for some time time past by the writer in his lectures to undergraduates; and although the problem proposed for solution is of little intrinsic interest, this opportunity is taken of making the methods known.

2. The type of equation considered is

$$(P_1 + P_2)dx + (Q_1 + Q_2)dy = 0, \tag{1}$$

where P_1, P_2, Q_1 and Q_2 are functions of x and y, continuous in both variables in the domain under consideration.

The first method depends upon the possibility of obtaining an integrating factor common to both differentials P_1dx+Q_1dy and P_2dx+Q_2dy.

If possible let μ be such a factor and λ its logarithm. Further let

$$D_r = \frac{\partial P_r}{\partial y} - \frac{\partial Q_r}{\partial x} \quad (r = 1, 2), \qquad \Delta = \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix} \neq 0,$$

$$\Delta_P = \begin{vmatrix} P_1 & D_1 \\ P_2 & D_2 \end{vmatrix}, \qquad \Delta_Q = \begin{vmatrix} Q_1 & D_1 \\ Q_2 & D_2 \end{vmatrix}.$$

On applying the Euler condition to the two differentials and solving the resulting equations for the derivatives of μ we obtain

$$\frac{\partial \lambda}{\partial x} = \frac{\Delta_P}{\Delta}, \tag{2}$$

$$\frac{\partial \lambda}{\partial y} = \frac{\Delta_Q}{\Delta}, \tag{3}$$

The commutative property then gives as the criterion for the existence of μ

$$\frac{\partial}{\partial y}\left(\frac{\Delta_P}{\Delta}\right) = \frac{\partial}{\partial x}\left(\frac{\Delta_Q}{\Delta}\right) \tag{4}$$

If this equation is satisfied, integration of (2) and (3) gives the required integrating factor.

3. The equation referred to in §1 is

$$(\sin y - m \tan x) \cos y\, dx - (\tan x + m \sin y)\, dy = 0, \tag{5}$$

where m is a constant. Writing

$$P_1 = \cos y \sin y, \quad P_2 = -m \tan x \cos y, \quad Q_1 = -\tan x, \quad Q_2 = -m \sin y,$$

[1] Vol. 34 (1927), p. 380.

we get

$$D_1 = \cos 2y + \sec^2 x, \qquad \Delta = -m \cos y(\sin^2 y + \tan^2 x),$$
$$D_2 = m \tan x \sin y, \qquad \Delta_P = m \tan x \cos y(\cos^2 y + \sec^2 x),$$
$$\Delta_Q = 2m \sin y \cos^2 y.$$

Hence

$$\frac{\partial}{\partial y}\left(\frac{\Delta_P}{\Delta}\right) = \frac{\sin 2x \sin 2y}{(1 - \cos^2 x \cos^2 y)^2} = \frac{\partial}{\partial x}\left(\frac{\Delta_Q}{\Delta}\right)$$

and condition (4) is satisfied.

On integrating

$$\frac{\partial \lambda}{\partial y} = \frac{\Delta_Q}{\Delta} = -\frac{\sin 2y}{(\sin^2 y + \tan^2 x)},$$

we obtain

$$\log \mu = -\log (\sin^2 y + \tan^2 x) + \log F(x),$$

where $F(x)$ is an arbitrary function

Substituting this in

$$\frac{\partial \lambda}{\partial x} = \frac{\Delta_P}{\Delta} = -\frac{\tan x(\cos^2 y + \sec^2 x)}{\sin^2 y + \tan^2 x},$$

we obtain $F(x) = \sec x$. Hence the desired integrating factor is

$$\mu = \frac{\cos x}{(1 - \cos^2 x \cos^2 y)}$$

and therefore the solution of equation (5) is

$$\arctan (\sin x \cot y) = \log C\left(\frac{1 - \cos x \cos y}{1 + \cos x \cos y}\right)^{m/2}.$$

4. The second method applicable to differentials of the type now under consideration is of a more tentative nature; nevertheless it is of some utility.

Let μ_1 be an integrating factor of the differential $P_1dx + Q_1dy$; and let $\zeta_1 = \text{constant}$ be the resulting integral. Let μ_2 and ζ_2 be similar functions relative to the differential $P_2dx + Q_2dy$.

Expressing μ_1 and μ_2 in terms of ζ_1 and ζ_2, we then determine arbitrary functions f and F such that

$$\mu_1 f(\zeta_1) = \mu_2 F(\zeta_2), \tag{6}$$

and this expressed in terms of x and y is an integrating factor for equation (1).

5. Applying this method to the problem under discussion we find, for the subdivisions of §3,

$$\mu_1 = \sec y \operatorname{cosec} y \cot x\ ; \qquad \zeta_1 = \log \sin x - \log \tan y\ ;$$
$$\mu_2 = \sec y\ ; \qquad \zeta_2 = \log \cos x + \log \cos y.$$

From these equations we find

$$\frac{\mu_2}{\mu_1} = -\frac{\sinh \zeta_2}{\cosh \zeta_1}$$

Hence from (6) $\mu_1 \operatorname{sech} \zeta_1$ is an integrating factor of (5) and this reduces to $\cos x[1-\cos^2 x \cos^2 y]^{-1}$ as before.

RECENT PUBLICATIONS

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REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Mengenlehre. By F. HAUSDORFF. Berlin-Leipzig, Walter de Gruyter and Co., 1927. 285 pages.

Any attempt to produce a book on the theory of aggregates and have it approximately up to date and complete at the time of its appearance must, on account of the present status of the subject, fall somewhat short of the mark. Each year there appear one or two volumes of Fundamenta Mathematicae, a journal which is devoted almost entirely to this subject; and perhaps an equal volume of papers are published each year in German and American journals. When one considers the vast number of new results being obtained all the time, and realizes the practical impossibility of foretelling of what importance a new proposition is going to be when first discovered, one readily appreciates the difficulties which confront an author in culling out the new material and selecting the results which will likely prove most useful in future research. This is especially true on account of the comparative youth of the subject and the scanty supply of books available on this subject. However, in consideration of the difficulties, Professor Hausdorff seems to have accomplished this feat with a fair degree of success in his new book, that is, in so far as it was attempted. Such does not seem to have been the author's chief purpose in writing the book. This book appeals to the reviewer as one which would be of much more interest and practical value either to a beginner in the subject or to the mathematician

whose main field lies in some other branch of mathematics but who has occasional opportunity in his work to use some elementary theorems on aggregate theory than it would be to the research specialist in this particular field.

The reviewer would hardly feel justified in criticizing the author for his choice of the material to include in his book, because that is the author's pleasure; and undoubtedly the material chosen is fairly representative of the work which has been done and is being done in the field of aggregate theory, and it is capable of giving the beginner a nice insight into some of the beauties of the subject. However, there are a number of places in the book where the treatment could have been somewhat clarified and the proofs of some of the theorems materially shortened if only the author had chosen to make use of some quite well known theorems which were not included in his book. One or two instances of this sort will be given below.

We do not purpose, in this review, to give a synopsis of the material covered in the book under review, nor to compare it with an earlier book, *Grundzüge der Mengenlehre*, by the same author. The reader who desires this is referred to an excellent review of this book by H. M. Gehman in the Bulletin of the American Mathematical Society.[1] Instead, some features of the book which are of particular interest to the reviewer and which perhaps will be of interest to readers of the Monthly will be commented upon.

A very notable feature of this book is the author's attempt to introduce a logical system of notation and terminology. In most cases the author has followed the modern tendencies in this respect. The problem of adopting a standard system of notation and terminology is squarely up to workers in this field at the present time. Some symbolic language clearly is necessary in order to prevent our papers being cumbersome. However, the question as to just how far one should go in this respect is a debatable one. Especially is this true in point set theory. It is quite possible, in fact it often happens, that an author uses so many signs and symbols in his paper that his article is made almost unreadable. In the reviewer's opinion, letters and symbolic notation are of no use whatever in any subject unless they simplify the presentation and very materially shorten it. Yet it quite frequently occurs that an author of a paper in the field goes to considerable trouble to introduce a host of letters to mean certain things, and sets up his hypothesis and conclusions in the form of very imposing appearing equations when, as a matter of fact, the same thought could be conveyed more clearly and intelligibly in words and this could actually be done in much less space. This may appear rather astounding, but if one considers the extra space required in printing an equation or a number of sigma signs, which usually have to be set off, together with the space required to

[1] Vol. 33 (1927), pp. 778–780.

define these symbols, such a statement appears less unreasonable. The chief objection to a complicated system of notation, let me repeat, is the fact that the paper thereby becomes much more tedious to read. This objection can be largely overcome through the functioning of two agencies, to wit, (1) the adoption by all writers in the field of a standard set of symbols and terms to stand for the more commonly used notations whose expression in words either is awkward or requires considerable space, and (2) the exercise of great care by each writer when introducing a new symbol—the symbol should be suggestive, if possible, of the idea for which it stands, and it should be defined in each and every paper in which it is used until its usage by other authors becomes sufficient to warrant the omission of the definition. The reviewer knows of nothing which can do more to bring these agencies to bear in the field than the appearance of books like Hausdorff's in which a very elegant system of notation is used which follows, in the main, the modern tendencies in the subject. Hausdorff's symbols are all clearly defined and are, for the most part, quite suggestive of the things for which they stand. In some cases it seems to the reviewer that too much notation is used and several unnecessary symbols are introduced, but of course that is questionable. The notation $a\epsilon A$ to mean "a is an element of A" and $a\bar{\epsilon}A$ to mean "a is not an element of A" seems objectionable in some respects. Particularly it is true since, just two pages further on, the author introduces the notation $A \leqq B$ to mean "A is a subset of B", and $A < B$ to mean "A is a proper subset of B". It is quite true that $a\epsilon A$ and $a \leqq A$ do not convey exactly the same meaning, since the first indicates that a is one of the elements of A, while the second merely indicates that a belongs to A (a might be either a single element or a collection of elements of A); nevertheless, it seems that we should be able to do without the former of these symbols. In general usage it would ordinarily be known whether or not a consisted of a single element or a collection of elements; and such being known, the distinction between the notations $a\epsilon A$ and $a \leqq A$ appears to the reviewer to become unimportant. The notation $a\bar{\epsilon}A$ is objectionable for the additional reason that the notation $\overline{X}$ means the set X plus all of its limiting elements, and to use $\bar{\epsilon}$ in an entirely different sense is confusing to a reader even though he can soon grasp the correct meaning. Another confusing thing is the author's use of the term "Entfernung," which we would probably translate as "distance", for one thing and the term "Distanz" for a different thing. The former of these is denoted by xy (for distance from x to y) and the latter by $\overline{xy}$. On the whole, however, the system of notation is very good and is one which authors in the subject would do well to study and follow wherever it is necessary or desirable to resort to symbolic language. The use of the term *component* to replace the rather awkward term *maximal connected subset* seems especially desirable to the reviewer.

We consider now §39, 1 of the book, which is entitled *Bedingungen für einfache Kurven*. Here the author lists the following five conditions as necessary conditions in order that a metric space C be a simple continuous arc.

(α) C is self-compact.

(β) C contains two points a and b between which it is an irreducible continuum.

(γ) C contains two points a and b between which it is irreducibly connected.

(δ) [a condition equivalent to (γ)].

(ϵ) C is locally connected, i.e., connected im kleinen. He then proceeds to state (cf. Theorems III and IV) that the sets of conditions (α, γ), (α, δ), and (α, β, ϵ) are necessary and sufficient in order that C should be a simple continuous arc. No proof is given for the sufficiency of the sets (α, γ) and (α, δ); instead, references are made to N. J. Lennes and W. Sierpinski, respectively. However, the author gives a complete proof for the sufficiency of the set of conditions (α, β, ϵ). This proof is very elementary in nature, and we do not criticize the author in the least for giving it. It is interesting, however, to see how the theorem follows by the following line of argument: Since C is compact, by (α), and metric and, by (ϵ), is locally connected, it is readily seen that C satisfies axioms 1, 2, and 4 of R. L. Moore's paper *On the foundations of plane analysis situs*.[1] Hence by Theorem 15 of that paper, the proof of which uses essentially only these axioms, it follows that C is arcwise connected. But, by (β), C is an irreducible continuum between some two of its points a and b; and since C contains an arc from a to b, it is clear that C must be identically this arc.

It is of interest in this connection to note that if C is a subset of a Euclidean space of any number of dimensions, then the condition α may be omitted from the set (α, β, ϵ), i.e., conditions (β, ϵ) characterize an arc in a Euclidean space. For, by (β), C is an irreducible continuum between some two of its points a and b; and since, by (ϵ), C is locally connected, then by a theorem of R. L. Moore's[2] C contains an arc t from a to b. Clearly C must be identical with t. It is also true[3] that, in a Euclidean space, conditions (γ, ϵ) characterize a simple continuous arc.

G. T. WHYBURN

[1] Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–164.

[2] *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 233–236.

[3] Cf. G. T. Whyburn, *Concerning connected and regular point sets*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 685–689. Although the proofs in this paper are worded for the Euclidean plane, it is obvious that they hold in a Euclidean space of any number of dimensions. Also it seems to be true that the results of this paper hold, and hence that conditions (γ, ϵ) characterize an arc, in any metric space which is locally compact.

Analytic Geometry (Revised). By CLYDE E. LOVE. The Macmillan Co., 1927. 257 pages.

In writing a text book, one should present something new, or he should present something old in a new way, or he should present something old in an old way as well or better than it has been presented before. The text under review is one which presents the usual subjects of analytic geometry in the usual way. However it should be added immediately that this usual way has been given new life and vitality by means of the arrangement of topics and well selected exercises, and by means of proper emphasis on the more important ideas. It is a text that will greatly please the majority of mathematics teachers and ought to enjoy a wide popularity. Approximately one third of the text is given to the discussion of coordinates, the straight line and the circle; one third to conic sections; and one third to solid geometry. The amount of space given to conics seems too large but the subject is well treated. The excellent exercises throughout the book are so chosen that the student will be encouraged in analytic methods and self-reliance. It is an excellent text.

R. P. STEPHENS

Analytic Geometry. By T. E. MASON and C. T. HAZARD. Boston, Ginn and Company, 1927. xi+224 pages. Price $2.40.

This is a very good elementary textbook on plane and solid analytic geometry, carefully written, published under the skillful editorial cooperation of Professor R. D. Carmichael, and well printed. It is hard to say with certainty, without having taught the book, but it seems to be a very teachable text. It is not cluttered with unimportant topics of study. The properties of conics are grouped together and not scattered. The eccentricity of conics is treated in detail. The book appears to be one from which it would be easy to select material for a course to suit most teachers. Logic and generality are not sacrificed to make the subject pleasant and easy.

The authors treat the straight line before they take up the general discussion of equation and locus, thus linking up the course with college algebra. The derivation of the general formula for the area of a triangle is given as the last problem in Chapter I, so all the problems on areas of triangles and of other polygons in this chapter are to be solved from the figures and not by a formula. No use is made of determinants in the book. The use of the projection of a line segment is carefully avoided in deriving the normal forms for the equations of straight lines in the plane and of planes in solid analytics.

The problems are numerous, excellent, and well graded as to difficulty, answers being given in the back of the book. Some of the problems would tax the best students. Many illustrative examples are worked out in the text, with most of the details given. This last is a good point, because students usually

will not fill in omitted steps in such problems. Besides many numerical problems there are plenty of a more general type, such as those about properties of geometrical figures.

No treatment is given of oblique coordinates or of the discriminants and invariants of conics and quadric surfaces. Polar coordinates and parametric coordinates are well and fully treated. There is a chapter on empirical equations. Poles and polars are discussed, also diameters. The discussion of locus problems and of curve tracing is quite systematic and not haphazard. In the treatment of these last topics, as well as in several other places in the book, the different steps of a discussion are outlined. Luckily this is not done too frequently, otherwise the student tends to follow these outlines mechanically. The ideas of function, continuity, asymptotes, plus and minus infinity could have been given with more detail and precision and not casually slipped into the text. It was a pleasure to find among the miscellaneous examples such a one as the derivation of the general formula for the centroid of a triangle. The authors show how to put the equation of a straight line in the point-slope form by purely algebraic manipulations. Why did they not do the same for the intercept form, instead of having the student substitute in a formula?

The thirty-three pages given over to solid analytics contain no general discussion of quadric surfaces, no transformation of axes, and just brief sections on cylindrical and spherical coordinates. Otherwise the treatment of lines, planes, direction cosines, and the discussion of surfaces are complete enough for an elementary book. There are not enough topics of an advanced nature either in plane or solid analytics in this book for it to be used beyond a first course.

This book might be said to hark back to the older (shall we say classic) type of analytic geometries like Fine and Thompson's *Coordinate Geometry*. This is quite a relief. For example, no calculus is used in the study of tangents and normals to conics. Also the conics and other curves are studied for their own sake. Although there are good problems from applied mathematics and although practical applications of the subject are pointed out, all these are unobtrusive. This text-book impresses one as a book on geometry and not as primarily an introduction to physics and engineering. A course chosen out of the topics there offered would lay solid foundations for more geometrical study.

ALAN D. CAMPBELL

The Mathematical Theory of Elasticity. Fourth Edition. By A. E. H. LOVE. Cambridge, England, University Press, 1927. xviii+643 pages. Price £2.

The fact that Professor Love's book has been for years the standard treatise on elasticity is in itself enough to arouse keen interest in a fourth edition. Among those who make more than casual reference to the book, the new

edition will excite genuine enthusiasm. For as one gains acquaintance with Love's treatise, the book becomes much more than a formal text of reference; one feels its charm, and partakes somewhat of the life and spirit which the author has breathed into this remarkable work.

As we turn to the new edition, we recall the first edition of 1892, the second edition of 1906, and the third edition of 1920, and ponder on the long period of years spent in the perfecting of the book. We visualize the plan and scope of the work, and appreciate the magnitude and the difficulties of the undertaking. And we note what levels of excellence the author attains in the consummation of his task, and recognize that in this last edition Professor Love has given us a treatise of exceptional finish such as can be produced only by years of painstaking and scholarly effort.

Any appraisal of Love's *Elasticity* must inevitably take into account the history of the subject itself. Since Navier, Cauchy, and Poisson laid the foundations of the mathematical theory, just a hundred years ago, the subject has developed with ever-lengthening strides. The early growth was marked by such comprehensive treatises as those of Lamé (1852), of Clebsch (1862), and of de Saint-Venant (1883), and was summarized in the exhaustive *History* of Todhunter and Pearson (1886–1893). By the 'nineties, researches were multiplying with such rapidity that it took rare courage to contemplate a further work of comprehensive character. Yet Professor Love, catching the inspiration of his teacher Webb, dreamed of a treatise that should present in the compass of a single volume a fair picture of this extensive field in its various aspects. The subject has grown so fast in the last decade, that it now seems impossible that the future should witness another attempt at a treatise as exhaustive as Love's. All the more reason, then, that we should be profoundly grateful to Professor Love for bringing his work to perfection in a fourth edition at this time.

The rapid advance of the subject in recent years prompts mention of a feature of the present edition which may give cause for regret: although the author plans to incorporate, under certain topics, some account of, or reference to, current researches, there appear to be omissions of some importance. But Professor Love can hardly be blamed for shortcomings in this direction. Added to the increased activity, there were the difficulties of tracing the literature published just before, and during, the years 1914–1918. Then, too, a paper of genuine mathematical interest is often effectively buried, nowadays, in some out-of-the-way technical journal. Indeed, it is understandable that one should find it well-nigh impossible to keep abreast of the rapidly expanding literature.

Particularly welcome are certain timely additions, mentions, and revisions which distinguish the present edition from its predecessors. Quoting from the

preface: "The most important additions are (i) a discussion of the theory of a rectangular plate, clamped at the edges, and bent by pressure applied to one face; (ii) a discussion of the theory of the resistance of a plate to pressure, when it is so thin that the extension of the middle plane, due to deformation of that plane into a curved surface, cannot be neglected; (iii) an account of the process by which stress-strain relations are deduced from the molecular theory of a crystalline solid. The most important revision concerns the theory of the equilibrium of a sphere. It has been found possible to simplify very considerably the easier parts of that theory, and thus to lead, by comparatively elementary methods, to the most important geophysical applications of the subject." As in previous editions, Professor Love continues to advance the theory by valuable contributions of his own. Moreover, when recasting the material of another, he invariably succeeds in bringing to the topic originality of treatment and vigor of exposition.

Finally, we are well reminded by this particular treatise that both author and publisher can, if they will, concern themselves with matters of detail which are too often slighted. We have here delightful evidence of painstaking editing, careful paging, faultless typography. As an example of a book of significant content brought to perfection by the combined and unselfish efforts of author and publisher, this indeed is a work that will long reflect credit upon Professor Love and the Cambridge University Press.

CARL A. GARABEDIAN

Applied Elasticity. By J. PRESCOTT. London, Longmans, Green, and Co., 1924. vii+666 pages.

Modern engineering is becoming more and more exacting. Among other things, it demands higher unit stresses, and this in turn demands a more accurate knowledge of the properties of materials and of the laws governing the distribution of the stresses in a body. Accordingly, there is a general tendency to use greater refinement in the experimental work in the laboratory and to use a more rigorous mathematical analysis in the solution of the problems that present themselves.

Not all of the problems in engineering lend themselves to a rigorous mathematical analysis.

Most of the results obtained by the use of the elementary theory of strength of materials are approximations, and it may be of decided interest to use a more rigorous mathematical analysis and derive more exact results. Thus we may fix the limitations that should be placed upon the approximate formulas in use.

The general tendency to put engineering on a more rigorous mathematical basis is illustrated by Prescott's *Applied Elasticity.* There is very little in this

Principia Mathematica. By A. N. WHITEHEAD and B. RUSSELL. Second Edition. Vol. I, 1925, lxvi+674 pages; vol. II, 1927, xxxi+742 pages; vol. III, 1927, vi+491 pages.

In the years since the first edition was published, (vol. I, 1910; II, 1912; III, 1913), the nature of the project undertaken in Principia Mathematica has become sufficiently understood. It is the demonstration that pure mathematics is rigorously deducible from correct definitions of its fundamental concepts alone, without other assumptions except such as are sufficient for a logic which provides an adequate canon of proof in general. For the completion of this project, it was necessary not only to dispense with the usual postulates of the various branches of mathematics but also to analyze the basic concepts, usually taken as undefined ideas, in such manner that these could all be defined solely in terms of the initial concepts of logic. That is, all such notions as ordinal and cardinal number (in general), the ordinal and cardinal numbers, 1, 2, 3, . . . , ω, $\aleph$, etc., the relations $+$, $\times$, etc., the ideas involved in serial order and in the various types of series, in linear, vector, and angular measure, and so on, must all be defined in terms, eventually, of such concepts as "either or", "elementary proposition," "negative of," and "propositional function."

This seemingly impossible task was carried out for the fundamental branches, excepting geometry, in the three volumes of the first edition. The fourth volume, on geometry, has not appeared, and probably is no longer to be expected. Although part VI, in volume III, covers concepts of quantity and measurement applicable in geometrical branches, it remains a matter of conjecture how far and by what procedures geometrical postulates could have been dispensed with in this fourth volume.

In the new edition, the text of the first edition is reprinted without change, except for the correction of misprints and similar errors. Any other procedure would be most difficult, since it would disturb the whole system of backward references in proofs. Accordingly, the desirable alterations are indicated and discussed in an "Introduction to the Second Edition" and in three appendices to volume I.

These alterations fall under two heads: economies of procedure which have been proved possible by work in logic done since the first edition, and discussion of a new and less comprehensive assumption in place of the "axiom of reducibility."

The alterations of the first sort are two: substitution of the Sheffer-Nicod "stroke-function" for other undefined ideas of logic, and elimination of "asserted propositional functions."

In the first edition, the undefined ideas are eight: (1) elementary propositions (those involving no variable or unspecified term) symbolized by p, q, r, . . . , (2) propositional functions (statements involving an undetermined or

variable constituent) symbolized by ϕx, $\psi(x, y)$, etc., (3) assertion of a proposition, (4) assertion of a propositional function, (5) negation of a proposition, p, symbolized by $\sim p$, (6) disjunction, $p \vee q$, meaning exactly "At least one of the two, p and q, is true," (7) the idea "ϕx is true for all values of x," symbolized by $(x).\ \phi x$, (8) "ϕx is true for some values of x," symbolized by $(\exists x).\ \phi x$. All the logical principles necessary for mathematical proof are derived from seven symbolic postulates in terms of the above, together with certain postulates (which are really principles of logical operation) which cannot be symbolized.

Professor H. M. Sheffer showed[1] that the two undefined ideas, negation and disjunction, can be replaced by one, p/q, meaning "p and q are, one or both, false." (Sheffer, Nicod, and Russell all read this "p and q are incompatible," which is misleading, since obviously two propositions which are both false, or one true and one false, may be compatible or consistent in the ordinary sense. This relation is to "material implication," on which the logic of Principia Mathematica is based, as the ordinary meaning of inconsistency is to the "implies" of ordinary inference. If we represent "p materially implies q" by $p \supset q$ and the ordinary meaning of "p implies q" by $p < q$, then

$$(p \supset q) = (p/\sim q) \quad \text{and} \quad (p < q) = \sim (p_0 \sim q),$$

where $p_0 q$ represents the usual meaning of "p is consistent with q.")

In terms of this stroke-function, the negation of p is definable as p/p, and the disjunction of p and q as $(p/p)/(q/q)$. Following this paper of Sheffer's, Jean Nicod[2] reduced five of the symbolic postulates of *Principia Mathematica* to one,

$$[p/(q/r)]/\{[t/(t/t)]/[(s/q)/((p/s)/(p/s))]\}.$$

The rather startling result is the proof that all the propositions of mathematics (unless we must except geometry) can be derived from three symbolic postulates in terms of the idea p/q.

The second economy of procedure results from the recognition that the idea of "asserted propositional function" is unnecessary. In the first edition, a theorem such as $p \supset \sim q.\ \supset .q \supset \sim p$, "If p implies that q is false, then q implies that p is false," is regarded as a propositional function, since p and q are undetermined or variable. Now propositional functions cannot in general be asserted, being neither true nor false (*e.g.*, "x is a rational fraction" is neither true nor false until the variable x is replaced by some one of its "values"). But certain propositional functions, such as this theorem, *can* be asserted: they are true *formulas*. However, in all such cases the function is assertable only because it holds for all values of its variables. Hence any statement of the form

[1] Transactions of the American Mathematical Society, vol. 16, pp. 481–488.

[2] Proceedings of the Cambridge Philosophical Society, vol. 19, pp. 32–41.

$$p \supset \sim q . \supset . q \supset \sim p$$

is equivalent to another, of the form

$$(p) : . (q) : p \supset \sim q . \supset . q \supset \sim p,$$

"for all values of p and all values of q, if p implies that q is false, then q implies that p is false." This last statement is not a propositional function but a proposition, of the general type $(x) \cdot \phi x$. Thus all "asserted propositional functions" are in reality propositions, and any theorem of the first edition, having the form of a function, should be regarded as proposition asserting this function to hold for all values of its variables.

The recognition of this fact works another improvement, which the book does not note. Since the idea "negation of p" is symbolized by $\sim p$, we ought to be able to express the fact that a deduction-formula (let us say, $p . \supset . p \supset q$) is false, by its symbolic negation, $\sim(p . \supset . p \supset q)$. But this is not possible, because $p . \supset . p \supset q$ covers the special case $p . \supset . p \supset p$, which is true. Hence we can assert neither $p . \supset . p \supset q$ nor its symbolic contradictory; which seems incompatible with the principle that one of every pair of contradictory assertions must be true. When $p . \supset . p \supset q$ becomes $(p) : . (q) : p . \supset . p \supset q$, this difficulty disappears, because its contradictory then is

$$(\exists p) : . (\exists q) : \sim (p . \supset . p \supset q),$$

"for some values of p and some values of q, $p . \supset . p \supset q$ is false." This last is, of course, true.

The emendation in the above fashion of the logic of elementary propositions, contained in sections *1 to *5 of the first edition, is very simple. It is merely sketched in the new Introduction. The logic of propositional functions, in terms of the stroke-function, is more involved and also more fundamental. The principles of it, sufficient for all later proofs, are developed in a new section *8, to replace section *9 of the first edition. This new section is printed as Appendix A.

The second main novelty of the new edition—limitation of the use of the "axiom of reducibility"—touches one of the most recondite features of Principia Mathematica. In ordinary developments of mathematics no need is felt for the axiom of reducibility or any substitute for it, because the existence of classes in general is taken for granted if defining properties of the class can be specified. (Existence of a class does not here mean existence of members of the class: null-classes may "exist.") But the authors desired to avoid such assumption, both because it seemed theoretically doubtful and because it contributed to "vicious-circle fallacies," which are encountered in the theory of assemblages. These are such as Burali-Forti's paradox about the ordinal number of all ordinals, and the paradox of the "least indefinable (transfinite) ordinal".

Accordingly, propositions which would ordinarily be interpreted as being about some class are, in Principia Mathematica, treated as statements about some propositional function which holds for every member of the class. Thus if ϕx be "x is an ordinal number," any proposition about the class of ordinals is rendered by some statement $f(\phi x)$ about the propositional function ϕx. Moreover, the "theory of types," adopted to avoid the vicious-circle fallacies, restricts the significance of any propositional function to some one type—to individuals, or classes of individuals, or classes of such classes and so on. The result of these two limitations—the absence of the assumption that classes exist, and the theory of types—is to render it difficult to prove anything about certain legitimate totalities such as all the classes to which a given term belongs, or all the properties of a term. For instance, "All those properties which belong to 1, and are such that if they belong to n then they belong to $n+1$, are properties of every finite number." The axiom of reducibility was assumed to obviate such difficulties. It has two cases (ed. 1, vol. I, p. 174):

$$*12\cdot 1 \vdash :(\exists f):\phi x.\equiv_{x}.f!x \qquad \text{and} \qquad *12\cdot 11 \vdash :(\exists f):\phi(x,y).\equiv_{x,y}.f!(x,y).$$

These differ only in that $*12\cdot 1$ is for functions of one variable, $*12\cdot 11$ for functions of two. The first may be read, "For any propositional function, ϕx, there is some predicative function, $f!x$, which is always true when ϕx is true, and false when ϕx is false." A "predicative function," as nearly as can be briefly put, is a single predicate; while the predicate ϕ in the axiom may be complex. Thus what the axiom assumes is that for any compound or complex statement about a term, x, there is some single predicate which holds when the complex statement is true and fails when it is false. This allows such statements as would ordinarily be made by reference to some totality of classes to which a term belongs (or the totality of properties of the term) without allowing the notion of any totality of propositional functions which the term satisfies. This last kind of totality is what leads to the vicious-circle fallacies, and is forbidden by the theory of types.

However, this axiom of reducibility is dubious. Hence the new edition proposes for consideration the substitute assumption that "A [propositional] function can only enter into a proposition through its values" (ed. 2, Appendix C, p. 659). This substitution requires the rewriting of many proofs, particularly those concerning Frege's "ancestral relation" by which the procedure of mathematical induction is shown to be completely deductive. But all the theorems essential to mathematical induction can still be demonstrated; the revision of these proofs is carried out in Appendix B. However, a similar revision for theorems concerning the real numbers and well-ordered series in general, is not found possible. "There is, ... so far as we can discover, no way by which our present primitive propositions can be made adequate to Dedekindian and well-

ordered relations. It is upon this that the theory of real numbers rests, real numbers being defined as segments of the series of rationals. If we were to regard as doubtful the proposition that the series of real numbers is Dedekindian, analysis would collapse." (Ed. 2, vol. I, p. xliv).

The real purpose both of the axiom of reducibility and of the proposed substitute is to be able to treat all the propositional functions which figure in mathematics in a manner which would be valid without further assumption if it were established that all such functions can be analyzed into purely logical relations of their ultimate constituents. The difficulty of establishing this, in Principia Mathematica, is due, I believe, in part at least, to the preconception that all logical functions are truth-functions—that is, such that their truth or falsity depends solely on the truth or falsity of their terms. (See Appendix C, *passim*) All the functions of logic are, by the method of Principia Mathematica, treated as such truth-functions. For example, the material implication, $p \supset q$, can be defined by the accompanying table, where + represents "true" and − "false." That is, "p materially implies q" fails when p is true and q false, and otherwise holds.

p	q	$p \supset q$
+	+	+
−	+	+
+	−	−
−	−	+

The conception that all mathematics is concerned only with such truth-functions, leads to difficulty, because a truth which depends, in any part, upon the *form* of propositions can *not* be determined in this fashion merely from the truth or falsity of its constituents; while if differences of form are not significant in mathematics, all equivalent propositions will be identical. If, further, logical equivalence means only equivalence of truth-value, this view leads inevitably to the conclusion that there are really only two propositions, one true and one false; and all mathematics will thus collapse into an immense tautology. Essentially this conclusion was accepted by Frege, and has been renewed by Wittgenstein: Mr. Russell[1] seems to draw back from it without being willing to abandon the premises which lead to it.

As a fact, I take it, form is of the essence of both logic and mathematics. Logical truth, such as what is ordinarily meant by "q is deducible from p," depends in part upon the form of p and q, and cannot be determined from their truth or falsity alone. If I should be correct in this, it is not a matter for surprise if a procedure which depends upon treating all mathematical assertions as logical functions of their constituents, while at the same time holding logical functions to be exclusively truth-functions, should stop short of complete success. Actually, in spite of the fact that the relations assumed as primitive are all truth-functions, the method of this work is largely dependent upon considerations of

[1] I refer to Mr. Russell because he is alone responsible for the revisions of this edition, and because I do not think Mr. Whitehead would subscribe to the content of this Appendix C.

form, and is, I believe, inconsistent with the theoretical conceptions of the nature of mathematics which are expressed in it.

In volume II, the prefatory statement of symbolic conventions is considerably expanded. In the remainder of the volume, and in volume III, I do not find any material alterations.

C. I. LEWIS

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activites should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE SIMPSON MATHEMATICS CLUB, University of Florida, Gainesville, Florida.

The Simpson Mathematics Club held seven meetings in the spring of 1927. The first meeting was taken up with the election of officers and the appointment of committees. The other meetings were as follows:

February 3. "Squaring the circle" by Mr. Brodmerkle. "Geometric properties of the parabola" by Mr. Rosser.

February 17. "History of logarithms" by Mr. Quade. "Geometric properties of the ellipse" by Mr. Rosser.

March 3. "A solution of the cubic equation" by Prof. Chandler. "Geometric properties of the hyperbola" by Mr. Rosser.

March 17 and March 31. "The transcendence of e" by Prof. Kusner.

April 14. "Mathematical logic" by Mr. Brodmerkle.

It was customary to have an open forum at the end of each meeting in which members were invited to put up interesting problems for discussion and solution.

(Report by Mr. Barkley Rosser)

THE TULANE MATHEMATICS CLUB, Tulane University, New Orleans, La.

The officers of the club for the year 1926–1927 were: Professor H. E. Buchanan, president; Mrs. H. L. Titsworth, secretary-treasurer; Professor H. L. Menuet, chairman of executive board; Mrs. W. G. Allee, Professor E. S. Kalin, members of executive board.

The program for the year 1926–1927 included the following papers:

December 2, 1926. "Problem analysis" by Miss Carrie Breehan. "The laboratory method of teaching arithmetic" by Miss Cecelia Brimna. "How we subtract" by Miss Adrienne Rayl.

February 17, 1927. "The shifting emphasis in the teaching of algebra" by Professor E. S. Calin.

May 12. "The early history of trigonometry" by Professor H. E. Buchanan.

(Report by Mrs. H. L. Titsworth)

THE ALBION MATHEMATICS CLUB, Albion College, Albion, Mich.

The following programs were given at the meetings of the club during the year 1926–1927:

October 19, 1926. "Sidelights on mathematics" by Mary Alice Parsons ('27).

November 2. "Influence of periodicals and associations on mathematics" by Dorothy Schliskey ('27).

November 16. "Transcendentals and higher degree equations" by Charles Jacobes ('27).

December 7. "Comparison of mathematics in the United States before 1800 and now" by Dorothy Kennedy ('28).

January 4, 1927. "The origin and existence of hyperspace" by Francis Moore ('28).

February 1. Election of officers. "Relation of mathematics to architecture" by Harold Wilson ('28).
February 15. "Heresy and orthodoxy in geometry" by Thelma Raidle ('29). "Fallacies" by Hugh Sebastian ('29).
March 1. "Numerical magic squares" by Raymond W. Bowers ('29).
March 15. "The development of our system of notation" by Professor Sleight.
April 5. "The Chinese calculator" by Lorraine Traut ('29). "The slide rule" by Mr. Hickox.
April 12. "Recreations in arithmetic" by Laurence Wylie ('29). "Recreations in algebra" by Luna Bachelor ('29).
May 18. Annual out-door party, and election of officers. (Report by Mr. R. W. Bowers)

THE WELLESLEY MATHEMATICS CLUB, Wellesley College, Wellesley, Mass.

The following programs were given in the year 1926–1927:
October 22, 1926. Discussion of books on the mathematics browsing shelf. "The philosophy of mathematics" by Helen Sawin. "Simplified theory, statistics, and mathematics in other lands" by Frances Hartman. "Recreation in mathematics" by Isabel MacKerracher.
November 19. Anniversary Topics: "Eratosthenes" by Blanche Weatherhead; "Edmund Gunter and Albert Girard" by Janet Durand; "Wilbur Snell" by Dorothy Beaton; "Early mathematics in American colleges" by Mary Parsons.
January 7, 1927. A study of the old mathematics books in the treasure room of the college library, conducted by Miss Lennie P. Copeland.
February 25. Problems of antiquity and their solution: "Geometric representation of the roots" by Elizabeth Mitchell; "Trisection of an Angle" by Frances Baume; "Duplication of a Cube" by Elizabeth Peek; "Spirals" by Doris Miller.
March 11. "The geometry of imaginaries" by Dr. William C. Graustein of Harvard University.
May 28. Social meeting and election of officers for the year 1927–1928: President, Leona Bayly; vice-president, Elizabeth Peek; treasurer, Mary Parsons; secretary, Doris Raine; Junior executive member, Esther Kirkbride; faculty advisor, Miss Marion Stark.
(Report by Miss Frances Hartman)

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I.

The following is the program for 1927–28:
November 1, 1927. "Omar Khayyám" by Leslie Thomas Fagan, Gr. "Constructions with a parallel ruler," by Eleanor Margerum, '29. "Musical scales" by Herbert Anthony Howard, '28.
November 29. "Geometrography" by Peter Shahdan, '30. "Cypher writing" by Ethel McKechnie, '28.
January 10, 1928. "Mathematics and business" by Theodore Henry Brown, Associate Professor of Business Statistics, Harvard University.
February 21. "Porisms and dimensional units" by Albert Arnold Bennett, Professor of Mathematics, Brown University.
March 27. "Sylvester" by Louise Henderson Woodman, '29. "Units of angular measurement" by Aubrey Henderson Smith, Gr. "Curiosities in numbers" by Silvio Carosella, '29.
April 24. "Linkages" by Allan Francis Nickerson, '30. "Probabilities in the game of shooting craps" by Mary Honor Cummings, '29.
May (*Date and place to be announced later*). Picnic.
Committee on Program: Professors Archibald and Bennett; Herbert Anthony Howard, '28; Ida Allenson Noble, '28; Homer Pine Smith, '29; Louise Henderson Woodman, '29.
Committee on Arrangements: Aubrey Wilfred Landers, Jr., Gr.; Silvio Carosella, '29; Elinor Margerum, '29; Allan Francis Nickerson, '30; Julia Ayer Oldham, '28.
(Report by Professor R. C. Archibald)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, Eugene, Oregon.

The program for the year of 1926–27 was the following:
October 28, 1926. "The calculus of variations" by Assistant Professor Davis.

November 18. "The life and works of Newton" by Hubert Yearian, '27.
December 16. Social meeting.
January 20, 1927. "The smoothing of curves" by Professor Milne.
February 17. "The many uses of graphs" by Gladys Beular, '26.
April 21. "Line coordinates and point equations" by Assistant Professor Bunch. Eight new members were elected.
May 12. Mathematics club picnic.
May 26. Election of officers for the next year: President, Edna English; vice-president, Robert Jackson; secretary, Beatrice Mason; Treasurer, Eric Peterson.

(Report by Miss Mason)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3315. Proposed by Harry Langman, Arverne, L. I.

Show that

$$x^2+y^2+z^2-yz-zx-xy+u^2+v^2+w^2-vw-wu-uv \geqq 3^{1/2}\begin{vmatrix} u & x & 1 \\ v & y & 1 \\ w & z & 1 \end{vmatrix}$$

for all real values of the variables. Also find the conditions under which the equality obtains.

3316. Proposed by Wm. B. Campbell, Colgate University.

A right circular cylinder with a base of radius r and with an altitude $2a$, closed top and bottom, contains a fixed volume of liquid, $\pi r^2 h$. It is rotated about a horizontal axis perpendicular to the axis of the cylinder at its center. It is required to determine the maximum torque and the maximum displacement of the centroid of the liquid.

3317. Proposed by Emma M. Gibson, High School, Springfield, Mo.

The coordinates of a point are expressed in terms of two parameters, α, β, by the formulae

$$\frac{x}{a}=\frac{(\alpha+\beta)}{(\alpha-\beta)}, \quad \frac{y}{b}=\frac{(1-\alpha\beta)}{(\alpha-\beta)}, \quad \frac{z}{c}=\frac{(1+\alpha\beta)}{(\alpha-\beta)},$$

where a, b, and c are constants.

Determine the locus of the point and determine the relations of the lines $\alpha=$constant and $\beta=$constant, to the surface Also show how to express the differential equation of the lines of curvature in terms of α and β.

3318. Proposed by C. N. Mills, Normal, Ill.

For the parabola $y^2=-2p(x-k)$ find the length (in terms of p and k) of the shortest segment of a line tangent to the curve included between the coordinate axes.

UNSOLVED PROBLEMS

Readers are requested to send in solutions for the following unsolved problems which were proposed in the years 1924, 1925, 1926. The number of each problem is printed in bold face type, with the page number following.

1924

3056, 101; **3057,** 101; **3060,** 101; **3073,** 206; **3075,** 206; **3076,** 254; **3077,** 254; **3082,** 305; **3085,** 305; **3087,** 306; **3089,** 353; **3094,** 353; **3097,** 402; **3099,** 455; **3106,** 499.

1925

3111, 46; **3112,** 46; **3119,** 95; **3124,** 138; **3126,** 204; **3127,** 204; **3133,** 261; **3140,** 315; **3141,** 315; **3144,** 385; **3147,** 385; **3148** (3144), 433; **3157** (3153), 520; **3158** (3154), 520.

1926

3162 (3158), 47; **3163** (3159), 47; **3167** (3163), 47; **3169** (3165), 104; **3172** (3168), 104; **3174** (3170), 104; **3180** (3176), 159; **3183** (3179), 159; **3184** (3180), 159; **3185** (3173), 228; **3186** (3174), 228; **3189** (3177), 228; **3196** (3184), 278; **3204** (3192), 338; **3209,** 385; **3229,** 525; **3231,** 525.

SOLUTIONS

3231 [1926, 525]. Proposed by R. B. Stone, Purdue University.

The root-mean-square (*R. M. S.*) of n numbers $x_1, x_2, \cdots, x_n$ is defined by the formula

$$R.M.S.=\left(\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}\right)^{1/2}.$$

(*a*) For what values of n is the *R. M. S.* of the first n integers also an integer?

(*b*) For what values of a and n is the *R. M. S.* of n successive integers

$$a,\ a+1,\ \cdots a+n-1$$

also an integer?

(*c*) Under what conditions is the *R. M. S.* of n integers also an integer?

SOLUTION BY ELIJAH SWIFT, University of Vermont.

(*a*) Using the well known formula for the sum of the squares of the first n integers and setting *R. M. S.*$=k$, an integer, we have

$$(n+1)(2n+1)=6k^2. \tag{1}$$

Here n must be an odd integer; and, if we set $n=6\alpha+\beta$, $\beta=1, 3, -1$, we find by trial $\beta=1$ or -1. Consider first, case (*A*), where $n=6\alpha-1$. Here $\alpha(12\alpha-1)=k^2$, and, since the two factors on the left are prime to each other, $\alpha=l^2$ and $12\alpha-1=m^2$. Hence $12l^2=m^2+1$; but this is impossible, since the square of an integer is either a multiple of 4 or such a multiple plus unity. Consider next case (*B*), $n=6\alpha+1$. Here $(3\alpha+1)(4\alpha+1)=k^2$, and the factors on the left are again prime to each other. Hence $3\alpha+1=l^2$, $4\alpha+1=m^2$, and $(2l)^2-3m^2=1$. This last equation may be solved by well known methods, and from the smallest solution all others may be derived. (See, for example, Carmichael's *Diophantine Analysis*, p. 26 ff). The three smallest solutions are

$$\begin{aligned} &l=1,\ 13,\ 181; \quad n=1,\ 337,\ 65521,\\ &m=1,\ 15,\ 209; \quad k=1,\ 13\times 15,\ 181\times 209. \end{aligned} \tag{2}$$

(*b*) Subtracting the sum of the squares of the first $a-1$ integers from that of the first $a+n-1$, dividing this difference by n and reducing the results, we find

$$2n^2 + (6a - 3)n + 6a^2 - 6a + 1 = 6k^2, \quad k \text{ being an integer.} \tag{3}$$

As before we find (*A*) $n=6\alpha+1$, (*B*) $n=6\alpha-1$. For (*A*) the above equation becomes

$$3\alpha^2 + \alpha = k^2 - l^2, \quad l = a + 3\alpha. \tag{4}$$

Since $k^2-l^2\equiv 0, 1$, or 3 (mod. 4), we must have $\alpha\equiv 0$ or 1 (mod. 4). In either case the left side of (4) is divisible by 4, and k and l are both odd or both even. Let $k-l=2m$; then (4) becomes

$$\alpha(3\alpha + 1) = 4m(l + m). \tag{5}$$

We are now in a position to find as many solutions as we desire. For, take any admissible value for α, compute the left side of (5), take for m any factor of one-fourth of the result, and we can then find l, k and a. We note that $l>3\alpha$ from (4). Any admissible value of α as large as 5 will give at least one solution and may give several. Below is given a table of a few results.

In case (*B*) we have

$$3\alpha^2 - \alpha = k^2 - l^2, \quad l = a - 1 + 3\alpha \tag{6}$$

by similar methods; and here $\alpha\equiv 0$ or 3 (mod. 4). Hence, if again we set $k-l=2m$, we have

$$\alpha(3\alpha - 1) = 4m(l + m). \tag{7}$$

(*A*)	$n =$	1,	31,	49,	55,	73,	79,	79,	97,	97, $\cdots$
	$a =$	*	4,	25,	35,	74,	90,	24,	48,	147, $\cdots$
(*B*)	$n =$	41,	47,	65,	65,	71,	89,	89,	95,	95, $\cdots$
	$a =$	14,	22,	10,	55,	69,	8,	120,	45,	140, $\cdots$

(*c*) Regarding this case apparently nothing can be said beyond the trivial remark that the sum of the squares of the n integers must be n times the square of an integer.

Also solved by H. L. Olson, F. L. Wilmer, and the Proposer.

3257 [1927, 217]. Proposed by Albert A. Bennett, Lehigh University.

Prove that if V is any real function, of three variables, x, y, z, satisfying the condition that $(\partial^2V/\partial x^2)+(\partial^2V/\partial y^2)+(\partial^2V/\partial z^2)=0$ and expressible as a term-wise differentiable power series, and if the magnitude of the vector $(\partial V/\partial x, \partial V/\partial y, \partial V/\partial z)$ is constant, say equal to unity, then the direction of this vector is also constant. Conversely, if the direction is constant, the magnitude is constant. Note that the restriction to real functions is essential for the first part of the problem but extraneous for the second part.

This is stated without proof as apparently obvious in an old physics book.

Solution by the Proposer.

Given V, a real function expressible as a power series in x, y, z, and differentiable term by term. To prove that if

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = 1 \text{ and also } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

then V is a linear function.

Proof. Express V as a series as follows: $V=P_0+P_1+P_2+\cdots+P_n+\cdots$, where P_n is a homogeneous polynomial of degree n, in x, y, z. Transform by an orthogonal transformation so that $P_0=0$, $P_1=x$. Let partial derivatives be denoted by subscripts.

Assume for induction for some $n>2$, that $P_i\equiv 0$, for each i, $1<i<n$. This is vacuously satisfied for $n=2$. Then equating coefficients of terms of like order in the expansion of $(\partial V/\partial x)^2+(\partial V/\partial y)^2+(\partial V/\partial z)^2=1$, we have

* Arbitrary.

$$P_{jx} \equiv 0, \quad n-1 < j < 2n-1; \qquad 2P_{2n-1\,x} + P_{ny}^2 + P_{nz}^2 \equiv 0,$$

and hence, using the second condition of the hypothesis, $P_{n\,yy}+P_{n\,zz}\equiv 0$. This in turn implies that P_n is the real part of an analytic function of $u=y+zi$, and since it is homogeneous of nth degree, the function must be $(a_0+a_1i)u^n$ where a_0, a_1 are real.

Thus $P_n + iQ_n = (a_0 + a_1i)u^n$, where $P_{ny} = Q_{nz}$, $P_{nz} = -Q_{ny}$. $\therefore P_{ny} - iP_{nz} = (a_0 + a_1i)nu^{n-1}$.

Also $P_{ny} + iP_{nz} = (a_0 - a_1i)n\bar{u}^{n-1}$, where $\bar{u} = y - zi$.

$$\therefore P_{ny}^2 + P_{nz}^2 = (a_0^2 + a_1^2)n^2(u\bar{u})^{n-1} = (a_0^2 + a_1^2)n^2(y^2 + z^2)^{n-1}.$$

Then $-2P_{2n-1}=n^2(a_0^2+a_1^2)\,(y^2+z^2)^{n-1}x+F(y,z)$. Now $P_{2n-1\,xx}+P_{2n-1\,yy}+P_{2n-1\,zz}\equiv 0$. But $P_{2n-1\,xx}\equiv 0$, and $P_{2n-1\,yy}+P_{2n-1\,zz}$ cannot vanish identically unless $a_0^2+a_1^2=0$. Since a_0 and a_1 are assumed real, $a_0=0$, and $a_1=0$. Hence $P_{2n-1\,x}\equiv 0$. This in turn implies that $P_{ny}^2+P_{nz}^2\equiv 0$, or that P_{nx}, P_{ny}, P_{nz} all vanish identically and $P_n\equiv 0$. Since the above conclusions apply also for $n=2$, the induction is complete. The transformed expression for V is merely x itself, and the original V was linear.

The converse theorem, that if $\partial V/\partial x:\partial V/\partial y:\partial V/\partial z=a:b:c$ where a, b, c are constants, and if further $\partial^2V/\partial x^2+\partial^2V/\partial y^2+\partial^2V/\partial z^2=0$, then $(\partial V/\partial x)^2+(\partial V/\partial y)^2+(\partial V/\partial z)^2$ is constant, is readily proved. Indeed, an orthogonal transformation may be used to make the continued ratio, of the form, 1:0:0. If then $\partial V/\partial y\equiv 0$, and $\partial V/\partial z\equiv 0$, V is of the form $f(x)$. Since further, $(\partial^2V/\partial x^2)+(\partial^2V/\partial y)^2+(\partial^2V/\partial z^2)=0$, it follows that this V is of the form $a+bx$, and the original function was linear in x, y, z, which serves to prove the theorem.

3258 [1927, 272]. Proposed by H. E. Arnold, Wesleyan University, Middletown, Connecticut.

The perpendiculars dropped from the vertices of a triangle upon the lines joining the feet of the angle bisectors to the orthocenter of the triangle meet the respective sides of the triangle in three collinear points. The line joining these three points is perpendicular to the line joining the orthocenter to the center of the inscribed circle.

Solution by N. A. Court, University of Oklahoma.

The proposition may be generalized as follows:

Let P be any point in the plane of a triangle ABC, and L, M, N, the traces of the lines APL, BPM, CPN, on the sides BC, CA, AB. The perpendiculars dropped from the vertices A, B, C, respectively, upon the lines joining the orthocenter H of ABC to the points L, M, N, meet the opposite corresponding sides of the triangle in three collinear points. The line joining these three points is perpendicular to the line PH.

Proof. The orthocenter H of ABC is the center of the circle (H) for which the triangle ABC is self-polar. Hence the perpendicular from A upon HL is the polar of L with respect to (H), and the trace L' of this polar on BC is the pole of AL with respect to the same circle. Similarly for the poles M', N' of the lines BM, CN. Now since the lines AL, BM, CN, are concurrent in P, by assumption, their poles L', M', N', are collinear, and the line $L'M'N'$ is the polar of P with respect to (H), and is therefore perpendicular to PH.

In the proposed problem the point P coincides with the incenter of the triangle ABC. Any number of similar propositions may be obtained by making P coincide with other remarkable points of the triangle. For instance, if P coincides with the centroid of the triangle ABC, the line $L'M'N'$ will coincide with the orthic axis of ABC (i.e., the axis of homology of ABC and its orthic triangle). This gives a proof of 3228 [1926, 525].

From the proposition proved above we may derive several other propositions. For lack of space we shall state only one of them:

The perpendiculars dropped from the orthocenter of a triangle upon the lines joining the vertices of the triangle to the traces of a transversal on the opposite sides will meet these sides in three points such that the lines projecting them from the vertices are concurrent.

Here again we may obtain any number of special cases by making the transversal coincide with some known line. For instance, we know that the external bisectors of the angles meet the sides of the triangle in three collinear points, hence the proposition:

The lines joining the vertices of a triangle to the traces, on the opposite sides, of the perpendiculars dropped from the orthocenter upon the external bisectors of the angles are concurrent.

Also solved by PAUL WERNICKE and the PROPOSER.

NOTE BY OTTO DUNKEL, Washington University.

This brief and interesting proof may at first sight appear to suffer from the fact that there is no real circle (H) for which ABC is self-polar if the triangle has only acute angles, and that it is not desirable in a geometrical proof to employ an imaginary circle. However, by means of a slight change for this case the form of the proof remains the same. Denote by $[H]$ the circle with center H and radius $r=(AH\cdot HH_a)^{1/2}=(BH\cdot HH_b)^{1/2}=(CH\cdot HH_c)^{1/2}$, where H_a is the foot of the altitude AH. If P is any point and if p' is its polar with respect to $[H]$, we shall say that the straight line p is the polar image of P if p and p' are symmetrically placed with respect to the center H. We may also say conversely that if p is any straight line P is its pole image. If Q is any point on p then it easily follows that its polar image q passes through P. It is obvious that the line joining any point P with the center H is perpendicular to its polar image p. The polar image of each vertex of ABC is the opposite side of the triangle. Hence with this circle $[H]$ the proof follows as above replacing the terms pole and polar by pole image and polar image.

3259 [1927, 272]. Proposed by Norman Anning, University of Michigan.

Solve the cyclic set of simultaneous equations,

$$x_1 - 2x_n = x_n^2 x_1; \qquad x_{j+1} - 2x_j = x_i^2 x_{j+1},$$

where n is any positive integer greater than unity and $j=1, 2, 3, \cdots, (n-1)$.

SOLUTION BY PAUL WERNICKE, Washington, D. C.

The cyclic set of equations

$$x_{j+1} = 2x_j/(1 - x_j^2), \text{ or } x_{j+1} = 2/(1/x_j - x_j), \text{ where } x_k = x_j \text{ whenever } k \equiv j \pmod{n},$$

has for any n the obvious solutions: all x_j equal 0, i, or $-i$.

By successive substitution of $2x_j/(1-x_j^2)$ for x_{j+1} etc. until only $x_{j+n}=x_j$ appears, we obtain a resulting equation of the (2^n+1)th degree. There must be the factor $x_j(x_j^2+1)$.

A goniometric solution may be obtained by putting $x_j=\tan\phi$, whence $x_{j+1}=\tan 2\phi, \cdots, x_{j+n}=x_j=\tan 2^n\phi$; whence $\phi=k\pi/(2^n-1)$.

The 2^n-1 real values of $\tan\phi$ for $k=0, 1, \cdots, (2^n-2)$ (repeating the value already known, $x_j=0$) together with $\pm i$ constitute the complete solution of our equation of degree 2^n+1.

Also solved by the PROPOSER.

3262 [1927, 272]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through two given points, real or conjugate imaginary, to draw a circle so that the ends of one of its diameters shall lie on two given lines.

SOLUTION BY OTTO DUNKEL, Washington University.

Let the two given points be A and B. The system of circles through these two points has the line AB as its radical axis, and each circle (C) has its center C on a line CM perpendicular to AB at M, and it cuts orthogonally a given fixed circle (O) with its center O on AB. Hence the points A and B are defined by such a system of circles (C) with their centers on a given line CM and each circle of the system orthogonal to a given circle (O). If a circle of the system does not cut OM, the two points A and B are conjugate imaginaries on OM.

If the two given lines, l and m, are each parallel to OM and cut CM at D and E, the center C of the required circle is the mid-point of DE. The circle (C) is then easily constructed, except for certain positions of D and E where there is no real solution.

We now consider the case where one line l is not parallel to OM but cuts it in L. Let a circle of the system cut l in R and S. Then these two points are pairs of points of the involution on l determined

by the system of circles, and therefore $S \perp R$. Draw the line l' symmetric to l with respect to MC and let it cut (C) in R' and S' and OM in L'. Thus R' and S' on l' are symmetric to R and S. Let the other end of the diameter RC be P. Then PR' is perpendicular to OM, and PS is perpendicular to l. Also PR' and PS are corresponding elements of two projective pencils with their centers at infinity. For $R' \perp R \perp S$ and $R'P \perp R'R \perp SP$. Hence $R'P$ and SP cut two projective ranges of points U and V on m, and the self-corresponding points of the two ranges determine the other end of the diameter, say $\overline{U}$, on m. Then $\overline{S}$ and $\overline{R}$ are determined on l and the required diameter is $\overline{U}\overline{R}$. To determine $\overline{U}$ we may construct any three points U_1, U_2, U_3 on m and their mates V_1, V_2, V_3; then the usual construction described in the Note to the solution of 3240 [1928, 46] determines $\overline{U}$. As explained in the note there may be two solutions, one solution or no solution.

The locus of P is evidently an hyperbola with asymptotes perpendicular to l and to OM. When R is at L, R' is at L', S and P are at infinity, and therefore the perpendicular to OM at L' is one asymptote. When S is at L, R and P are at infinity, and the perpendicular to l at L is the other asymptote.

In the case where l is parallel to OM the hyperbola reduces to a parabola.

Also solved by PAUL WERNICKE.

3264 [1927, 335]. Proposed by L. S. Shively, Mount Morris College.

A square is subdivided into n^2 equal squares by drawing lines parallel to its sides, and all the diagonals of the squares thus formed are drawn. How many triangles are there in the configuration?

SOLUTION BY ENRIQUE LINARES, JR., Panama.

On inspection it is found that the number of triangles which have their hypothenuses parallel to the sides of the square, and which are smallest in size, is $4n^2$. The number of those next in size is $4(n-1)n$; those next, $4(n-2)(n-1)$; then $4(n-3)(n-1)$; and so on.

The number of triangles having their hypothenuses parallel to the diagonal of the square, and smallest in size, is $4n^2$. The number of those next in size is $4(n-1)^2$; those next, $4(n-2)^2$ and so on. Hence

$$N = 4[n^2 + (n-1)n \cdots + (n-2a)(n-a) + (n-2a-1)(n-a) \cdots + n^2 + (n-1)^2 \cdots + b^2 \cdots + 1];$$

$$N = 4\Big[\sum_{a=0}^{a=A}(n-2a)(n-a) + \sum_{a=0}^{a=A}(n-2a-1)(n-a) + \sum_{b=0}^{b=n} b^2\Big],$$

where $A = n/2 - 1$ if n is even, and $A = (n-1)/2$ if n is odd;

$$N = 4\sum_{a=0}^{a=A}(2n^2 - n) - 4\sum_{a=0}^{a=A}(6n-1)a + 4\sum_{a=0}^{a=A}4a^2 + 4\frac{n}{6}(n+1)(2n+1).$$

If n is even,
$$N = 4\Big[(2n^2 - n)\frac{n}{2} - (6n-1)\frac{n}{4}\Big(\frac{n}{2} - 1\Big) + \frac{4}{6}\Big(\frac{n}{2} - 1\Big)\frac{n}{2}(n-1) + \frac{n}{6}(n+1)(2n+1)\Big].$$

If n is odd,
$$N = 4\Big[(2n^2 - n)\Big(\frac{n+1}{2}\Big) - (6n-1)\Big(\frac{n+1}{4}\Big)\Big(\frac{n-1}{2}\Big) + \frac{4}{6}\Big(\frac{n-1}{2}\Big)\Big(\frac{n+1}{2}\Big)n + \frac{n}{6}(n+1)(2n+1)\Big].$$

On expanding and collecting terms,

$$N = 3n^3 + \frac{9}{2}n^2 + n - \frac{1}{2} \text{ when } n \text{ is odd; and } N = 3n^3 + \frac{9}{2}n^2 + n \text{ when } n \text{ is even.}$$

These two results give the single formula,

$$N = 3n^3 + \frac{9n^2}{2} + n - \frac{1}{4}[1 + (-1)^{n+1}].$$

Also solved by MICHAEL GOLDBERG, EDWARD S. SMITH, PAUL WERNICKE, and the PROPOSER.

3267 [1927, 335]. Proposed by H. A. DoBell, Colgate University.

In a given triangle the line joining the ninepoint center and the orthocenter of the pedal triangle is parallel to the Euler line of the tangential triangle.

SOLUTION BY N. A. COURT, University of Oklahoma.

The sides of the pedal, or better, the orthic triangle (i.e., the triangle formed by the feet of the altitudes) are parallel to the sides of the tangential triangle;[1] hence the two triangles are homothetic. Now the line joining the ninepoint center of the given triangle and the orthocenter of the orthic triangle is the Euler line of the orthic triangle; therefore this line is parallel to the Euler line of the tangential triangle.

Also solved by the PROPOSER.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The ninth annual meeting of the National Council of Teachers of Mathematics was held at Boston, February 24–25, 1928. On the general program were President George B. Olds, of Amherst College, who spoke on "Mathematics and Modern Life;" Professor H. W. Tyler, of Massachusetts Institute of Technology, who spoke on "Mathematics in Modern Science;" and Professor H. E. Slaught, of the University of Chicago, who spoke on "Mathematics and Sunshine." Other speakers were: Miss Edith Clarke, of the General Electric Company, on "Mathematics in Modern Business;" Walter F. Downey, English High School, Boston; Olive A. Kee, Teachers College, Boston; Harry C. Barber, Phillips Exeter Academy; W. S. Slauch, High School of Commerce, New York; and Dr. W. D. Reeve, of Teachers College, Columbia University.

Important actions of the National Council were (1) the presentation of its third yearbook, an imposing volume of over 200 pages; (2) authorization of a committee under the chairmanship of Professor Slaught to effect the incorporation of this body under the laws of Illinois; (3) authorization for the publication of a register of members early next autumn; (4) provision in the new by-laws for branches of the National Council, to consist of clubs or associations now existing or to be formed, so as to constitute a federation of all such organizations of teachers of secondary mathematics; and (5) the definite determination to increase the membership of the Council to at least ten thousand by 1930, to which end every one of the present forty-three hundred members was urged to bring in at least ten new ones.

[1] See, for instance, N. A. Court, *College Geometry*, p. 85.

The new officers are: President, HARRY C. BARBER, Exeter, N. H.; Secretary, J. A. FOBERG, California, Pa.; Editorial Committee, W. D. REEVE, Columbia University, JOHN R. CLARK, New York University, and H. E. SLAUGHT, University of Chicago.

A large and enthusiastic club of teachers of secondary mathematics in New York City is engaged, under the leadership of Professor W. D. Reeve, of Columbia University, in the study of curriculum and teaching improvement. On the evening of March 3 a meeting of one-hundred-fifty members of the club was addressed by Professors David Eugene Smith and Herbert Ellsworth Slaught.

Professor H. E. Slaught of the University of Chicago has just returned from an eastern trip where he had the pleasure of seeing the final proof sheets of most of the plates for the second volume of the Ahmes Papyrus. He says: "It was interesting to see at first hand just what the difficulties have been in making these plates practically perfect. It could be done only by a highly expert Egyptologist such as Dr. Bull of the Metropolitan Museum in New York, and while it has taken far longer than we anticipated, nevertheless it seems very certain that in the end we shall not so greatly regret this delay in view of the fact that the outcome promises to be so highly satisfactory. It is even now not possible to name a definite date for the actual publication of this volume, but we have reason to believe that it may be ready before the end of the present academic year. (The first volume is already printed.) Further announcement will be made as the work progresses."

The fourth annual meeting of The History of Science Society was held at the American Museum of Natural History, New York, on November 25–26, 1927, to commemorate the bi-centenary of the death of Sir Isaac Newton. This meeting was shared in by the affiliated organizations: The American Mathematical Society, The Mathematical Association of America, The American Astronomical Society, and the American Physical Society.

The two days' program consisted of papers on various phases of Sir Issac Newton's contributions to astronomy, mathematics, physics, chemistry, religion, problems of the mint, and the development of science since his day, and on Newton's influence on the early science in the American Colonies from 1687 to 1779. Two papers were devoted to each subject and these were given by scholars of distinction—from both the United States and Canada. The program for the meeting was as follows:

"Introductory address on Newton's life and work" by Dr. DAVID EUGENE SMITH.

"Newton and optics" by Professor D. C. MILLER.

"Newton's philosophy of gravitation, with special reference to modern relativity ideas" by Professor G. D. BIRKHOFF.

"Newton's influence on the development of astro-physics" by President W. W. CAMPBELL.

"Newton and dynamics" by Professor M. I. PUPIN.

"Newton as an experimental philosopher" by Dr. P. R. HEYL.

"Developments following from Newton's work" by Professor E. W. BROWN.

"Newton's twenty years' delay in announcing the law of gravitation" by Professor FLORIAN CAJORI.

"Newton's work in alchemy and chemistry" by Doctor L. C. NEWELL.

"Newton's place in the history of religious thought" by Doctor G. S. BRETT.

"Mint problems of Newton" by Mr. G. E. ROBERTS.

"Newton's first disciple in America" by Mr. F. E. BRASCH.

There was a large exhibition of books in which portrait prints, letters, autographed documents and medallic illustrations of Sir Issac Newton and of his mathematical contemporaries, which supplemented the addresses, were arranged in suitable cases, besides twelve items of Newtonian collected works and single volumes, one of these being Newton's copy of a journal with his signature and numerous notes.

These addresses are to be published in a memorial volume under the auspices of The History of Science Society. An account of the commemoration together with abstracts of all the papers has been published in *Popular Astronomy*, and in the issue of *Science* for March 9, 1928.

The Carnegie Corporation has presented the sum of $10,000 to the American Philosophical Society, for the preparation of a series of "Source Books in the History of the Sciences." Professor D. E. SMITH, of Columbia University, is in charge of the volume of this series that is devoted to mathematics.

Professor M. I. PUPIN has been given the Washington award of the Western Society of Engineers, in recognition of his work in long distance telephony and radio communication.

The following courses in mathematics are announced for the summer of 1928:

Stanford University, June 21 to August 31. By Dr. ALFRED ERRERA (Belgium): Topics selected from the foundations of mathematics; Geometry and the elements of axiomatics. By Professor E. HILLE (Princeton): Differential equations; Theory of functions of a complex variable.

Syracuse University. In addition to the usual courses in mathematics through the calculus, the following advanced courses are offered. By Professor F. F. DECKER: The teaching of algebra and geometry in secondary schools;

Introduction to the theory of groups or Introduction to modern algebra. By Professor A. D. CAMPBELL; Projective geometry or Theory of functions of a complex variable.

Dr. ETHEL L. ANDERTON of Smith College has been appointed assistant professor of mathematics at Mount Holyoke College.

Assistant Professor GRACE BERRY has been promoted to an associate professorship of mathematics at Pomona College. This semester she is on sabbatical leave, studying at the University of Illinois.

Mr. J. ALSTON CLARK of Princeton has been appointed instructor of mathematics at Rutgers University for the coming academic year.

Dr. H. N. DAVIS, professor of mechanical engineering at Harvard University, has been elected president of the Stevens Institute of Technology.

Dr. HAZEL G. SCHOONMAKER of Cornell University has been appointed assistant professor of mathematics at New Jersey College for Women.

Assistant Professor ARTHUR K. SOLUM of St. Olaf College is continuing his graduate studies at the University of Minnesota. Mr. CLARENCE CARLSON is taking Mr. Solum's place at St. Olaf College.

Professor THOMAS E. MCKINNEY of the department of mathematics and astronomy of the University of South Dakota will retire from service at the end of the present academic year on account of blindness.

Doctor C. H. RICHARDSON, professor of mathematics at Georgetown College, Kentucky, has been appointed head of the department of mathematics at Bucknell University. Professor Richardson will take the chair formerly filled by Doctor W. C. BARTOL who retires after forty-seven years of teaching service at Bucknell.

CORRIGENDUM

The following correction should be made in the March issue of the Monthly: On page 134, in line 24, replace $f(t)dt(t-z)$ by $f(t)dt/(t-z)$.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 3.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, May.

OHIO, Columbus, Ohio, April 6.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Durham, N. C., March.

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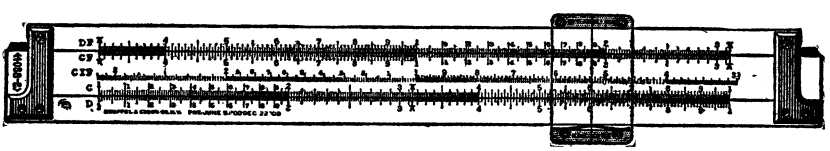
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THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The seventh regular meeting of the Southern California Section was held at Whittier College, Whittier, California, on Saturday, March 10, 1928. Professor W. P. Russell presided.

The attendance was forty-one, including the following thirty members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, Harry Bateman, Clifford Bell, E. T. Bell, W. N. Birchby, E. W. Brown, Jessie R. Campbell, Myrtie Collier, P. H. Daus, Harriet E. Glazier, E. R. Hedrick, G. H. Hunt, Mary M. Keith, G. R. Livingston, W. E. Mason, W. B. Orange, Lena E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, Marcus Skarstedt, D. V. Steed, H. C. Van Buskirk, Mabel G. Whiting, H. C. Willett, Clyde Wolfe, Euphemia R. Worthington.

The meeting began with a luncheon at Redwood Cottage, and the Association was welcomed to Whittier College by President Dexter. The following officers were elected for the academic year 1928–29; Chairman, E. T. BELL, California Institute of Technology: Vice-Chairman, E. E. ALLEN, Occidental College: Program committee, D. V. STEED, University of Southern California and LENA E. REYNOLDS, Fullerton Junior College. The meetings for the coming year were set for November 3, 1928 at Fullerton Junior College and March 9, 1929 at the University of Redlands.

The following five papers were presented:

1. "Reflections of a mathematics professor" by Professor MARCUS SKARSTEDT, Whittier College.

2. "Observations on the constructibility of regular polygons" by Professor O. W. ALBERT, University of Redlands.

3. "A new derivation of the Lorentz transformation" by Mr. P. Y. CHOU, California Institute of Technology. (Introduced by Professor E. T. Bell.)

4. "A porism in hyperspace" by Professor D. V. STEED, University of Southern California.

5. "A problem in calculation" (by invitation) by Professor E. W. BROWN, Yale University.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles. Professor E. W. Brown of Yale University, who addressed the section by invitation, was introduced by Professor E. R. Hedrick.

2. This paper illustrated the different methods used for constructing certain regular polygons by applying them to the decagon. These methods included a trigonometric one, which led to proofs of the impossibility of constructing regular polygons of 7 or 9 sides and to an approximate construction of the regular polygon of 17 sides.

3. By assuming that the kinematical relations between the two sets of space and time measurements in two reference systems S and S′ in uniform motion with respect to each other leave the wave equation invariant, it is possible to prove that the velocities of light in the two systems are equal and that the equations of transformation are the Lorentz transformation.

4. Professor Steed considered a method of proving that the cubic surface in 3-space has in general a finite number of lines on it, the method depending upon finding four conditions to determine the four parameters of the line. By an analogous argument, six conditions were found, to determine the six parameters of a plane in 4-space, which lies on a hyperquadric. This led to a porism, since the six conditions did not determine a finite number of such planes.

5. Professor Brown discussed the problem of the expansion of $(1-2\alpha\cos x+\alpha^2)^{-1/2}=a_0+2\sum_1^\infty a_i\cos ix$, from the standpoint of practical calculation. By means of certain auxiliary calculations, he showed how the coefficients of this Fourier expansion could be calculated in descending rather than ascending order. This method has the advantage that the degree of accuracy is increased rather than decreased as successive coefficients are computed.

P. H. DAUS, *Secretary-Treasurer*

THE THIRTEENTH MEETING OF THE KANSAS SECTION

The thirteenth regular meeting of the Kansas Section of the Mathematical Association of America was held at Topeka, on Saturday, February 4, 1928, in connection with the annual meeting of the Kansas Association of Mathematics Teachers. Two sessions were held, and the luncheon between the sessions provided a pleasant social hour. The morning meeting was a joint session of the two organizations. In the absence of Professor W. T. Stratton, president of the Kansas Association, Professor U. G. Mitchell, chairman of the Kansas Section, presided at both sessions.

The attendance was about sixty-five, including the following twenty-four members of the Association: Wealthy Babcock, Florence Black, Lucy Dougherty, E. F. Farner, W. H. Garrett, W. A. Harshbarger, W. H. Hill, Ina Holroyd, Emma Hyde, W. C. Janes, C. F. Lewis, Nina McLatchey, Anna Marm, U. G. Mitchell, Thirza Mossman, H. S. Myers, A. W. Philips, W. L. Remick, Ethel Rumney, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, H. S. MYERS, Vice-Chairman, C. H. ASHTON, Secretary-Treasurer, LUCY T. DOUGHERTY.

The following papers were presented:

1. "The Nashville meeting," by Professor J. A. G. SHIRK, Kansas State Teachers College, Pittsburg.

2. "Should the essentials of solid geometry be included in plane geometry?" by Miss MILDRED NEWMAN, Lawrence High School.

3. "Mathematics in Italian Universities," by Professor E. B. STOUFFER, University of Kansas.

4. "The problem of two bodies and a related sextic," by Professor W. H. GARRETT, Baker University.

5. "The linear section of a five-point," by Miss FERNA WRESTLER, Junior College, Eldorado, Kansas.

6. "What college mathematics is desirable for the student of business?" by Professor H. F. HOLTZCLAW, School of Business, University of Kansas. By invitation.

7. "Frequency curves," by Professor A. E. WHITE, Kansas State Agricultural College.

Papers 1, 2, and 3 were presented before the joint session, a complete report of which, with summaries of the papers, will be found in the Bulletin of the Kansas Association of Mathematics Teachers. (Edited by Professor Ina E. Holroyd, and published at Manhattan, Kansas.)

4. From the differential equations of motion of two bodies

$$\frac{d^2x}{dt^2} = -k^2M\frac{x}{r^3}, \qquad \frac{d^2y}{dt^2} = -k^2M\frac{y}{r^3},$$

Professor Garrett obtained the equation of the orbit of the first with respect to the second and derived the well known relation ($e<1$)

$$a = k^2Mr/(2k^2M - V^2r),$$

where a represents the semi-major axis of the ellipse, r the distance between the bodies and M the sum of the masses.

From this relation it follows that the members of the family of ellipses obtained by projecting the first body in different directions with the same velocity will have the same length of major axis. The problem of the locus of the perihelion and aphelion points of members of the family was discussed and the following polar equation of the locus obtained:

$$\rho^2 + 2a\rho + ar - r(\rho + a)\cos\theta = 0.$$

This becomes in rectangular coordinates:

$$x^6 + 2rx^5 + 3x^4y^2 + (r^2 + 2ar - 4a^2)x^4 - 4rx^3y^2 + (4a^2r - 2ar^2)x^3 + 3x^2y^4 + (4ar - 8a^2 + r^2)x^2y^2 - 2rxy^4 + (4a^2r - 2ar^2)xy^2 + y^6 + (2ar - 4a^2)y^2 = 0,$$

which in the special case, $a=r$, factors into

$$(x^2 + y^2 - r^2)(x^4 - 2rx^3 + x^2y^2 - 2ryx^2 - r^2y^2 + y^4) = 0,$$

the equation of the circle and the cardioid.

At the conclusion of the mathematical discussion, Professor Garrett illustrated the cases $a>r$, $a=r$, $a<r$ by charts, and showed the simple apparatus by which he had constructed the graphs of several members of the family of sextics mechanically.

5. The linear section of a complete plane five-point determined by any line l not on a given point or a diagonal point consists of ten points. If two complete five points correspond in such a way that any eight pairs of homologous sides meet in points of a line l, the other two pairs of homologous sides meet in points of l. If the correspondence is such that two homologous simple five-points are perspective from the line l and any two other pairs of homologous sides of the complete five-point meet in points of l, the other three pairs of corresponding sides meet in points of l. It should be noticed that the theorem is still valid if l passes through one or even two of the diagonal points of the figure. If l is chosen as a line joining two of the given points, $P_i(i=1,2,3,4,5)$, the double ratio of one of the points P_i and any three collinear diagonal points is the same as the double ratio of the pencil of lines having P_i as center and connecting P_i to each of the other four points. The five points P_i determine a conic K. Any tangent $t_i(i=1,2,3,4,5)$ with point of contact P_i determines a linear section of three pairs of points of an involution of which P_i is a double point.

6. There is a growing demand among business men for students trained in mathematics. Given a body of students so trained, it is found that all business relations and operations are readily grasped and skill easily attained. Every student of business should have a good course in college algebra, and those who major in statistics, accounting, or insurance should also have trigonometry and calculus as tool subjects, and as much more mathematics as they can get.

7. Pearson's twelve types of frequency curves are discussed in this paper. The usual form of development is explained. The parameters of the Type III are found. Pearson's method of determining the kind of curve to use on any particular case is considered. The Gram-Charlier system of frequency curves with their generating functions for types A and B are considered.

LUCY T. DOUGHERTY, *Secretary*

REPORT OF THE COMMITTEE ON ASSIGNED COLLATERAL READING IN MATHEMATICS

A committee of three, A. A. Bennett, Chairman, R. C. Archibald, J. A. Nyswander, was appointed in February, 1927, by the Mathematical Association

of America in response to a suggestion by Professor Bennett, "to draw up a list of prescribed assignments in collateral reading in mathematics for freshmen and sophomore students in American Colleges". This committee submits the following report:

It is generally recognized that there has been a strong tendency to decry the traditional course of instruction in mathematics for freshmen and sophomores in the American colleges. The objections urged are familiar, and although worded in many ways are essentially analogous: "that the drill designed for the specialists is wasted on those who do not continue mathematical study; that no glimpse of the philosophy or history of the subject is given to students who can really assimilate only these features; that what is needed is a survey of what mathematics aims to accomplish, and not manipulative speed nor problem-solving ingenuity; that the same disciplinary training can be acquired in any other study, while the added gain of lively interest and ready application makes other subjects more suitable than mathematics; that the classical languages are being laid aside save for the professional scholar, and mathematics which partakes of their character should share their fate," and so forth. This tendency has resulted in the dropping of mathematics as a required freshman subject from many colleges. Many teachers of mathematics are appreciating the force of a part of the criticism as is seen for example in the wide-spread experimentation with unified courses which aim with a minimum of routine drill to cover as many as possible of the important concepts within the grasp of the student.

The committee believes that it expresses the conviction of most experienced teachers of mathematics when it insists upon the importance in freshman mathematics of regular class exercises with exposition, problem work, theory and demonstration. The average student cannot be expected to attain any adequate ideas about mathematical abstractions without systematic and prolonged practice. A "survey course in appreciation" may be interesting, but it must leave distorted and hopelessly superficial traces upon the less scholarly students who come to college with such preparation as is at present customary. An illustrated lecture in natural science cannot replace the vivid impressions gained in a laboratory, nor can historical and philosophical readings supersede the concrete convictions arising from actual mathematical work. A rational point of view is not to be engrafted, but develops irresistibly by the individual discovery and personal contact with truth which classroom work is intended to afford, granted the native intelligence required.

The committee suggests that despite the prime necessity in required mathematical courses of classroom instruction in the traditional manner, certain additional mathematical contacts may well be introduced when other circumstances are favorable. Even a mediocre student may reasonably be expected

to acquire under proper direction, outside the classroom, a certain amount of historical and philosophical background, and at worst cannot be injured by exposure to these less measurable influences.

In some cases at least it would not seem wise to increase the burden of routine mathematical drill at present assigned to the freshmen. It takes time for new ideas to become part of the student's mental equipment. To give more hours a week of classroom exercise might not result in more of the subject being mastered. Still less might it seem feasible to add new unconnected topics to the freshman curriculum. In too many instances there are evidences of mental indigestion that perhaps are attributable to the number of unrelated subjects which the student is urged to master. However, the question as to whether the average student is devoting too many hours a week to his studies, or whether the time he is spending in literary pursuit in the library is excessive, is of another sort. Readings which will explain and relate his work, which do not increase his hours of drill nor seek to condense to a briefer interval his allotted days of grace in mastering what the race took centuries to evolve, seem not to be open to these objections.

Doubtless an intelligent and conscientious student will always find interest in any subject over which he is conscious of growing mastery, but there seems to be no reason why the poorer student should not have his attention aroused by varied appeals so long as these do not descend to triviality nor induce a dissipation of his energies and time. The arguments that may be adduced for undergraduate mathematics clubs might also be urged in some institutions with perhaps greater force for the inclusion of mathematical view points in the schedule of all students, for which regular classroom teaching can ill-afford an extensive time allotment.

To make a constructive contribution, the committee offers the following suggestion as a basis, upon which some departments of mathematics might plan to develop a definite program, with variations suggested by experience or required by local conditions.

A list of student readings is here collected, all of them in English in books that may be easily secured and in no case (it is believed) hopelessly beyond the understanding of the students, although only a superficial impression may result in most cases. The plan, most economical of books, would be to divide the set of readers into a number of groups, say fifteen, numbered from one to fifteen. The first week's assignments would consist of Group One reading Assignment One, Group Two reading Assignment Two, and so on. The following week, Group One would read Assignment Two, Group Two would read Assignment Three, and finally Group Fifteen would read Assignment One. In fifteen weeks each group would have read each assignment. In this way there would be not such a great congestion in the demand for a specified book and the entire

semester's reading could be completed without a hurried rush at the end of the term. Such a plan would, however, make impossible any progressive continuity of logical development. It would break up the class into small groups whose members could not, if they would, discuss intelligibly the subject matter of outside reading. It would make any uniform tests for the class inoperative previous to the close of the semester. If a single text is to serve for more than one assignment, even the economy is not so striking. The assignments are arranged roughly in logical order. When used as required reading it would, of course, be necessary to have a number of copies available of all assigned books in order to be fair to the student who may legitimately be rushed up to the last moment in any week. For this reason references to journal articles have not been made for the student readings, since it is usually undesirable to have a large number of duplicates of some particular serial publication on hand when only a few pages are to be used. Library experience in other similar cases seems to indicate that it suffices to have about one copy per twelve students, of each reserved book. The idea is to give the students in any single group for any single week, only one assignment. No option is here offered, partly for this reason, but chiefly because of the scarcity of books which seem to satisfy all the requirements of the situation.

A practicable plan might be to devote occasionally a brief time, perhaps a half hour once a week, to these non-routine topics. The instructor would briefly summarize the facts covered by the reading, but supposedly from a different point of view. He would try to bring out a unity that scattered assignments might fail to suggest. The student might be held responsible not merely for outside reading, but also for the substance of these brief surveys, upon which he is expected to take such notes as may seem necessary.

If this plan is followed, the instructor might need also some definite suggestions for preparing his remarks. References for his discussions might be expected to be less restricted as to topic, and difficulty of literary treatment, and of course one copy each of the source material would be sufficient. Hence journal articles, and more serious and extensive references are regarded as suitable in the list of "instructor-readings," here given after the list of student readings.

Most mathematical literature is beyong the grasp of the freshman because of the technical preparation assumed. Again, there are many special subjects which require accurate knowledge of little beyond high school topics, but which are usually unsuitable for assigned freshman reading, because of the general maturity assumed, and the abstractness of the presentation. Much that an intelligent college graduate with scientific interests and developed powers of concentration, although with a minimum of mathematical credits, might profitably study, would require for freshmen too much explanation and guidance

to merit a place in the plan of outside reading proposed here. One might mention map making, nomography, non-Euclidean geometry, projective geometry, number theory, the elements of all of which demand only a high school background, and any one of which could be handled in an elementary way with freshmen if sufficient care and patience were used. For casual outside reading, most freshmen would gain nothing by references to these topics.

It is hoped that this list itself may prove of value in its present form. There is obviously much room for taste and judgment in the naming of items for any such collection, and a department staff that may examine this list may make substitutions of other texts that would result in unquestioned improvement. The plan itself may not seem attractive for most institutions, but the plan is more significant than the particular list. Comments or suggestions would be welcomed on any part of this suggestion.

STUDENT'S LIST

I. *History. Antiquity.*

1. D. E. SMITH. *History of Mathematics*, vol. 1. Ginn and Company, 1923. (22)+547+(49) pages. Assignment: pp. 1–53.

2. F. S. MARVIN. *The Living Past*, Fourth edition. Oxford Clarendon Press, 1920. (16)+280+(16) pages. Assignment: pp. 1–45.

3. I. B. HART. *Makers of Science: Mathematics, Physics, Astronomy.* Oxford University Press, 1923. (18)+298+(6) pages. Assignment: pp. 19–51.

4. F. CAJORI. *A History of Elementary Mathematics*, Revised Edition. Macmillan, 1917. (8)+309+(15) pages. Assignment: pp. 1–42.

5. D. E. SMITH. *Mathematics* (Series: Our Debt to Greece and Rome). Marshall Jones Company, Boston, 1923. (10)+164+(11) pages. Assignment: pp. 1–89.

6. D. LARRETT. *The Story of Mathematics.* E. Benn, London, 1926. 88 pages. Assignment: pp. 1–36.

7. Sir T. L. HEATH. *Archimedes* (Series: Pioneers of Progress, Men of Science). Soc. for Prom. Chr. Knowl., London, 1920. 56+(2) pages. Assignment: Entire book.

8. Same book as 1. Assignment: pp. 54–95.

9. Same book as 2. Assignment: pp. 48–90.

10. Same book as 4. Assignment: pp. 43–93.

II. *History. Middle Ages and Modern Times.*

11. J. W. N. SULLIVAN. *The History of Mathematics in Europe* (Series: Chapters in the History of Science, IV). Oxford University Press, 1925. 109 pages. Assignment: pp. 1–48.

12. FLORENCE A. YELDHAM. *The Story of Reckoning in the Middle Ages.* G. G. Harrap and Co., London, 1926. 96 pages. Assignment: pp. 19–72.

13. Same book as 2. Assignment: pp. 120–166.

14. Same book as 3. Assignment: pp. 52–92.

15. Sir OLIVER LODGE. *Pioneers of Science* (Reprint). Macmillan, 1919. (15)+397+(8) pages. Assignment: pp. 5–83.

16. W. W. BRYANT. *Galileo* (Series: Pioneers of Progress, Men of Science). Soc. for Prom. Chr. Knowl., London, 1918. 64 pages. Assignment: Entire book.

17. Same book as 4. Assignment: pp. 93–138.

18. Same book as 6. Assignment: pp. 37–59.

19. V. E. PULLEN. Sir Isaac Newton. (Benn's Sixpenny Library). E. Benn, London, 1927. 80 pages. Assignment: Entire pamphlet.

III. *Problems, Miscellany, Philosophy.*

20. (E. A. ABBOTT.) "A-SQUARE." *Flatland,* Third edition. Basil Blackwell, Oxford, 1926. (16)+102 pages. Assignment: Entire book.

21. H. P. MANNING, Ed. *The Fourth Dimension Simply Explained.* Munn and Co., New York, 1910. (6)+246 pages. Assignment: pp. 7–51.

22. CLAUDE BRAGDON. *The Beautiful Necessity,* Second edition. Alfred Knopf, New York, 1922. 111 pages. Assignment: Entire book.

23. D' A. W. THOMPSON. *Growth and Form.* Cambridge University Press, 1917. (16)+779+(14) pages. Assignment: pp. 16–45; 493–528.

24. H. E. DUDENEY. *The Canterbury Puzzles,* Second edition. Thomas Nelson, London, 1919. 255 pages. Assignment: pp. 11–57.

25. W. W. R. BALL. *Mathematical Recreations and Essays,* Tenth edition. Macmillan, 1922. (13)+359+(7) pages. Assignment: pp. 44–61, 170–187.

26. A. N. WHITEHEAD. *Science and the Modern World.* Macmillan, New York, 1925. (11)+296 pages. Assignment: pp. 28–54.

27. J. B. SHAW. *Lectures on the Philosophy of Mathematics.* Open Court, 1918. (7)+197+(9) pages. Assignment: pp. 1–13, 154–195.

28. C. J. KEYSER. *The Human Worth of Rigorous Thinking.* Columbia University Press, 1916. 314 pages. Assignment: pp. 1–60.

29. J. W. N. SULLIVAN. *Aspects of Science* (Traveller's Library). Jonathan Cape, London, 1927. 223 pages. Assignment: pp. 11–26, 58–89, 185–205.

30. W. F. WHITE. *A Scrap-Book of Elementary Mathematics.* Open Court, 1910. 248 pages. Assignment: pp. 203–233.

ADDITIONAL REFERENCES SUGGESTED FOR INSTRUCTORS

I

F. S. MARVIN. *The Living Past, a Sketch of Western Progress,* Fourth edition, Oxford, 1920. (Many references to mathematics.)

W. W. R. BALL. *Short Account of the History of Mathematics*, Fifth edition. Macmillan, London, 1912.

J. L. HEIBERG. *Mathematics and Physical Science in Classical Antiquity*, translated into English. Oxford University Press, 1922. 110 pages.

E. W. BROWN. *Mathematics*, pp. 1–42 of *The Development of the Sciences*. Edited by L. L. Woodruff. Yale Univ. Press, New Haven, 1923. (In the same volume, pp. 129–167, F. Schlesinger writes on *Astronomy*.)

C. SNYDER. *The World Machine*. Longmans, London, 1907. 488 pp. (VI; Eratosthenes and the earliest measures of the earth; X: Archimedes and the first ideas of gravitation; XX: Newton and the motor force of the world machine; etc.)

F. CAJORI. *The History of Zeno's Arguments on Motion*, American Mathematical Monthly, vol. 22 (1915), pp. 1–6; 39–47.

II

R. BROWNING. *Rabbi Ben Ezra*. See also D. E. Smith and J. Ginsburg, *Rabbi ben Ezra and the Hindu-Arabic problem*, American Mathematical Monthly, vol. 25 (1918), pp. 99–108.

W. W. BRYANT. *Galileo* (Pioneers of Progress, Men of Science). London, 1918. 64 pages.

A. NOYES. *The Torch Bearers*, 2 vols. New York, 1922–25. (Copernicus, Tycho Brahe, Kepler, Newton, William Herschel, Pythagoras, Leonardo da Vinci, etc.).

III

H. POINCARÉ. *Science and Method*, translated by Francis Maitland. Nelson, London, 1914. (*Mathematical discovery*, pp. 46–63; *Mathematical definitions and education*, pp. 117–142; etc.)

P. E. B. JOURDAIN. *The Nature of Mathematics*, Revised edition, Jack, London, 1919. 126 pages.

F. D. MITCHELL. *Mathematical Prodigies*, American Journal of Psychology, vol. 18 (1907), pp. 61–143.

T. A. COOK. *The Curves of Life*. London, 1914. 509 pages.

C. BRAGDON. *Projective Ornament*. Rochester, N. Y., 1915.

J. A. ZAHM. *Woman in Science*. Appleton, New York, 1913. *Women in mathematics and astronomy*, pp. 136–196.

R. C. ARCHIBALD. *Mathematics and Music*, American Mathematical Monthly, vol. 31 (1924), pp. 1–25.

F. C. BOON. *A Companion to Elementary School Mathematics*. Longmans, London, 1924. 302 pages. (Chap. I: Biographical notes; II: Historical Outline; III: Euclid's Postulates; IV: Squaring the Circle; XXIII: Paradoxes and Fallacies; etc.)

H. SCHUBERT (T. J. McCormack, tr.). *Mathematical Essays and Recreations.* Open Court, 1903. 143+(6) pages.

E. W. HOBSON. *Squaring the Circle.* Cambridge University Press, 1913. 57 pages.

E. V. HUNTINGTON. *The Continuum*, second edition. Harvard University Press, 1917. (7)+80+(2) pages.

G. ST. L. CARSON. *Essays on Mathematical Education.* Ginn and Co., 1913. (5)+139 pages.

LAPLACE'S CALCULUS OF GENERATRIX FUNCTIONS[1]

By IRENE PRICE, Indiana University

1. Introduction. The aim of this paper is to give a brief explanation of the theory and application of an almost extinct branch of mathematics, the calculus of generatrix functions, which should be revived because:

a. It is a powerful and elegant method of attacking the analysis and problems of probability.

b. It gives an asymptotic expansion for the finite integral of probability.

c. It has wider application than any other theory connected with probability.

These functions were the result of Laplace's attempts to generalize the analysis and problems in the theory of probability and were made the basis of his "Théorie des Probabilités."

It has been said that there is no book equal to Laplace's "Théorie des Probabilités" for a comprehensive and masterly treatment of probability, but this "ne plus ultra of mathematical skill and power" as it is called is not easy reading. Much of its difficulty is connected with the use of a mathematical method which is now almost superseded, namely "Generatrix Functions."[2]

This calculus has become extinct because mathematicians have centered their interests on other methods.

2. Definition of the Operators G and D. A generatrix function in its most elementary form is any function which when expanded into an infinite power series which converges will give coefficients formed according to a given law. The coefficient of the general term, that is of x^n, is called the "déterminante" of the function. For example e^x is a function which when expanded gives coefficients which are $1/n!$ where n is the exponent of x in the term considered.

[1] Presented to the Indiana Section of the Mathematical Association of America at De Pauw University, April 30, 1927.

[2] Encyclopedia Britannica, 13 edition, vol. 22, p. 403, "Probability."

Hence e^x is called the generatrix function of $1/n!$, and $1/n!$ is called the "déterminante" of e^x:

$$e^x = G1/n! \; ; \; 1/n! = De^x.$$

In the general case, if

$$u(x) = \phi(0) + \phi(1)x + \phi(2)x^2 + \cdots + \phi(n)x^n + \cdots$$

and $\phi(n)$ is a function of n, a positive integer, then

$$u(x) = G\phi(n) \; ; \; \phi(n) = Du(x).$$

The series may begin with any power of x; hence $\sum_{n=-\infty}^{\infty}\phi(n)x^n$ is also a generatrix function of $\phi(n)$ where n is taken negative as well as positive.

We may think of the use of a generatrix function as giving a method of treating the successive values of some function as the coefficients of a function with reference to a different variable.

Since $u/x^r = \phi(0)x^{-r} + \phi(1)x^{1-r} + \cdots + \phi(n+r)x^n + \cdots$, and

$$(x^{-1} - 1)u = \phi(0)x^{-1} + [\phi(1) - \phi(0)] + \cdots + [\phi(n+1) - \phi(n)]x^n + \cdots,$$

we obtain

$$u/x^r = G\phi(n+r) \text{ and } (x^{-1} - 1)u = G[\phi(n+1) - \phi(n)] = G\Delta\phi(n),$$

$\Delta\phi(n)$ being the symbol which denotes the finite difference, $\phi(n+1) - \phi(n)$, of $\phi(n)$. Similarly

$$(x^{-1} - 1)^2u = G[\Delta\phi(n+1) - \Delta\phi(n)] = G\Delta^2\phi(n) \; ; \; \cdots (x^{-1} - 1)^r u = G\Delta^r\phi(n).$$

From this we derive the rule: *The generatrix function of a finite difference of a variable is equal to the generatrix function of the variable multiplied by* $(x^{-1} - 1)$.

Example: To find $G\Delta\phi(n)$ for $\phi(n) = 1/n!$

$$\Delta\phi(n) = \frac{1}{(n+1)!} - \frac{1}{n!};$$

$$G\Delta\phi(n) = \sum_{n=0}^{\infty}\left[\frac{1}{(n+1)!} - \frac{1}{n!}\right]x^n = 1 - 1 + \left(\frac{1}{2!} - \frac{1}{1!}\right)x + \left(\frac{1}{3!} - \frac{1}{2!}\right)x^2 + \cdots + \left[\frac{1}{(n+1)!} - \frac{1}{n!}\right]x^n + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{x} - 1\right)x^n,$$

which is the same as the result which would have been obtained by using the rule, since the generatrix function of $\phi(n)$ is $\sum_{n=0}^{\infty}x^n/n!$.

If r is a negative number the finite differences become finite integrals.

No difficulties are encountered in finding the "déterminante" if the generatrix function is given, but trying to find a generatrix function from the "déterminante" requires indirect methods, namely, the solution of a finite difference equation. *The process of passing from the generatrix functions to their "déterminantes" and from the "déterminantes" to their generatrix functions constitutes the calculus of generatrix functions.*

Prof. Boole, whose *Calculus of Operations* has partially replaced the methods of generatrix functions, says with reference to them:

G is from its very nature a symbol of inversion. In applying it, we do not perform a definite and direct operation, but seek the answer to a question, i.e., What is that function which on performing the direct operation of development, produces terms possessing coefficients of a certain form? and this is a question which admits of an infinite variety of answers according to the extent of the development and the kind of indices supposed admissable."

3. The Problem of Interpolation. Interpolation is nothing more than the process of finding $\phi(n+r)$ whatever be the value of r from the terms which precede or follow. But $\phi(n+r)$ is the coefficient of x^{n+r} in the expansion of $u(x)$ and the coefficient of x^n in u/x^r. We may write

$$u/x^r = (1 + x^{-1} - 1)^r u = \left[1 + r(x^{-1} - 1) + \frac{r(r-1)}{2!}(x^{-1} - 1)^2 + \cdots\right] u.$$

If r is an integer we get on passing from the generatrix functions to their "déterminantes,"

$$\phi(n + r) = \phi(n) + r\Delta\phi(n) + \frac{r(r-1)}{2!}\Delta^2\phi(n) + \cdots,$$

which we recognize as Newton's formula for interpolating in a series. This formula may also be shown to be true if r is a fraction.

Numerous other formulas may be obtained by expressing u/x^r in another form and expanding by the binomial theorem. One of the formulas for interpolating easily obtained by this method is

$$\phi(n + r) = \phi(n) + \frac{r^2}{2!}\Delta^2\phi(n-1) + \frac{r^2(r^2-1)}{4!}\Delta^4\phi(n-2) + \cdots$$
$$+ \frac{r}{2}\left\{[\Delta\phi(n) + \Delta\phi(n-1)] + \frac{r^2-1}{3!}[\Delta^3\phi(n-1) + \Delta^3\phi(n-2)]\right.$$
$$\left. + \frac{(r^2-1)(r^2-4)}{5!}[\Delta^5\phi(n-2) + \Delta^5\phi(n-3)] + \cdots\right\}.$$

This formula is used for interpolating in a series containing an odd number of terms, where n is taken as the mean and r is the distance from this mean.

4. Expansion of Functions in a Series. One of the most important applications of these functions is the expansion of functions and their differences into a series. To illustrate this method $\Delta^r{}_\alpha\phi(n)$ will be expanded into a series in $\Delta\phi(n)$ and higher ordered differences of $\phi(n)$, where $\Delta_\alpha\phi(n)$ represents the finite difference $\phi(n+\alpha)-\phi(n)$. If we take the function

$$(A) \qquad (x^{-\alpha}-1)^r u = [(1+x^{-1}-1)^\alpha - 1]^r u,$$

u being the generatrix function of $\phi(n)$, it is clear that the "déterminante" of the first term is $\Delta^r{}_\alpha\phi(n)$, since

$$D(x^{-\alpha}-1)u = \phi(n+\alpha)-\phi(n) = \Delta_\alpha\phi(n),$$
$$D(x^{-\alpha}-1)^2 u = Du(x^{-2\alpha}-2x^{-\alpha}+1) = \phi(n+2\alpha)-2\phi(n+\alpha)+\phi(n) = \Delta_\alpha{}^2\phi(n).$$

By mathematical induction it follows that

$$(B) \qquad D(x^{-\alpha}-1)^r u = \Delta_\alpha{}^r\phi(n).$$

It then remains to find the "déterminante" of the right member of (A) in terms of $\Delta\phi(n)$. This may be done by expanding it in a power series of $(x^{-1}-1)$. This gives

$$[(1+x^{-1}-1)^\alpha - 1]^r u = u\cdot\Big[(1+x^{-1}-1)^{\alpha r} - r(1+x^{-1}-1)^{\alpha(r-1)} + \frac{r(r-1)}{2!}(1+x^{-1}-1)^{\alpha(r-2)} - \cdots\Big]$$
$$= u\cdot\Big\{\Big[1+\alpha r(x^{-1}-1)+\frac{\alpha r(\alpha r-1)}{2!}(x^{-1}-1)^2+\cdots\Big]$$
$$-r\Big[1+(\alpha r-\alpha)(x^{-1}-1)+\frac{(\alpha r-\alpha)(\alpha r-\alpha-1)}{2!}(x^{-1}-1)^2+\cdots\Big]$$
$$+\frac{r(r-1)}{2!}\Big[1+(\alpha r-2\alpha)(x^{-1}-1)+\frac{(\alpha r-2\alpha)(\alpha r-2\alpha-1)}{2!}(x^{-1}-1)^2+\cdots\Big]$$
$$-\cdots\Big\}.$$

Taking the "déterminante" of each term in this expression, we get

$$\Delta_\alpha{}^r\phi(n) = \Big[\phi(n)+\alpha r\Delta\phi(n)+\frac{\alpha r(\alpha r-1)}{2!}\Delta^2\phi(n)+\cdots\Big]$$
$$(C) \qquad -r\Big[\phi(n)+(\alpha r-\alpha)\Delta\phi(n)+\frac{(\alpha r-\alpha)(\alpha r-\alpha-1)}{2!}\Delta^2\phi(n)+\cdots\Big]$$
$$+\frac{r(r-1)}{2!}\Big[\phi(n)+(\alpha r-2\alpha)\Delta\phi(n)+\frac{(\alpha r-2\alpha)(\alpha r-2\alpha-1)}{2!}\Delta^2\phi(n)+\cdots\Big]$$
$$-\cdots.$$

We may write (*C*) as

$$(D)\quad \Delta_\alpha{}^r\phi(n) = [1+\Delta\phi(n)]^{\alpha r} - r[1+\Delta\phi(n)]^{\alpha(r-1)} + \frac{r(r-1)}{2!}[1+\Delta\phi(n)]^{\alpha(r-2)} - \cdots = \{[1+\Delta\phi(n)]^\alpha - 1\}^r,$$

it being understood that the exponent of $\Delta\phi(n)$ in the expansion is the order of the difference, i.e., $[\Delta\phi(n)]^i$ means $\Delta^i\phi(n)$.

Similarly it may be shown that

$$(E)\qquad \sum_\alpha^r\phi(n) = \{[1+\Delta\phi(n)]^\alpha - 1\}^{-r},$$

where $\sum_\alpha^r\phi(n)$ is the rth finite sum when n varies by α.

5. Asymptotic Expansion of the Finite Integral of Probability. After obtaining the expressions (*D*) and (*E*), these functions may be expressed in series which converge rapidly. The great advantage of this transformation from algebraic expressions into series which converge rapidly is that we can obtain a close approximation to the sum of a series containing a great many terms and factors. Such expressions occur frequently in the theory of probability when a large number of events are considered. Since only an approximation of the results is usually desired, this method is very valuable for the solution of such problems.

We wish to prove that

$$\Delta_h{}^r\phi(y) = [e^\mu - 1]^r,\quad \mu = h\frac{d\phi(y)}{dy},$$

where Δ_h^r is the symbol for the rth finite difference of $\phi(y)$ when y varies by h and the symbol on the right is

$$[e^\mu - 1]^r = \left[1 + h\frac{d}{dy} + \frac{h^2}{2!}\frac{d^2}{dy^2} + \frac{h^3}{3!}\frac{d^3}{dy^3} + \cdots - 1\right]^r \phi(y).$$

Laplace proved it in the following manner. If in $\phi(n)$ we let $n = y/\omega$, y will vary by ω when n varies by unity; $\Delta\phi(n)$ will become $\Delta_\omega\phi(y)$; and $\Delta_\alpha\phi(n)$ will become $\Delta_{\alpha\omega}\phi(y)$. Now let ω become an infinitesimal equal to dy; then $\Delta\phi(n) = d\phi(y)$. Let α become infinitely large, so $\alpha\omega = \alpha dy = h$, where h is a finite quantity. Then the variation of y in $\Delta_{\alpha\omega}\phi(y)$ is h. Then (*D*) may be written

$$(1)\quad \Delta_{\alpha\omega}^r\phi(y) = \{[1+d\phi(y)]^\alpha - 1\}^r = \{[1+d\phi(y)]^{\mu/d\phi(y)} - 1\}^r = [e^\mu - 1]^r.$$

This proof is lacking in rigor. As given it holds for only infinitely small values of y when ω becomes infinitely small, since n is a finite quantity. That

this formula holds for any real values of y and h may be shown as follows. By definition

$$\Delta_h\phi(y) = \phi(y + h) - \phi(y).$$

Expanding $\phi(y+h)$ by Taylor's formula, we may write

$$\Delta_h\phi(y) = \phi(y) + h\frac{d\phi(y)}{dy} + \frac{h^2}{2!}\frac{d^2\phi(y)}{dy^2} + \cdots - \phi(y).$$

or we may express this symbolically as

$$\Delta_h\phi(y) = (e^\nu - 1)\phi(y), \text{ where } \nu = h\frac{d}{dy}.$$

Hence the operator Δ_h is equivalent to the operator $e^\nu - 1$. Professor Boole[1] showed that these operators have many of the properties of algebraic numbers, one of which is the index law. Therefore

$$\Delta_h\Delta_h = \Delta_h^2, \ (e^\nu - 1)(e^\nu - 1) = (e^\nu - 1)_h^2, \text{ and } \Delta_h^2 = (e^\nu - 1)^2.$$

Similarly $\Delta_h^r = (e^\nu - 1)^r$. Thus we have

$$\Delta_h{}^r\phi(y) = (e^\nu - 1)^r\phi(y), \tag{1'}$$

which is the same as Laplace developed in (1) except that the quantity $\phi(y)$ which is to be operated upon has been placed after the operator instead of being in the exponent of e.

Upon expanding (1) or (1') we have

$$\Delta_h{}^r\phi(y) = \left[A_0\frac{d^r}{dy^r} + A_1\frac{d^{r+1}}{dy^{r+1}} + A_2\frac{d^{r+2}}{dy^{r+2}} + A_3\frac{d^{r+3}}{dy^{r+3}} + \cdots\right]\phi(y),$$

where $A_0, A_1, A_2, \cdots$, the coefficients of the derivatives, may be determined by expanding $(e^\nu - 1)^r$ and collecting terms. Thus

$$A_0 = h^r, \ A_1 = \frac{rh^{r+1}}{2!}, \quad A_2 = h^{r+2}\left(\frac{r}{6} + \frac{r(r-1)}{8}\right), \text{ etc.}$$

In a similar manner

$$\sum\nolimits_h^r\phi(y) = [e^\mu - 1]^{-r}. \tag{2}$$

If $h = r = 1$, then $\sum_h^r\phi(y) = [e^\mu - 1]^{-r}$ becomes

$$\sum\phi(y) = [e^\mu - 1]^{-1} = \left[\frac{d\phi(y)}{dy}\right]^{-1} - \frac{1}{2}\left[\frac{d\phi(y)}{dy}\right]^0 + \frac{B_1}{2!}\left[\frac{d\phi(y)}{dy}\right]^1 - \frac{B_3}{4!}\left[\frac{d\phi(y)}{dy}\right]^3 + \cdots,$$

[1] Boole's *Finite Differences*, pp. 16–17.

it being understood that the exponent of $d\phi(y)/dy$ is the order of the derivative. Hence

$$(3) \qquad \sum\phi(y) = \int \phi(y)dy - \frac{1}{2}\phi(y) + \frac{B_1}{2!}\frac{d\phi(y)}{dy} - \frac{B_3}{4!}\frac{d^3\phi(y)}{dy^3} + \cdots + C,$$

in which $B_1, B_3, B_5, \cdots$ are the Bernoulli numbers.[1]

A finite integral often occurring in the theory of probability is

$$\sum_{b=0}^{b}[2n^{1/2}e^{-nb^2/(2xx_1)} \div \pi^{1/2}(2xx_1)^{1/2}],$$

where n represents the total number of trials, x the most frequent number of successes, x_1 the most frequent number of failures in n trials, and b the number of the term counting from the largest. This integral gives the sum of all terms of the binomial $[p+(1-p)]^n$ between the two terms containing p^{x-b} and p^{x+b} as factors and which are equidistant from the greatest term, but the largest term must be subtracted since it is added twice. If L represents the largest term it can be shown that the desired sum

$$\sum\phi(y) - L = \int \phi(y)dy - \frac{1}{2}\phi(y) + \frac{B_1}{2!}\frac{d\phi(y)}{dy} - \cdots + C - L$$

may be expressed thus,

$$(4) \qquad \sum\phi(y) - L = \int \phi(y)dy + \frac{1}{2}\phi(y),$$

all terms after the second becoming small enough to be neglected if n is very large.

Example: We know the probability of getting a head when a penny is tossed is 1/2 and the probability of failure is 1/2. What is the probability of getting at least 48 and not more than 52 heads in 100 throws?

It is known that $xx_1^{-1} = p(1-p)^{-1}$. In this case $p = 1/2$, $x = 50$, $x_1 = 50$, $n = 100$, $b = 2$.

The required sum is

$$\sum_{b=0}^{2} \frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}} - L, \text{ where } \rho = \frac{-100b^2}{2\cdot 50\cdot 50}.$$

Applying formula (4), we get

$$(5) \qquad \sum_{b=0}^{2} \frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}} - L = \int_0^2 \frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}}db + \frac{1}{2}\frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}}.$$

Let $t = (-\rho)^{1/2}$ and (5) is equal to

[1] $B_1 = 1/6$, $B_3 = 1/30$, $B_5 = 1/42$, etc. See B. O. Peirce's *Short Table of Integrals*, p. 90.

$$\int_0^{(2/25)^{1/2}} \frac{2}{\pi^{1/2}} e^{-t^2} dt + \frac{1}{2} \frac{2e^{-4/50}}{\pi^{1/2}(50)^{1/2}} = 0.30997 + 0.073705 = 0.38268.$$

The correct value of

$$_{100}C_{48}(1/2)^{52}(1/2)^{48} + {}_{100}C_{49}(1/2)^{51}(1/2)^{49} + {}_{100}C_{50}(1/2)^{50}(1/2)^{50} + {}_{100}C_{51}(1/2)^{49}(1/2)^{51} + {}_{100}C_{52}(1/2)^{48}(1/2)^{52}$$

is 0.38266, which shows what a close approximation to the correct results may be obtained by this method when n is as small as 100.

6. Generatrix Function of the Solution of a Finite Difference Equation. We are able to find the generatrix function of a quantity if we know a finite difference equation which it satisfies.

It may be shown that if n is a positive integer the function $\phi(n)$ given by the finite difference equation

$$0 = a\phi(n) + b\phi(n+1) + c\phi(n+2) + \cdots + q\phi(n+r)$$

is the coefficient of x^n in

$$\frac{A + Bx + Cx^2 + \cdots + Hx^{r-1}}{ax^r + bx^{r-1} + cx^{r-2} + \cdots + px + q}, \tag{6}$$

$A, B, C, \cdots, H$ being constants which are to be determined. Since (6) is the generatrix function of $\phi(n)$, the values of the constants may be calculated from the identity

$$\frac{A + Bx + \cdots + Hx^{r-1}}{ax^r + bx^{r-1} + \cdots + mx^2 + px + q} = \phi(0) + \phi(1)x + \cdots + \phi(n)x^n + \cdots.$$

Clearing the equation of fractions and equating coefficients of like powers, we get

$$A = q\phi(0) \; ; \; B = q\phi(1) + p\phi(0) \; ; \; C = q\phi(2) + p\phi(1) + m\phi(0).$$

Replacing $A, B, C, \cdots, H$ in (6) by these values, we have the generatrix function of $\phi(n)$.

Example: If $\phi(n)$ is $n^2 - 1$, it satisfies the finite difference equation

$$0 = \Delta^3\phi(n) \text{ or } 0 = -\phi(n) + 3\phi(n+1) - 3\phi(n+2) + \phi(n+3).$$

The generatrix function of $\phi(n)$ will be

$$\frac{A + Bx + Cx^2}{-x^3 + 3x^2 - 3x + 1} = -1 + 3x^2 + 8x^3 + \cdots.$$

Then

$$A + Bx + Cx^2 = (-x^3 + 3x^2 - 3x + 1)(-1 + 3x^2 + 8x^3 + \cdots).$$

Therefore $A = -1$, $B = +3$, $C = -3+3 = 0$.

A GEOMETRICAL METHOD FOR SOLVING THE BIQUADRATIC EQUATION[1]

By W. C. GRAUSTEIN, Harvard University

1. The biquadratic equation,

$$t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 = 0,$$

is usually solved by factoring the polynomial

$$f(t) \equiv t^4 + a_1t^3 + a_2t^2 + a_3t + a_4$$

into two quadratic factors. The algebraic devices by means of which the factorization is effected are not easily kept in mind. This fact has led to the development of a method whereby the problem of the factorization of the biquadratic polynomial in t is replaced by the simpler and more familiar problem of factoring into linear factors a quadratic polynomial, $F(x, y)$, whose discriminant vanishes. The method is essentially geometrical, standing in close relationship to the theory of pencils of conics. We see fit, however, to describe it first in purely algebraic terms.

2. Algebraic description of the method. The quadratic polynomial $F(x, y)$ is obtained from $f(t)$ by replacing the powers of t in $f(t)$ by properly chosen products of powers of x and y. Each of the powers of t, except t^2, can be written in just one way as the product of at most two factors of the forms t^2 and t: $t^4 = t^2 \cdot t^2$, $t^3 = t^2 \cdot t$, $t = t$. On the other hand, t^2 can be expressed either as t^2 or as $t \cdot t$. If, then, we write

$$f(t) \equiv t^2 \cdot t^2 + a_1t^2 \cdot t + \lambda t \cdot t + (a_2 - \lambda)t^2 + a_3t + a_4,$$

and replace t^2 and t respectively by x and y ($x = t^2$, $y = t$), we obtain the desired polynomial

$$F(x, y) \equiv x^2 + a_1xy + \lambda y^2 + (a_2 - \lambda)x + a_3y + a_4.$$

This polynomial has two properties which insure the functioning of the proposed method. It reduces identically to $f(t)$ for $x = t^2$, $y = t$, and it contains a parameter, λ, which can be determined so that the polynomial can be factored into linear factors.

THEOREM. *If λ is so chosen that $F(x, y)$ is factorable into linear factors, these factors become the quadratic factors of $f(t)$ when x and y are replaced by t^2 and t respectively.*

For, if λ is so chosen that $F(x, y) \equiv (x + b_1y + c_1)(x + b_2y + c_2)$, then, since

$$F(t^2, t) \equiv f(t),$$

$$f(t) \equiv (t^2 + b_1t + c_1)(t^2 + b_2t + c_2).$$

[1] Presented at the meeting of the American Mathematical Association in Madison, Wis., Sept. 5, 1927.

The values of λ for which $F(x, y)$ is factorable are the solutions of the equation obtained by equating the discriminant of $F(x, y)$ to zero:

$$\lambda^3 - 2a_2\lambda^2 + (a_1a_3 + a_2^2 - 4a_4)\lambda + (a_1^2a_4 - a_1a_2a_3 + a_3^2) = 0.$$

This is the form, then, which the cubic resolvent of the biquadratic equation here assumes.

3. Geometrical interpretation. The equation

$$F(x,y) \equiv (x^2 + a_1xy + a_2x + a_3y + a_4) + \lambda(y^2 - x) = 0,$$

where λ is arbitrary, represents the pencil of conics whose base conics are

$$x^2 + a_1xy + a_2x + a_3y + a_4 = 0, \quad y^2 - x = 0.$$

The roots of the cubic resolvent determine the degenerate conics of the pencil.

The equation which results from the elimination of x from the equations of the base conics of the pencil is evidently $f(y) = 0$. Hence:

THEOREM. *The solutions of the biquadratic equation are the y-coordinates of the points common to all the conics of the pencil.*

There are five types of pencils of conics and they correspond here to the five possibilities as to the multiplicities of the roots of the biquadratic equation. For example, if the pencil of conics consists of all the conics each two of which have three-point contact at a certain point and intersect simply in a second point, the biquadratic equation has one triple root and one simple root.

It is essential in this connection to note that, inasmuch as one of the base conics of the pencil is the parabola $y^2 - x = 0$, two distinct points common to the conics of the pencil can never have the same y-coordinate. Otherwise, a double root, let us say, of the biquadratic equation might correspond to two distinct points of intersection of the conics of the pencil and the correspondence, just described and illustrated, between the five possibilities for the roots of the biquadratic equation and the five types of pencils of conics would thus be vitiated.

In the general case in which the pencil of conics consists of all the conics through four points, the roots of the biquadratic equation are distinct. The polynomial $f(t)$ can then be factored into quadratic factors in three essentially different ways, which correspond respectively to the three distinct degenerate conics of the pencil and to the three corresponding solutions of the cubic in λ which determine these degenerate conics.

4. Example. Let the given biquadratic equation be

$$f(t) \equiv t^4 + t^3 - 7t^2 + 2t + 4 = 0.$$

Then

$$F(x,y) \equiv x^2 + xy + \lambda y^2 - (7 + \lambda)x + 2y + 4,$$

and the discriminant of $F(x, y)$, equated to zero, is

$$\lambda^3 + 14\lambda^2 + 35\lambda + 22 = 0.$$

This equation has as its roots -1, -2, -11. Of these three values of λ, only the second, $\lambda = -2$, renders the quadratic terms in $F(x, y)$ factorable into linear factors with *rational* coefficients. Setting $\lambda = -2$ in $F(x, y)$ and factoring, we have

$$F(x, y) \equiv (x-y-1)\ (x+2y-4).$$

Hence

$$f(t) \equiv (t^2 - t - 1)(t^2 + 2t - 4),$$

and the roots of $f(t) = 0$ are $\frac{1}{2}(1+\sqrt{5})$, $\frac{1}{2}(1-\sqrt{5})$, $-1-\sqrt{5}$, $-1+\sqrt{5}$.

5. Graphical solution. Thus far we have made no distinction between real and imaginary roots of our biquadratic equation. We now assume that the coefficients in the equation are real numbers and restrict ourselves to the problem of finding the real roots.

By the usual method of adding a suitable quantity to the roots, a_1 can be made zero. We assume that thereby a_3 is not also made zero; otherwise $f(t)$ would be a quadratic in t^2. Since $a_3 \neq 0$, the roots can now be multiplied by a quantity so chosen[1] as to make $a_3 = -1$. Thus, $f(t)$ is reduced to the form:

$$f(t) = t^4 + 2a_2t^2 - t + a_4.$$

The base conics of the pencil are, then, the two congruent parabolas,

$$[x + a_2]^2 - [y - (a_4 - a_2^2)] = 0, \quad y^2 - x = 0,$$

and the y-coordinates of the points of intersection of these two parabolas are the roots of the reduced equation.

The two following methods of finding these roots graphically immediately present themselves.

Method 1. On a sheet of coordinate paper construct the parabola $y^2 - x = 0$. On a sheet of tracing paper draw, to the same scale, the parabola $x'^2 - y' = 0$. Place the tracing paper on the coordinate paper so that the axes of the two parabolas are perpendicular and the vertex of the parabola on the tracing paper covers the point $(-a_2, a_4 - a_2^2)$ on the coordinate paper, and then read off the ordinates of the points of intersection of the two parabolas.

Method 2. (For use in the classroom when a blackboard coordinate system is available.) Make a templet of a parabola to a scale which is suited to that of the coordinate system. Draw the two parabolas in their proper positions on the blackboard and read off the results.

[1] The possibility of this second reduction was suggested to me by C. N. Haskins.

CERTAIN COMPLETELY SOLVABLE SYSTEMS OF SIMULTANEOUS DIOPHANTINE EQUATIONS

By E. T. BELL, California Institute of Technology

1. Introduction. If an algebraic theorem can be attained by two or more distinct algorithms, the fact that the final results are necessarily equivalent frequently leads to interesting systems of diophantine equations completely solvable by parameters. For example, Bounaikowsky made such a use of partial integration.[1] Elementary algebra and the theory of algebraic equations offer a rich store of such results. The process used here differs in principle from Bounaikowsky's, in that it compares two sets of necessary and sufficient conditions instead of two algorithms; it is therefore *complete.*

Let r, s and $n=r+s$ be arbitrary constant integers >0. We shall show how to construct a system of r simultaneous, homogeneous, isobaric diophantine equations, all of degree $s+2$, with rational integral coefficients, of the respective weights $n, n-1, \cdots, n-r+1$, in the $n+r$ indeterminates

$$u_i (i = 1,2, \cdots, r), \quad v_j (j = 1,2, \cdots, n),$$

and we shall exhibit a parametric solution of the system in the form

$$\begin{aligned} u_0 &= \phi, \quad u_i = (-1)^i \phi U_i (i = 1,2, \cdots, r), \\ v_0 &= \psi, \quad v_j = (-1)^j \psi V_j (j = 1,2, \cdots, n), \end{aligned} \tag{1}$$

where ϕ, ψ are parameters, and U_i, V_j are homogeneous polynomials, with rational integral coefficients, of the respective degrees i, j, in some or all of n parameters $\theta_1, \theta_2, \cdots, \theta_n$. Precisely, U_i in the final solution is the ith elementary symmetric function of $\theta_1, \theta_2, \cdots, \theta_r$, and V_j is the jth elementary symmetric function of $\theta_1, \theta_2, \cdots, \theta_n$. The simultaneous system will therefore have an $(n+2)$-fold infinity of solutions beyond the trivial one in which each indeterminate is zero.

The existence of such a solvable system is evident from two obvious remarks. The polynomial $F(\theta)$ in the indeterminate θ,

$$F(\theta) \equiv \theta^r - U_1 \theta^{r-1} + \cdots + (-1)^r U_r$$

vanishes identically for $\theta = \theta_i$ $(i = 1, \cdots, r)$, and

$$G(\theta) \equiv \theta^n - V_1 \theta^{n-1} + \cdots + (-1)^n V_n$$

vanishes identically for $\theta = \theta_j$ $(j = 1, 2, \cdots, n)$. This remark will give us the solution of the simultaneous system obtained as follows. Write

$$\begin{aligned} f(\theta) &\equiv v_0 \theta^r + v_1 \theta^{r-1} + \cdots + v_r, \\ g(\theta) &\equiv u_0 \theta^n + u_1 \theta^{n-1} + \cdots + u_n. \end{aligned}$$

[1] Bulletin de l'Académie des Sciences, St. Petersbourg, 1853, p. 72.

In order that $g(\theta)$ be exactly divisible (algebraically) by $f(\theta)$, it is necessary and sufficient that the remainder on division vanish identically. But it is also necessary and sufficient that the r values of θ for which $f(\theta)$ vanishes identically shall be r of the n values for which $g(\theta)$ vanishes identically. Equating to zero the several coefficients of $\theta^r, \theta^{r-1}, \cdots, 1$ in the remainder, we get the simultaneous diophantine system described; applying the second remark, we have its parametric solution.

In the actual construction and solution of the system it is sufficient to postulate that the indeterminates and parameters are elements of an abstract ring (≡set of marks closed under addition, subtraction, multiplication), with the single modification that division by u_0 is defined for elements of the ring generated from $u_0, u_1, \cdots, u_n$. This modification is, however, unnecessary if mathematical induction alone be used to justify all the steps. Thus finally it is necessary to postulate only that the elements and indeterminates are elements of an abstract ring. Otherwise stated, we reach a system of identities between marks that can be proved by mathematical induction on using only addition, subtraction and multiplication.

2. Solvable system. With n, r, s and the u's, v's as in §1, define the $R_{t,j}$ by

$$R_{0,j} = v_{n-j} \quad (j = 0,1, \cdots, n) ; \tag{2}$$

$$R_{t,j} = 0 \quad (0 > j > n - t) ; \tag{3}$$

$$u_0 R_{t+1,n-t-j} = u_0 R_{t,n-t-j} - u_j R_{t,n-t} \quad (j = 1,2, \cdots, r) ; \tag{4}$$

$$R_{t+1,n-t-r-j} = R_{t,n-t-r-j} \quad (j = 1,2, \cdots n - , t - r), \tag{5}$$

of which (3), (4), (5) are to hold for $t=0, 1, \cdots, s$.

Having calculated for a given value of t the successive $R_{t,j}$ by these recurrences, we construct the $P_{t,j}$ defined by

$$P_{t,n-j} \equiv u_0{}^t R_{t,n-j} \quad (j = 1,2, \cdots, r) ; \tag{6}$$

$$P_{t,n-r-j} = R_{t,n-r-j} \quad (j = 1,2, \cdots, n - r). \tag{7}$$

From (2)–(5), either by mathematical induction, or by the algebraic division of $g(x)$ by $f(x)$ as suggested in §1, it follows that $P_{t,n-j}$ is a homogeneous polynomial of degree $t+1$ and isobaric of weight j in (some or all of) $u_0, u_1, \cdots, u_r, v_0, v_1, \cdots, v_n$. Counting $f(x)$ as the *zero*th stage, we see that $R_{t,n-j}$ is the coefficient of θ^{n-j} in the remainder at the tth stage of the division, in which, by (3) or otherwise, the highest power of θ is θ^{n-t}.

The division will terminate when $t=s+1$. The last remainder is therefore

$$\sum_{j=1}^{r} R_{s+1,n-s-j}\, \theta^{n-s-j} \equiv \sum_{j=0}^{r-1} R_{s+1,j}\theta^j .$$

Hence the r polynomials $R_{s+1,j}$ $(j=0, 1, \cdots, r-1)$ are homogeneous of degree $s+2$ and isobaric of the respective weights $n, n-1, \cdots, n-r+1$.

It follows, as outlined in §1, that the parametric solution of the simultaneous system

$$R_{s+1,j} = 0 \quad (j = 0,1, \cdots, r-1) \tag{8}$$

is as given by §1(1). The system (8) is that described in §1. Similar systems solvable in terms of parameters arise in the same way from the H.C.F. and from resultants.

When $r=2$, the final system consists of two equations which, by elimination, reduce to a single equation. For example, if $r=s=2$, $n=4$, the equation is homogeneous of degree 4, isobaric of weight 5, in 7 indeterminates, namely,

$$u_0^2(u_2v_3 - u_1v_4) - u_2^2(u_0v_1 - u_1v_0) = 0 ;$$

of which the parametric solution, by (1), is

$$u_0 = \phi ; \; -u_1 = \phi(\theta_1+\theta_2) ; \; u_2 = \phi\theta_1\theta_2 ; \; v_0 = \psi ; \; -v_1 = \psi(\theta_1+\theta_2+\theta_3+\theta_4) ;$$
$$-v_3 = \psi(\theta_2\theta_3\theta_4 + \theta_1\theta_3\theta_4 + \theta_1\theta_2\theta_4 + \theta_1\theta_2\theta_3) ; \; v_4 = \psi\theta_1\theta_2\theta_3\theta_4 ;$$

as may be verified by direct substitution.

From the necessity and sufficiency of the two sets of conditions by which the system (8) and its solution (1) were obtained, it follows that (1) is the general solution of (8).

SRINIVASA RAMANUJAN

By B. M. SRIKANTIA, Mysore, India

Srinivasa Ramanujan was born on the 22nd of December, 1887 at Erode, a picturesque spot in South India at the confluence of the Bhavani and the Cauvery. His parentage was humble. He was put to school at the early age of five, and before he was seven years old found himself in the Town High School, Kumbakonam. Even at that young age, he used to puzzle his teachers with intricate questions about "zeros and imaginary quantities," the "distances of stars", and the like.

From his boyhood, Ramanujan was of a contemplative mood and sedentary in habits. He seldom mixed with his schoolmates at play. His early life at school may be summed up in the words of a classmate of his at Kumbakonam as follows:

"His teachers had already noticed his extraordinary and precocious intellect. He used to borrow Carr's *Synopsis* from the College Library and took delight in verifying some of the formulas given there. He would clear in half the time

examination papers in algebra and geometry, and a few seconds' thought always used to suggest to him the solution to any question however difficult. He used often to entertain his friends with his 'theorems' and 'formulas' even in those early days, which doubtless appeared to his hearers as mathematical tricks. He had a distinctly religious turn of mind and was a staunch believer in Hinduism. He had an extraordinary memory and could repeat the complete lists of Sanskrit roots. He could give the values of $\sqrt{2}$, π, e, etc., to any number of decimal places. In manners he was simplicity itself."

After passing the matriculation examination from the Town High School in 1903, he entered the Government College at Kumbakonam in 1904 and secured the Junior Subramaniam Scholarship as a result of a competitive test in "Mathematics and English Composition." Owing to weakness in English, he failed to get promotion from the Junior F. A. to the Senior F. A. class and lost his scholarship. Leaving Kumbakonam he went first to Vizagapatam and then to Madras. He presented himself for the F. A. Examination from the Pachaiyappa's College in December, 1906, without success and never tried again. For the next three years he was working at mathematics in his own way, jotting down his results in two good-sized note-books, one of which is now with Professor G. H. Hardy.

In 1909, he married and his college career came to an abrupt end.

He went to Tirukoilur in search of employment and met Mr. T. Ramaswami Aiyar, founder of the Indian Mathematical Society. The latter, seeing his wonderful gifts, persuaded him to return to Madras so as to be in touch with mathematics. Dewan Bahadur R. Ramachandra Rao generously helped Ramanujan to stay at Madras for a time and finally, in 1911, unwilling to be a burden on anybody, Ramanujan took up a small appointment under the Madras Trust Board.

He never lost touch with mathematics all this time. His earliest contribution to the Journal of the Indian Mathematical Society was published in the December number of volume 3. The editor had to return the article in question to the author more than once to be rewritten in a form suitable for publication. Ramanujan's methods were extremely slip-shod and no reader could ever follow him. Though his methods were not rigorous, the formulas behaved correctly and were the wonder of mathematicians. It was during this period that he went one day to Professor Seshu Aiyar with some theorems on prime numbers and when asked to refer to G. H. Hardy's tract on *Orders of Infinity*, he is said to have observed that on page 36 of that tract, Hardy had said: "the exact order of $p(x) = \phi(x) - \int_2^x dt/\log(t)$ where $\phi(x)$ denotes the number of primes less than x, has not yet been determined," and that he himself had discovered a result giving the order of $p(x)$. On this, Professor Seshu Aiyar suggested to Mr. Ramanujan that he open correspondence with Professor

Hardy. The suggestion was at once acted upon, and proved the turning point in his career. Hardy took a keen interest in Ramanujan and his work and proposed that he should be induced to go to Cambridge. Meanwhile Dr. G. T. Walker, F. R. S., Head of the Meteorological Department (formerly, Fellow and Lecturer of Trinity College, Cambridge) while on a visit to Madras in 1912, noticed the quality of Ramanujan's work and at once drew the attention of the Government and the University of Madras to its excellence. A research studentship of Rs.75/- per mensem was awarded him for two years. Ramanujan thus became a professional mathematician and remained so for the rest of his life.

Though Ramanujan declined to go to England at first on religious grounds, the joint efforts of Mr. E. H. Neville (then of Trinity College, who had come to Madras to deliver a course of University Lectures), Sir Francis Spring, President of the Port Trust, and the Honorable R. Littlehailes succeeded in inducing him to sail for England in March, 1914, on a liberal Government scholarship of £250 for three years.

This event is graphically described by Dewan Bahadur Ramachandra Rao in the following words:

"And slowly they (Cambridge mathematicians) came to see that there was something in Ramanujan they are not aware of—results true and unknown and methods irregular. This latter did not suit their training. So efforts were made to induce Ramanujan to go to Cambridge to study proper methods and to publish his results in the accepted strait-coat moulds."

"Ramanujan consulted me and, perhaps unfortunately, I lent all the weight of my influence to induce him to go. I was tempted to do so because, in the first place, I knew that unless he published in English journals, his results would not be advertised; and the only way of benefiting India, if not himself, was by this course. And, in the second place, there was no place in India stocked with up-to-date mathematical literature where a higher mathematician could easily find books of reference. Ramanujan demurred a long time. And it is well known to his friends that his ultimate decision he always attributed to divine inspiration."

"The last two or three days of his stay in India before he proceeded to England he spent with me in a mofussil station. I took him to the house of a friend living in European style to initiate him in the mysteries of the fork and the knife under the strict stipulation that nothing but vegetable food should be served. He was not happy. He did not relish food being served by strange servants. He was not very jubilant over his future journey and over his assured distinctions to come."

As regards Ramanujan's work in England we cannot do better than quote from Professor Hardy's report to the University of Madras (1917):

"I need hardly say that he has in his possession, as he had before he came to England, a mass of unpublished material containing hundreds of most curious and interesting formulas, some definitely proved and others only conjectured"

"In one respect Mr. Ramanujan has been most unfortunate. The war has naturally had disastrous results on the progress of mathematical research. It has distracted three-quarters of the interest that would otherwise have been taken in his work, and has made it almost impossible to bring his results to the notice of the continental mathematicians most certain to appreciate it. It has moreover deprived him of the teaching of Mr. Littlewood, one of the great benefits which his visit in England was intended to secure. All this will pass; and in spite of it, it is already safe to say that Mr. Ramanujan has justified abundantly all the hopes that were based upon his work in India, and has shown that he possesses powers as remarkable in their way as those of any living mathematician. His work is only the more valuable because his abilities and methods are of so unusual a kind and so unlike those of a European mathematician trained in the orthodox school."

"India has produced many talented mathematicians in recent years, a number of whom have come to Cambridge and attained high academical distinction. They will be the first to recognize that Mr. Ramanujan's work is of a different category. In him India now possesses a pure mathematician of the first order, whose achievements suggest the brightest hopes for its scientific future."

At Cambridge, Ramanujan was given the research degree of B.A. and in recognition of his great ability in mathematics he was elected a Fellow of the Royal Society in 1917. In the year 1918, he was granted a Fellowship of Trinity. The Madras University gave him an annual grant of £250/– (without imposing any conditions regarding residence or work) for five years. But he did not live long to enjoy this annuity.

An incurable disease, impossible of diagnosis, was undermining his health during the period of his stay at Cambridge. He never permitted strangers to cook his food for him and he scrupulously stuck to his accustomed Indian food. In spite of the best medical aid, his frail physical body never regained strength, but his mental vigour and his intellect were as keen as ever. His was truly a life of philosophic resignation and self-sacrifice.

Here again we quote from Professor Hardy:

"It was in the spring of 1917 that Ramanujan first appeared to be unwell. He went into the Nursing Home at Cambridge in the early summer, and was never out of bed for any length of time again. He was in sanatoria at Wells, at Matlock, and in London, and it was not until the autumn of 1918 that he showed any decided symptom of improvement. He had then resumed active

work, stimulated, perhaps, by his election to the Royal Society, and some of his most beautiful theorems were discovered about this time. His election to a Trinity Fellowship was a further encouragement; and those famous Societies may well congratulate themselves that they recognized his claims before it was too late. Early in 1919, he had recovered, it seemed, sufficiently for the voyage home to India, and the best medical opinion held out hopes of a permanent restoration. I was rather alarmed by not hearing from him for a considerable time; but a letter reached me in February, 1920 from which it appeared that he was still active in research."

Poor Ramanujan returned to India only to die. He came full of honours and distinctions; but he had been reduced to a skeleton and was in the grip of that dire disease, tuberculosis. Ramanujan knew that his end was near and all that his friends and relations could do was "to lighten his pains and keep alive the lingering flame of his life for as long a while as possible." He passed away peacefully on the 26th of April, 1920, surrounded by a limited circle of friends, but mourned by an ever-increasing circle of admirers.

Ramanujan's activities[1] were chiefly confined to topics of pure mathematics, such as the theory of numbers, continued fractions, and partitions[2]. He knew the idiosyncracies of integers in a remarkable degree. When he was lying ill at Putney, Professor Hardy went to see him and told him that he had ridden a taxi-cab No. 1729 ($7\times13\times19$) and remarked that the number seemed rather a dull one. "No" he replied, "It is a very interesting number. It is the smallest number expressible as a sum of two cubes in two different ways." If he had been trained in the conventional orthodox style, he would not have been the genius he was. "Drilling" and inventiveness are perhaps antagonistic faculties and cannot thrive together. That is the reason why the greatest inventors are generally non-University men.

Ramanujan was full of puzzles, and inconsistencies. He could solve modular equations of any order, and he had discovered independently the fundamental equations satisfied by the zeta-functions. At the same time, he had no idea of the essentials of a mathematical proof. His theorems and formulas were mostly the result of "intuition" or "induction."

What struck one most was his extreme simplicity and retiring disposition. He never sought publicity. He did his work unostentatiously without caring for wealth or title. He was a true Indian Saint in his outlook.

[1] Ramanujan's *Collected Papers*, edited by G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson, (xxxvi+335 pages) were published in 1927 by the Cambridge University Press. EDITOR.

[2] See Aubrey J. Kempner, *The development of "partitio numerorum," with special reference to the work of Messrs. Hardy, Littlewood, and Ramanujan*, this Monthly, vol. 30 (1923), pp. 354-369, 416-425.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. NEW METHODS IN EXTERIOR BALLISTICS

By F. R. MOULTON, Chicago, Illinois

In his review, published in the June–July (1927) number of this magazine, Mr. Roger Sherman Hoar extravagantly praises my recent work on "Exterior Ballistics" for reasons that do not seem to me important; and, on the other hand, he criticizes it for reasons that seem to me trivial. Although I was one of the first to contribute to the development in ballistics that has been made during the last ten years, I have not heretofore taken part in the rather abundant discussions of the subject. In order primarily to place on record the value I put on what has been accomplished, and in part to answer some of Mr. Hoar's criticisms, I present these few comments on the subject.

The first thing I shall consider is the computation of a trajectory. The methods in use before the war had little to recommend them, either from the theoretical or the practical point of view. Immediately on entering the service I introduced a general and convenient method of obtaining the numerical solution of the problem in any particular case. This work contained an explicit exposition of the whole problem, including the tabulation in convenient form of the G-function and the H-function and detailed printed computing schedules, and it was applied by computers under my direction to many hundreds of special cases. The method is applicable equally whatever be the variables in terms of which the problem is expressed.

The statement has often been made that I, being familiar with the mathematical methods used by astronomers, naturally introduced this method of computing trajectories. There seems to be a widespread belief that astronomers use it in determining the orbits of comets, in the lunar theory, and in perturbation theories. Nothing could be more remote from the facts. Astronomers as a class are as innocent of any acquaintanceship with this method of solving differential equations as most mathematicians were before the war. Probably not one out of twenty of them ever heard that such a method exists; and though the underlying principles of it have been used by a few mathematicians in computing special perturbations, the general procedure they employed has been so different in form from that used in ballistics that one would not easily detect the connection. So far as I am concerned, I did not learn the method from

astronomy, but from a general survey of the various processes that are known for solving differential equations.

The introduction of the method of solving numerical differential equations is so simple and obvious that any one familiar with the general field of differential equations would hardly fail to do substantially what I did. But I did make a contribution connected with it which I regard, from the mathematical point of view, as being very important. I laid down for the first time explicit conditions under which the process is valid in a strict mathematical sense. This proof was issued in blue print form (No. 69) in the autumn of 1918 by the Ballistic Branch of the Technical Staff of the Army Ordnance Department, and it constitutes Chapter V of my recent book on ballistics. One having any considerable degree of mathematical sophistication would not feel at liberty to ignore the question of the validity of the process on which he bases all his conclusions. Unfortunately for those interested in the applications, the proof of the validity of the method involves many of the niceties and much of the close reasoning characteristic of modern analysis, and it would be wholly beyond the appreciation of a beginner. Most of those who have commented on my work either have been blind to the necessity of proving that the process is mathematically sound or have assumed that its validity was established in some indefinite past.

The second question on which I shall make a few remarks is that of determining differential variations. In the first blue print (No. 1) issued very shortly after I entered the service in the spring of 1918, I laid the complete foundation for this theory and worked out all the details necessary for numerical applications. The only essential addition to this work that I have seen is the introduction of the adjoint system of differential equations, by Professor Bliss, for computing the terminal variations. Not only did I completely develop the problem so far as it depends upon the terms of the first order defined by linear differential equations, but in a supplementary blue print (No. 68) I gave the general theory for terms of all orders. This is given briefly in section 29 of my book on ballistics. Dr. Ritt made a computation for a special trajectory of the terms of the second order.

The third question which I shall mention is the theory of a rotating projectile describing a curved trajectory in a medium of varying density. The last chapter of my book gives a complete treatment of the problem and constitutes what I regard as my most important contribution to the subject of ballistics. It is my opinion that all essential future improvements in projectile design and firing control will be guided by such theory. Incidentally, I may remark that this part of the work cost me at least three times as much effort as all the remainder. While this work was in preparation, a noteworthy paper on the same subject was published by Fowler, Gallop, Lock, and Richmond in the Philo-

sophical Transactions (London), Series A, vol. 221, pp. 295–387. The methods used by these authors are entirely different from those which I employed.

In Mr. Hoar's comments upon my book he makes much of the fact that I did not use the notations that he adopted in his book, "A Course in Exterior Ballistics," in 1920. He says that his book is more up-to-date than the one that I have recently published. The purposes of the two books are so different that it would be difficult even for an impartial person to compare them justly. Mr. Hoar's book was prepared for use in a course on ballistics at the Aberdeen Proving Ground. In the introduction the author says, "It is assumed that the student is thoroughly grounded in algebra and plane trigonometry, and knows enough calculus to appreciate the meaning of a derivative, a differential, and a definite integral." It seems to me that the sentence quoted excellently defines the scientific level of the book.

Perhaps an illustration of the reasoning employed in Mr. Hoar's book will be illuminating. On page 28 the existence of a definite integral under certain conditions is inferred as follows:

"Let us now imagine a plotted curve with u for its ordinates and t for its abscissas. Then $\int_{t_n}^{t_m} u dt$ is the area between the curve, the $t=$axis, and the ordinates at $t=t_n$ and $t=t_m$. Such an area exists for every continuous curve, and hence every continuous curve has an integral, even though that integral is not expressible by means of the ordinary elementary functions."

So far as the computation of trajectories is concerned, Mr. Hoar gives an explanation, with no reference to proofs, of the method that had been used in a great number of cases. He seemed to think what he called "integrating forward" possessed some novelty. It involved only the substitution of certain differences for others by means of the linear relation that exists among them, and the variation was employed by certain of the computers in Washington and has been well known since before any of us were born. Another point on which Mr. Hoar lays stress is his choice of coordinates such that the y coordinate is the vertical height above the surface of the spherical earth. It is true that I made no mention of these coordinates. If one were interested in computing a single or a few unrelated individual trajectories, these coordinates might be used, but even in this case I think the cost would be much greater than the gain. My treatment, however, aimed at the construction of general ballistic tables by the aid of the machinery developed in Chapter IV of my book. With this broad and important problem in mind, and the enormous advantage of obtaining from a single computation a one-parameter family of trajectories, and the equal advantage of using fundamental sets of solutions of the differential equations for variations, it would have been inexcusable stupidity to have adopted Mr. Hoar's system of coordinates and thus to have sacrificed these advantages.

The theory of differential variations depends upon linear differential equations. The very heart of any adequate theory of them obviously is attached to that of a fundamental set of solutions of the differential equations. In my treatment of the problem I went straightway to a fundamental set of solutions and made essential use of them in most of the subsequent discussions of the chapter. Mr. Hoar uses no such general and powerful machinery—he does not have a syllable on it. Yet he writes, "The differential corrections, to which Professor Moulton devotes nearly the whole of Chapter IV, have been obsolete since 1919; being superseded by the Bliss system, to which Moulton devotes but one section." It is difficult for me to see how one who is not wholly innocent of a real understanding of my work and that of Professor Bliss on the subject could write such a statement. In the first place, I gave a complete exposition of Professor Bliss's use of the adjoint system for computing terminal variations. In order to be doubly sure that nothing important was overlooked, I placed in the hands of Professor Bliss the manuscript, and later the proofs, of the entire chapter on differential variations, which includes his use of the adjoint system. He read the manuscript and the proofs and returned them with comments that can only be taken to mean that he does not agree with the statement of Mr. Hoar. Mr. Hoar objects to my statement that the method of using the adjoint system is "not adapted to determining all the terminal variations of a trajectory." Perhaps I should have modified the word "adapted" by the adverb "conveniently." By virtue of the relations between the original set of variables and the adjoint set, it is possible, as Professor Bliss showed, to determine all the characteristics of a trajectory, but that does not mean that it is desirable to do so. One can, if he wishes to insist on the possibilities, go from New York to Boston by way of Buffalo, but it is not the desirable route.

Mr. Hoar has a flair for superlatives, which is illustrated by, "Gronwall discovered a new first integral which revolutionized the computations." Perhaps he was misled by the great detail with which Dr. Gronwall set forth his results in his paper in the Transactions of the American Mathematical Society, vol. 22. He devotes many pages to writing out the elementary transformations to derive my equations for the differential variations and those of Professor Bliss for the adjoint system. As for the "discovered new first integral," it is a direct consequence in this particular problem of general principles which have been well known for at least fifty years. The word "discovered" actually means no more than substituting in a general formula coefficients from the particular equations under consideration, and the statement that this integral has "revolutionized the computations" is somewhat rhetorical.

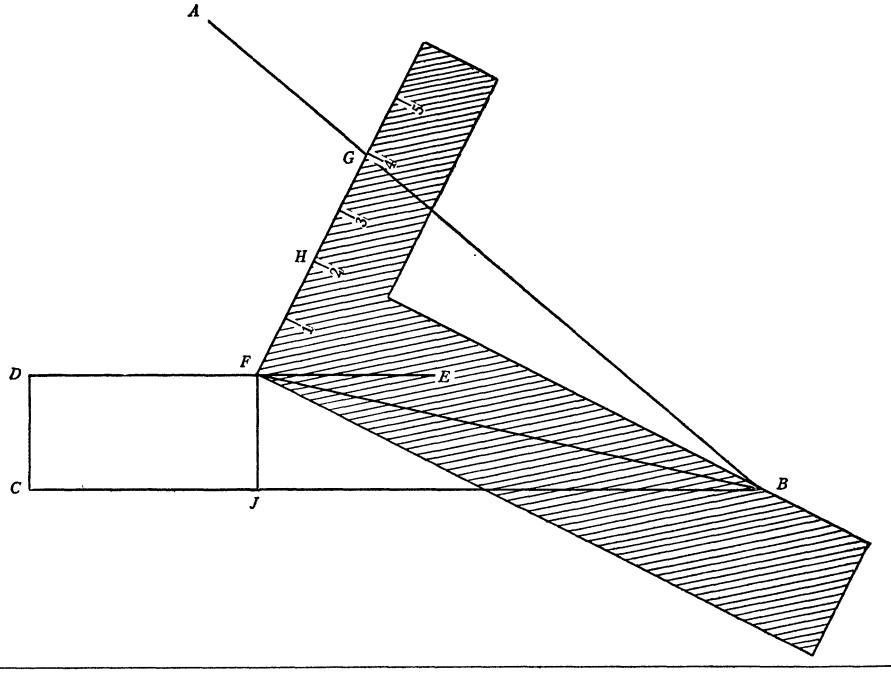
As has been said in an earlier paragraph, the problem of the motion of a rotating projectile along a curved path in a medium of varying density is by far the most difficult and the most important one of exterior ballistics. The

writings of Mr. Hoar contain no reference to it; yet he would lead the reader to infer that his book is the last word on the subject, rather than one which deals crudely with only the more elementary parts of it. Mr. Hoar lays great stress on what he calls "modern symbols," which consists largely in changing the function I called F to E, adopting a left-hand system of coordinates in place of the right-hand system which I used, and counting azimuth from the north rather than from the south. Apparently these conventions were "officially adopted" after a correspondence among three interested individuals. It is quite likely that their successors will officially adopt variations from their notations just as they adopted variations from mine, particulary as the advantages are all with the right-hand system.

II. How to Trisect an Angle with a Carpenter's Square[1]

By Henry T. Scudder, Marinette, Wisconsin

Let ABC be the given angle. By means of the larger arm of a carpenter's square, which is two inches wide, draw the line DE parallel to and two inches distant from the line CB. Adjust the square (as indicated in the diagram)



[1] Note by the Editor: This construction for the trisection of an angle is of the familiar type involving three-point contact of an instrument with a drawing. For other constructions of that type, the reader is referred to August Adler's *Theorie der geometrischen Konstruktionen;* to F. Enriques' *Fragen der Elementargeometrie*, vol. 2; and to K. Th. Vahlen's *Konstruktionen und Approximationen.*

so that the inner edge of its larger arm is on the point B, its outer corner is on the line DE (at F), and the mark for four inches on the shorter arm is on the line AB at G. Make dots at F and G and half way between them (at the mark for 2 inches) at H. Remove the square. Draw lines from the dots at F and H to the point B. These lines, FB and HB, trisect the given angle ABC: because if we draw the line FG and the line FJ perpendicular to CB, we shall have three congruent triangles with their equal angles at B.

III. Solid Geometry and the New Curricula

By Elizabeth B. Cowley, Pittsburgh Public Schools[1]

The insertion or omission of a required course in the curriculum has an important and far-reaching effect on the success, and sometimes even on the very existence, of a subject. On the other hand, it must not be forgotten that the reason that certain courses, formerly required, are elective in the new curricula is to be found, partly in the nature of the subjects themselves, partly in the type of clientele served, and partly in the inefficiency of some of those persons who have been highly paid to teach these subjects.

Although a few of the recent college curricula are so bizarre that we wonder who made them, nevertheless, there seems to be a tendency at present to allow the student to choose one from a number of courses. Usually each course has certain required subjects and a liberal allowance of electives.

Why has the former requirement of solid geometry in the freshman year been abolished by practically every college? Obviously no one reason can be given which will be valid in all cases. College administrators report that there have been many complaints against this subject from students. We must agree that in some cases these complaints are fully justified. For example, a student who comes from an excellent high school where he had a course in plane geometry taught by an alert and inspiring and capable teacher who regarded teaching as a worthy pursuit will naturally complain if he is assigned in college to a dull and untrained instructor or to a professor who scorns teaching and spends his time and energy upon his own private research to the neglect of his students. But there are other complaints against solid geometry that really condemn the students rather than the subject. These pupils, who object to any course that attempts to do any serious work, would like someone to "take the solid out of solid geometry."

The omission of the solid geometry requirement does not prove beyond a doubt that the members of the faculty honestly disapprove of the subject. If a college curriculum is the result of faculty legislation, it is apt to show the

[1] Dr. Cowley is on leave of absence from Vassar College.

scars of battle, for in the fierce competition between departments the large aims and ideals of education are lost sight of in the struggles of the departments for self-preservation.

Many who really wish to abolish the college requirement in this subject hold the theory that as knowledge advances it is inevitable that some subjects, once considered as all-important, should gradually be shoved aside to make way for other branches. They recall the disappearance of spherical trigonometry. These critics would replace college solid geometry by an introduction to the calculus. There is much to be said in favor of this point of view. While the calculus is working its way down into the freshman year in college (and, in a simplified form, into the high school), there is a natural tendency to place more stress upon solid geometry in the high school. Why should this subject be held back till the thirteenth school year? Demonstrative plane geometry is usually considered a subject for the tenth school year. Solid geometry is offered as an elective in many high schools and is available to students in the eleventh or the twelfth school year. Those members of college faculties who adhere strictly to academic caste, usually regard such a course as inferior to the college course. Has any one of these "standard-bearers" taught this subject in college and also in a present day high school? The writer, who taught various branches of mathematics in a large college for many consecutive years, is now spending a leave of absence teaching plane and solid geometry in a coeducational high school in a large city and doing research work on the mathematics curriculum for the senior high schools of that city.

When comparisons are made between the high school and the college courses in this subject, much emphasis is often placed upon the difference in the ages of the students. Although the high school students are one or two years younger, there is a compensation in the fact that they spend more than twice as many hours under the teacher's guidance. (In the city high schools there are twenty weeks each with five fifty-minute periods. The nominal college semester of sixteen weeks seldom provides forty class periods, even if the teacher does not "give cuts.") Students who have studied this subject in a good high school compete successfully with those who have had the college course when they all take uniform examinations in the subject or when they pursue college courses in analytic geometry and in the calculus.

There is a tendency to-day to urge that the tenth year course in demonstrative geometry include some solid geometry—omitting parts of the plane if necessary. If the sections are arranged according to the ability of the students, it is entirely possible to do this with the best section. But in many schools where there are Commercial Courses and Practical Arts Courses as well as the academic it is not possible for the administration to section the geometry classes in this way. However, any live teacher of plane geometry can bring in a bit of

the solid incidentally here and there as a means of emphasizing certain plane concepts. The writer has been experimenting this year in another way, too. If one or two boys mastered an assignment before the other members of the class had made much progress, they were given some carefully selected bit of solid geometry (without a text). For instance, the theorem about a straight line perpendicular to each of two intersecting lines is especially adapted to this purpose, for the boys are quick to pick out the congruent triangles. The aim is not to cover a predetermined set of theorems or to coach for an examination, but to give the boys a taste of solid geometry and to awaken a desire to elect the subject later.

The writer has also been experimenting in the matter of tests and examinations. Tests of the new type have been constructed for use at various stages in each subject. In this high school there are no compulsory final examinations at the end of the semester, but there is a "report period" with an "honor roll" at the end of the sixth, thirteenth, and twentieth week. Toward the end of each course, the students were given the option of taking a routine review (following an outline) or answering one (or more) College Entrance Board Examinations of previous years. As it was not possible to have the two or three consecutive hours required, it was necessary for each student to keep strict account of his time while working on the paper. The writer's experience in marking the papers of college freshmen taking examinations set by the department and the experience as a reader on the College Entrance Examination Board furnished a basis of comparison in judging these voluntary answer books in high school. The freshness and freedom from tension were in marked contrast to that jaded and weary tone too often noticeable in answer books where examinations are compulsory and all-important. Other experiments are in progress.

This controversy over solid geometry will eventually be beneficial to the subject, if it arouses the teachers and challenges them to investigate the possibilities and the needs of their students, and to study the subject, not merely for itself, but as a means of developing the students.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

Non-Riemannian Geometry. By LUTHER PFAHLER EISENHART. New York, The American Mathematical Society, Colloquium Publications, Volume VIII. viii+184 pages.

This book contains, with subsequent developments, the material presented by the author at the

* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

Ithaca Colloquium, in September 1925, under the title, The New Differential Geometry. "The generalization consists in using general functions of the coordinates in the formulas of covariant differentiation in place of the Christoffel symbols formed with respect to the fundamental tensor of a Riemannian space."

Plane Trigonometry. By N. J. LENNES and A. S. MERRILL, with the editorial cooperation of H. E. SLAUGHT. New York, Harper and Brothers. x+180 +72 pages.

"A flexible book adaptable to various types of courses and for teachers with varied methods of teaching." There is valuable supplementary material and a historical sketch.

College Algebra, by ARTHUR M. HARDING and GEORGE W. MULLINS. New York, The Macmillan Company. vi+324 pages.

"No radical departure from the standard material usually included..... Frequent use of the graphical method for the purpose of clarifying the processes of algebra rather than on its own account. Introduction of the derivative direct and simple, and the graphical illustration will make it easily understood."

College Algebra. By KENNETH P. WILLIAMS. Boston, Ginn and Company. xvi+312 pages. $2.00.

"In this book emphasis is given to [the fact that] algebra is not a mere matter of technique, and the effort is made to keep before the student the more estimable nature of the study."

Plane and Solid Analytic Geometry. By JAMES MCGIFFERT. Boston, Ginn and Company. xiv+338 pages. $2.48.

Analytic Geometry. By EDWIN S. CRAWLEY and HENRY B. EVANS. Revised Edition. New York, F. S. Crofts & Co. xiv+244 pages. $2.00.

Modern Solid Geometry. By JOHN R. CLARK and ARTHUR S. OTIS. Yonkers, New York, World Book Company. xx+138 pages.

This is a companion volume to the *Modern Plane Geometry* of the same authors, reviewed in this Monthly in the March, 1928 issue (p. 141).

REVIEWS

The Logarithmic Potential. By G. C. EVANS. American Mathematical Society Colloquium Publications, Vol. VI. New York, Published by the Society, 1927. viii+150 pages.

The work before us is a beautiful treatment of some modern results in potential theory, with the Dirichlet problem as fundamental, particularly as generalized to include certain discontinuous boundary values. The central result is a necessary and sufficient condition that a function harmonic in the unit circle can be represented as a Poisson-Stieltjes integral, and a necessary and sufficient condition that such a function can be represented as a Poisson-Lebesgue integral. Both these results are capable of wide and important generalizations and applications.

The first chapter is introductory, containing proofs of the usual properties of functions of bounded variation, and in addition useful theorems on the

(deleted) neighborhood of a point, the function has at most a removable singularity at that point. It is rather surprising that Evans ascribes this theorem to Lebesgue,[1] 1923. The theorem is given in Osgood's *Funktionentheorie*,[2] 1912, to which Evans makes several references in other parts of his book, and Bôcher indicated a proof of his own and published Osgood's original proof as early as 1903, in a paper[3] to which Evans refers in another place (p. 111). The reviewer is not aware of any statement of the theorem in the literature prior to 1903.

Further detailed consideration of the text would be out of place in so short a review as the present one. Professor Evans is to be congratulated on having written so elegant and so useful a book, with much of the material original with him, and the American Mathematical Society is to be congratulated on the continued high standard of its Colloquium Publications.

J. L. WALSH

Plane Analytic Geometry. By I. A. BARNETT. John Wiley & Sons, 1926. 269 pages.

This text, written for a half-year course in plane analytic geometry, contains the subject matter usual in such a text. However the author has endeavored to present his material in a new way and the success of the text will depend upon the success of his method of presentation. He follows a plan somewhat similar to that of a beginner's Latin book; that is, he gives from time to time what he calls a "dictionary." This dictionary, or vocabulary, is gradually extended until, in the closing summary, there are thirty-eight geometrical terms defined algebraically. This will furnish an excellent drill for review work and will also greatly assist the student who is taking his first steps in the solution of problems. The author has kept in view the development of analytical power rather than the mere application of rules. Another feature of merit is the unusual emphasis given to the geometric method in the solution of algebraic problems. On the other hand, there are several features of the book which are unsatisfactory. In defining coordinates, for instance, in his very first illustration

[1] Comptes Rendus, t. 176, p. 1097.

[2] Leipzig, 1912, p. 647. The theorem in question is also given in the first edition of Osgood's *Funktionentheorie*, Leipzig, 1907, p. 565.

[3] Bulletin of the American Mathematical Society (2) vol. 9, p. 460.

It seems probable that this theorem for harmonic functions in two dimensions was discovered independently and nearly simultaneously by Osgood and Bôcher. Bôcher (loc. cit.) states the theorem in question, indicates very briefly a proof of his own which is in an entirely different spirit from the rest of his paper, and then gives Osgood's proof. This proof of Osgood's holds for n dimensions, while Bôcher's does not. Osgood gives later a different proof of the theorem for two dimensions, in the first edition of his *Funktionentheorie* (loc. cit.) without reference to Bôcher.

Kellogg has recently indicated the importance of Bôcher's paper in connection with this same theorem, Bulletin of the American Mathematical Society, vol. 32 (1926) pp. 664–668.

and figure he uses a case where the abscissa is negative. Again, on the third page, before the reader has a proper idea of what coordinates are, he introduces translation of axes. The early introduction of transformation of coordinates is excellent but it ought not to precede an understanding of what coordinates are. The idea of locus is one that the pupil brings to analytics from his geometry and possibly from his algebra, and yet the author does not introduce *locus* until page 70, near the end of the discussion of straight line. The text contains a new convention in regard to the distance of a line from the origin and of a point from a straight line, "the value of f is always positive when the given point P_1 is above the given line L and d is always negative when P_1 is below L." This seems unfortunate, even though it is a good and useful convention, since it goes contrary to the accepted convention already in use. The book has many good features and the teacher will find it helpful for reference.

R. P. STEPHENS

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE MATHEMATICS CLUB OF THE COLLEGE OF THE CITY OF DETROIT, Detroit, Michigan.

The Mathematics Club of the College of the City of Detroit, organized January, 1926, presented to the students of the college an interesting and varied program during the year 1926–1927. The club is sponsored by the senior mathematics students and the faculty of the mathematics department, and the meetings are open to anyone interested.

The programs presented by the club during the past year included papers on the following subjects:
December 16, 1926. "Imaginary elements in geometry" by Mr. Max Coral.
February 23, 1927. "Geometric exercises in paper folding" by Miss Gladys Eesley.
March 23. "Sir Isaac Newton, his life and work" by Miss Elinor Batie.
April 5. "The construction of the icosahedron" by Professor J. W. Bradshaw, University of Michigan.
May 5. "Line geometry" by Professor Alfred S. Nelson.

Officers for the first part of the year were Mr. M. Coral, president and Miss Frances Garvey, secretary. For the latter part of the year Mr. Coral was president and Miss E. Batie was secretary.

(Report by Miss Elinor M. Batie)

THE MATHEMATICS CLUB OF SMITH COLLEGE, Northampton, Massachusetts.

The officers for the year were: Margaret Hilferty ('27), president; Madeline Bang ('27), secretary; Carolyn Gibby ('27), treasurer; Pauline Hitchcock ('27), historian.

The program for the year 1926–27 was as follows:
October 11, 1926. An article on "Some angles of a right triangle" was read by Evelyn Puffer.
November 1. An article on "Some applications of mathematics to architecture—Gothic tracery curves" was read by Carolyn Gibby ('27).
November 22. "The fourth dimension" by Frances Gilbert ('28).
December 13. A Christmas party at which mathematical games were played.
February 14, 1927. "Dynamic symmetry—the relation of mathematics to art" by Madeline Bang ('27).

March 7. "The life of Sir Isaac Newton" by Rachel Howe ('28). "Newton the scientist" by Pauline Hitchcock ('27). "The controversy between the Leibnitz and Newton discovery of the calculus" by Mary Doran ('27).

April 18. Mathematical games and puzzles were explained by Margaret Hilferty ('27).

May 9. Spring party. Mathematical games.

The officers for the year 1927–28 are: Evelyn Niemann ('28), president; Evelyn Puffer ('28), secretary; Victorian Pederson ('28), treasurer; Faith Curtis ('28), historian.

(Report by Miss Evelyn Puffer)

Pi Mu Epsilon, Syracuse University, Syracuse, N. Y.

During the year 1926–27 there were eight meetings of the Syracuse chapter of Pi Mu Epsilon. Aside from the regular business of the chapter, the following papers were presented.

1. "Life of Steinmetz" by Gordon Garnhart.
2. "The work of Steinmetz" by George Kern.
3. "The aims of mathematics" by Reuben Fields.
4. "One way in which mathematics is used in physics" by Miss Marjorie Howe.
5. "The conic section" by Charles Jacobs.
6. "The relation between forestry and mathematics" by Laurence Lee.
7. "Index numbers" by Edith Schneckenburger.
8. "Spiritualism and the fourth dimension" by Herbert Keating.
9. "Index stresses" by Henry Stearns.
10. "Entropy" by Cornelius Sutton.
11. "Hyperbolic functions" by Victor Hilaron.

(Report by Miss Eva Richards)

The Denison Mathematics Club of Denison University, Granville, Ohio.

October 5, 1926. Social meeting at the home of Dr. F. B. Wiley. Debate: Should freshman be admitted to the club?

October 26. Business meeting. Membership policy.

November 9. "Large numbers" by Dr. F. B. Wiley.

November 23. "Magic circles" by Mr. Ronald R. Fitch. "Magic squares" by Miss Thelma Weimer ('27) and Albert Bakeman ('29).

January 4, 1927. "The Pythagorean theorem and its proofs" by Professor Edson Rupp.

January 14. Classification of mathematics club members.

March 1. "Duodecimal system" by Miss Mattie Tippet.

April 19. "The fourth dimension" by Leland Powell ('27).

April 26. Discussion of a problem from the American Mathematical Monthly. Election of officers for 1927–28: Lillian Dallman, president; Thomas Parks, vice president; Albert Bakeman, treasurer; Ethyl Lewellyn, secretary.

May 14. Annual banquet: Speaker, Professor Kuhn, Ohio State University. The officers for the year 1926–1927 were: President, Leland Powell; vice president, Thelma Weimer; secretary, Ruth Garrett; treasurer, Thomas Parks.

(Report by Miss Ethel Llewellyn)

Pi Mu Epsilon, Washington University, St. Louis, Mo.

Director, Paul Reece Rider, associate professor of mathematics; vice director, P. Sheldon Remington, Jr., department of mathematics at the Principia; secretary, Jessica M. Young, assistant professor of mathematics and astronomy; assistant secretary, Edwin Ashley Lamke, junior; treasurer, Harry Bauer, senior; librarian, Herbert R. Grummann, instructor in mathematics; student members of executive committee, Walter D. Claus, Bertha T. Hirshstein, Robert C. Napier, Jr., Florence A. Schade.

Eight meetings were held during the year as follows:

October 28, 1926. Business Meeting—Vote on "officers general" of the fraternity.

November 18. "The devil's curve" by Dr. P. R. Rider. "The number e defined by a probability" by Dr. Otto Dunkel. Refreshments.

December 10. Business and program. "The astronomical telescope" by James Kamp. "Mathematics for recreation" by Sol Gleser. Refreshments.

January 27, 1927. Social Meeting. Games and refreshments.

March 4. Open Meeting. "The weight field of force of the earth" by Dr. W. H. Roever. (Presented before Section A of AAAS by Professor Roever as retiring vice-president of Section A, December, 1926.)

March 29. Business Meeting. Vote on amendments to constitution. Election of new members.

April 29. Initiation and annual dinner. Twenty eight persons were initiated: One faculty member, four graduate students, one alumnus, nine undergraduates from the schools of engineering and architecture, and one business man from the Southwestern Bell Telephone Company.

May 19. Business, program, and social meeting. "The phi progression" by W. D. Claus. "A particular method of treating conics analytically" by N. Dunford.

Officers for the academic year 1927–28 were elected as follows: Director, Jessica M. Young, assistant professor of mathematics and astronomy; vice director, John L. Yates, instructor in applied mathematics and engineering drawing; secretary, Herbert R. Grummann, instructor in mathematics; assistant secretary, Edmond Siroky, associate professor of applied mathematics; treasurer, Harry Bauer, graduate student; librarian, Pearl C. Miller, instructor in mathematics; student members of the executive committee, Ralph Belshe, Amy R. Claus, Edwin A. Lamke, Eugene J. Muench.

(Report by Professor Young)

Mattoon Mathematics Club of Park College, Parkville, Mo.

The following was the program during the year 1926–27:

October 14, 1926. "The geometric proof of trigonometric formulae" by Neal Firth.

October 28. "The triangle and its related circles" by Grace Giblin.

November 4. "The determination of π" by Newell Smith.

November 18. "The planimeter and slide rule" by Verdis Miller.

December 2. "De Moivre's theorem" by Professor R. A. Wells, head of the mathematics department of Park College.

February 12, 1927. "The theory of relativity" by John Waterman.

February 24. "History of teaching mathematics" by Louise Ehrstein.

March 3. Party at the home of Professor R. A. Wells.

May 19. "Life of Sir Isaac Newton" by Joseph Graham.

(Report by Miss Herma B. Hudson)

Gamma Eta Mu, University of Redlands, Redlands, California.

The program for 1926–27 was as follows:

November 8, 1926. "Work in Big Creek in surveying for pipe lines" by George Shinn. "Mathematical tricks" by Nelson Painter.

December 13. "Projective geometry" by Professor O. W. Albert. "Einstein and light" by Professor H. E. Marsh.

February 7, 1927. "Architecture in Europe" by Herbert Powell.

March 14. "The purpose of the club," for the benefit of new members by Nelson Painter.

April 9. "A physics problem" by Francis Horton. "Paper folding in the teaching of geometry in high schools" by Hubert Smith. "Artistic paper folding" by Mary Lewis.

The officers for 1926–27 were: President, June O'Brien; vice president, Jack Boren; secretary-treasurer, Mary Lewis.

(Report by Mary Lewis)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3319. Proposed by J. Rosenbaum, Milford, Conn.

Given the sides of a quadrilateral inscribed in a circle, to construct the quadrilateral.

3320. Proposed by N. A. Court, University of Oklahoma.

Through a given point to draw a line which shall form with the sides of a given angle a triangle of given perimeter.

3321. Proposed by Roger A. Johnson, Hunter College of the City of New York.

Through A', B', C', the mid-points of the sides of a triangle ABC, lines are drawn perpendicular to the bisectors of the opposite angles. Show that these lines are sides of a triangle whose circumscribed circle equals that of the given triangle.

3322. Proposed by Harry Langman, Brooklyn, N. Y.

1. Let $ABCD$ be a quadrilateral inscribed in a circle such that the adjacent sides BC and CD are equal. Then $AC^2=AB\cdot AD+BC^2$.

2. Using this result, show that the ratio of the smallest diagonal of a regular heptagon to the side is given by a root of $y^3-y^2-2y+1=0$.

3323. Proposed by C. N. Schmall, New York City.

In elementary geometry it is shown that two circles will touch, cut each other, or have no point in common according as the sum of their radii is equal to, greater than, or less than the distance between their centers. Prove this by analytical geometry.

3324. Proposed by H. Campbell, St. Johnsburg, Vt.

Given the difference of the segments of the base made by the perpendicular to the base from the vertical angle, the difference of the base angles, and the difference of the sides including the vertical angle, to construct the triangle.

3325. Proposed by Paul Capron, U. S. Naval Academy.

Two circles, S_1 and S_2 with centers O_1 and O_2 intersect at A; O_1A meets S_2 at K_2, O_2A meets S_1 K_1. Show that the triangle K_1AK_2 is similar to the triangle O_1AO_2.

SOLUTIONS

477[1915, 339]. Proposed by N. P. Pandya, Sojitra, India.

A village A_0 is equidistant from ten villages $A_1, A_2, \cdots, A_{10}$. The distances $A_1A_2, A_2A_3, \cdots, A_{10}A_1$ are in arithmetic progression. A person, starting from A_0 has to go to four of these villages consecutively, and has then to return to A_0. What four villages should he select so that the total distance traveled by him may be a minimum?

Discussion by J. H. Neelley, Carnegie Institute of Technology.

The ten villages $A_1, A_2, \cdots, A_{10}$ are on a circle with A_0 as its center. If Mr. Pandya means the villages are ordered as he states them the shortest distance must be either A_1A_2 or $A_{10}A_1$. So the person may travel to A_1 to A_2 to A_3 to A_4 and back to A_0 or to A_1 to A_{10} to A_9 to A_8 and back to A_0. It certainly seems that if a sequence is said to be in arithmetic progression it would mean the terms are taken in the order they occur. There appears to be no reason for choice of order for an ascending series.

494[1916, 210]. Proposed by David F. Barrow, University of Texas.

Students of geometry are very apt to assume that a theorem, true in general, will hold in all limiting cases. This trustfulness leads to frequent errors. An example is the following: Let C_1, C_2, C_3, C_4 denote four circles, and P_{ij}, P'_{ij} denote the two points in which C_i and C_j intersect. If P_{12}, P_{23}, P_{34}, P_{41} are concyclic on a circle C, then P'_{12}, P'_{23}, P'_{34}, P'_{41} will be concyclic on a circle C'. This is still true if C is very small. Hence we might hastily conclude that: If four circles are concurrent, then their other intersections, taken in pairs in a cyclic order, are concyclic. Why is not this true?

Note by Otto Dunkel, Washington University.

The theorem in the first part of the problem would lead by a properly arranged limiting process to the result: It is possible to have four circles concurrent in such a way that one set of four of their other intersections taken in cyclic order are concyclic. But no such conclusion may be inferred by this reasoning for any four concurrent circles. If four circles are concurrent there may be no set of four other intersections taken in cyclic order which lie on the same circle. For let $ABCD$ be a complete quadrilateral such that AB and DC meet in E while AD and BC meet in F. Then the four circles determined by EBC, EAD, FBA, FCD meet in a point P. Let the quadrilateral be chosen so that C lies within the triangle AEF, and so that the angles EBC and CDF are neither equal nor supplementary. Then E, C, F, A do not lie on a circle, and the same is true of the sets A, B, C, D and E, B, F, D. If on the other hand we make $\angle EBF = \angle EDF \neq 90°$, then E, B, F, D will be concyclic, but not so for the other two sets. If $\angle EBF = \angle EDF = 90°$, then there are two and only two concyclic sets.

Also solved by J. H. Neelley.

2968[1922, 179]. Proposed by Malcolm Foster, Yale University.

A curve C is the directrix of a ruled surface, and g is any ruling. Relative to the trihedral of C at the point of intersection with g, the direction-cosines of g are α,β,γ, expressed as functions of the arc of C. Prove that the distance t along g from C to the line of striction is given by

$$t = \frac{\dfrac{\beta}{\rho} - \alpha'}{\Sigma\alpha'^2 + \dfrac{2}{\rho}(\alpha\beta' - \alpha'\beta) + \dfrac{2}{\tau}(\beta'\gamma - \beta\gamma') + \dfrac{1}{\rho^2}(1 - \gamma^2) + \dfrac{1}{\tau^2}(1 - \alpha^2) + \dfrac{2}{\rho\tau}\alpha\gamma},$$

where ρ and τ are the radii of first and second curvature of C.

Note by the Editors.

A solution of this problem is contained in the paper by the proposer, *Ruled Surfaces Referred to the Trihedral of a Directrix*, in this Monthly, vol. 34, pp. 303, 304.

3263[1927, 272]. Proposed by Otto Dunkel, Washington University.

Give an elementary proof, without the use of the calculus, of the theorem that the regular tetrahedron has the greatest volume of all tetrahedrons having the same surface.

Solution by the proposer.

In Crelle's Journal, vol. 24 (1842) p. 220, Steiner proved as a result of two theorems the necessary

condition for a maximum volume, but apparently he did not consider a proof that the condition sufficed to obtain a maximum. Such a proof will be supplied below. The proof of the first theorem below follows pretty closely Steiner's proof.

THEOREM I. *Of all triangular pyramids having the same base and equal total lateral surfaces, the one for which the foot of its altitude is the center of the inscribed circle of the base has the greatest volume.*

Let ABC be the given base, I the center of its inscribed circle and V the vertex of the pyramid with the altitude VI. If the second pyramid having the same base and the same total lateral surface is such that its vertex V' lies on the side of the base opposite V, we may replace it by the symmetrical figure with respect to the base. We shall assume then that V and V' lie on the same side of the plane of ABC. Let IL, IM, IN be the radii of the inscribed circle to its points of contact with BC, CA, AB. Then $VL=VM=VN$ and each of these lines is perpendicular to a corresponding side of the base. Prolong VI to V_1 so that V_1L is perpendicular to LV, and consider the auxiliary pyramid V_1—ABC. Set $p=V_1L=V_1M=V_1N$. Each lateral face of this pyramid is perpendicular to a corresponding face of $V-ABC$. Let p', q', r' be the lengths of the perpendiculars dropped from V_1 to the lateral faces of V', and let the areas of these faces be α', β', γ', respectively. Then

$$3(\text{vol. } V' + \text{vol. } V_1) = \pm p'\alpha' \pm q'\beta' \pm r'\gamma',$$

where not all of the signs can be negative. Also

$$3(\text{vol. } V + \text{vol. } V_1) = p(\alpha + \beta + \gamma) = p(\alpha' + \beta' + \gamma'),$$

where the second equation follows from the equality of the lateral surfaces. Hence

$$3(\text{vol. } V - \text{vol. } V') = (p \mp p')\,\alpha' + (p \mp q')\,\beta' + (p \mp r')\,\gamma' > 0.$$

The inequality follows from the fact that p is not less than any one of the lengths p', q', r', and it is greater than at least one of them if $V' \neq V$.

THEOREM II. *Of all triangular pyramids having equivalent bases and equal lateral areas, the regular pyramid has the greatest volume.*

Let R be the volume of the regular pyramid and W, the volume of any other pyramid with the same lateral area S and with a base of the same area as that of R. Let V be the volume of the pyramid with the same base as W, with the lateral area S, and with the foot of its altitude at the center of the inscribed circle of the base. Then by Theorem I, $V \geqq W$. Let $2s$ be the perimeter of the base of R, which is an equilateral triangle, and let ϵ be the radius of its inscribed circle. Let $2s'$ and ϵ' be the corresponding symbols for V. Then $\epsilon s = \epsilon' s'$. Since the areas of the bases are equal, the equilateral triangle has the smaller perimeter, i.e., $s < s'$, and hence $\epsilon > \epsilon'$. Since the lateral areas are the same $sl = s'l'$, where l and l' are the slant heights. Hence $l/\epsilon = l'/\epsilon'$, and the two right triangles formed by the altitudes h, h'; l, l'; ϵ, ϵ' are similar. But since $\epsilon > \epsilon'$, $h > h'$, and hence $R > V \geqq W$.

If V is a regular tetrahedron the above suffices. If it is not, suppose that E is the volume of the regular tetrahedron having the given total area (including now the base), and let e be its edge. Each face has the area $3^{1/2}e^2/4$ and the altitude of E is $(2/3)^{1/2}e$. Let the equal sides of the base of V be denoted by c; then this base has the area $3^{1/2}c^2/4$. It will simplify the proof to write the altitude of V in the form $(2/3)^{1/2}h$. The corresponding volumes are then $2^{1/2} \cdot e^3/12$ and $2^{1/2} \cdot c^2h/12$. Equating the areas of the surfaces of the two figures, we have

$$c^2 + c(8h^2 + c^2)^{1/2} = 4e^2,$$

and this when cleared of the radical gives

$$2e^4 = c^2(h^2 + e^2).$$

But $h^2 + e^2 > 2he$, if $h \neq e$. Hence $e^3 > c^2h$, if $h \neq e$, and therefore $E > V$ and the theorem of the problem is proved.

3265[1927, 335]. Proposed by Norman Anning, University of Michigan.

Prove, or improve, the scout's rule for reading bicycle spoor: an inflexional tangent to the path of the hind wheel is an ordinary tangent to the path of the front wheel.

SOLUTION BY OTTO DUNKEL, Washington University.

A straight line MP of constant length l moves so that it is tangent at M to a given curve (M) and the other end P describes a curve (P). Let ψ be the angle between MP produced and the tangent to (P) at P, and let θ be the inclination of MP to a fixed line. Also let σ and s be the lengths of arcs of (M) and (P) to M and P, respectively, measured in the directions of positive motion, which we shall take as MP for (M) at the point M. If two neighboring positions of MP are considered, a figure gives easily the two results

$$d\sigma = \cos\psi ds, \quad ld\theta = \sin\psi ds, \tag{1}$$

and hence

$$l\frac{d\theta}{d\sigma} = \tan\psi. \tag{2}$$

Suppose now that M is at a point of inflection, then $d\theta/d\sigma=0$ and $d^2\theta/d\sigma^2\neq 0$ at this point. Let $d^2\theta/d\sigma^2>0$, and take the inflection tangent as the fixed line of reference for θ. Just before the point of inflection $d\theta/d\sigma$ is negative and the curve is below the inflection tangent, while just after, the reverse is true. At the corresponding point P_1 of (P), $\tan\psi=0$; just before this point it is negative, and just after it, positive. Hence ψ passes from a small negative value to zero and then becomes positive. From (2) and (1) we have

$$l\frac{d^2\theta}{d\sigma^2} = \sec^3\psi\frac{d\psi}{ds}. \tag{3}$$

If τ is the inclination of the tangent at P to the inflection tangent, then $\tau=\psi+\theta$. At P_1, $d\sigma=ds$ from (1) and hence $d\tau/ds=d\psi/ds$, since $d\theta/ds=d\theta/d\sigma=0$. Hence at P_1 we have

$$l\frac{d^2\theta}{d\sigma^2} = \frac{d\tau}{ds} = \frac{1}{R} > 0,$$

where R is the radius of curvature of (P) at P_1. Hence the inflection tangent is an ordinary tangent at P_1.

NOTE. The second equation of (1) is true whether MP is constant or variable. Let the normals at M and P meet in Q, then this equation may be written in the simple form $ds/d\theta=PQ$. If MP is of constant length, then from (2) $d\sigma/d\theta=MQ$. Hence if MP is of constant length the normal at P passes through the center of curvature of (M) at M. The reasoning above may be extended to show that the order of contact of (P) with MP is one unit less than the order of contact of (M) with MP at the corresponding point P. For, if $d^{n+1}\theta/d\sigma^{n+1}\neq 0$ but all the derivatives of lower order are zero, it is easily shown that $d^i\psi/ds^i=0$, $i<n$, and $ld^{n+1}\theta/d\sigma^{n+1}=d^n\psi/ds^n=d^n\tau/ds^n$.

Also solved by MICHAEL GOLDBERG.

3266[1927, 335]. Proposed by W. A. Jenkins, Chicago, Ill.

Cards are exposed one at a time, without being replaced, from an ordinary deck of playing cards. The draw ceases as soon as four cards of any one number (or Jack, Queen, or King) are exposed. Find the probability that the ith card exposed concludes the draw.

SOLUTION BY MICHAEL GOLDBERG, Washington, D. C.

The number of combinations of i cards which contain one four-set, and only one, is equal to thirteen times the number of combinations of $(i\text{-}4)$ cards chosen from 48 (that is, twelve four-sets) such that no combination contains more than three cards of any one set. Let this be represented by $13A[i\text{-}4]$. Then

$$A[n] = \binom{48}{n} - \binom{12}{1}\binom{44}{n-4} + \binom{12}{2}\binom{40}{n-8} - \binom{12}{3}\binom{36}{n-12} + \binom{12}{4}\binom{32}{n-16} - \cdots$$

The first term is the number of combinations of 48 cards taken n at a time. The second term includes all combinations which have at least one four-set, but it has counted twice all those combinations which

have at least two four-sets. The third term includes all those combinations which have at least two four-sets, but it has counted three times all those combinations which have at least three four-sets. Etc.

$$A[n] = \sum_{k=0}^{k=r}(-1)^k\binom{12}{k}\binom{48-4k}{n-4k}, \text{ where } r \text{ is the integral part of } \frac{n}{4}.$$

$$A[i-4] = \sum_{k=0}^{k=r}(-1)^k\binom{12}{k}\binom{48-4k}{i-4-4k}.$$

For any of these combinations, the probability that the last card drawn is one of the four-set is $4/i$. Hence, the required probability is

$$P = \frac{13A[i-4]}{\binom{52}{i}} \times \frac{4}{i} = \frac{52A[i-4]}{\binom{52}{i}i}.$$

NOTES AND NEWS

Readers are invited to contribute to the general interest of this depatment by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor M. H. Ingraham, Associate Secretary of the American Mathematical Society, has sent the following communication to the Mathematical Association of America:

"At the meeting of the American Mathematical Society in Chicago held Saturday, April 7, the following resolution was passed:

> "The American Mathematical Society wishes to express its particular appreciation of the constant and strong support that the Mathematical Association of America has given it, and especially of its gracious action in becoming a sustaining member of the Society. It hopes that this unity may strengthen the efforts of the two organizations to further Mathematics in America.

"The interrelations of the Society and the Association are so deep and the feeling of the two organizations so cordial that it is with a peculiar sense of pleasure and gratitude that I transmit to you the above resolution. I trust that the two organizations which have so many interests and so many members in common may always work together for the advancement of Mathematics in America."

The Bruce medal of the Astronomical Society of the Pacific has been awarded to Dr. W. S. ADAMS, of Mount Wilson Observatory.

Professor A. B. COBLE, of the Johns Hopkins University, has accepted a professorship of mathematics at the University of Illinois, where he had been prior to the present academic year.

Professor LEONARD EUGENE DICKSON, of the University of Chicago, has been awarded the "Frank Nelson Cole Prize" of the American Mathematical Society for "the most notable advance in the theory of linear algebras" during the years 1925, 1926, 1927. The award was made by President Virgil Snyder at the April meeting of the Society in Chicago.

Professor G. H. HARDY, of Oxford, will be inr esidence at Princeton during the first half of the academic year 1928–29, and will give a course entitled "Chapters in the theory of functions." During the same period, Professor OSWALD VEBLEN, of Princeton University, will be in residence at Oxford, and will give a course entitled "Tensors and other differential invariants."

At the University of Illinois, Professors ARNOLD EMCH and OLIVE C. HAZLETT have been granted leave of absence for the coming academic year, for study in Europe.

Mr. FRANCIS E. JOHNSTON has been appointed assistant professor of mathematics at George Washington University.

Dr. CHARLES F. ROOS has been appointed assistant professor of mathematics at Cornell University.

The following 41 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1927; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

R. G. ARCHIBALD, Chicago, June, *Diophantine equations in division algebras.*

W. L. AYRES, Pennsylvania, June, *Concerning continuous curves and correspondences.*

F. R. BAMFORTH, Chicago, December, *A classification of boundary value problems for a system of ordinary differential equations of the second order.*

L. M. BLUMENTHAL, Johns Hopkins, June, *Lagrange resolvents in Euclidean geometry.*

W. F. CHENEY, JR., Massachusetts Institute of Technology, June, physics, *Infinitesimal deformation of surfaces in Riemannian space.*

ALONZO CHURCH, Princeton, June, *Alternatives to Zermelo's assumption.*

T. F. COPE, Chicago, December, *An analogue of Jacobi's condition for the problem of Mayer with variable end points.*

C. C. CRAIG, Michigan, June, *An application of Thiele's semi-invariants to the sampling problem.*

D. R. DAVIS, Chicago, June, *The inverse problem of the calculus of variations in higher space.*

D. A. FLANDERS, Pennsylvania, June, *Double elliptic geometry in terms of point, order, and congruences.*

P. A. FRALEIGH, Cornell, June, physics, *Regular bilinear transformations.*

H. H. GERMOND, Wisconsin, June, physics, *The effect of space charge on a three element vacuum tube.*

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 31.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, May.

OHIO, Columbus, Ohio, April 6.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Durham, N. C., March.

SOUTHERN CALIFORNIA, Whittier College, March 10. Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

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6. The problem of finding the characteristics of the frequency function of y/x, in case the characteristics of the correlation or frequency function of x and y are known, has been studied only by Karl Pearson. He gave up his results as "practically unworkable" in case x and y are correlated. Under certain "almost necessary" restrictions the author developed formulas which are workable using the method of semi-invariants. The application of the results was illustrated numerically in two cases in which the correlation surface $z=F(x, y)$ is normal.

7. Professor Hopkins gave a report of an investigation now being conducted under his direction on periodic orbits for asteroids which revolve about the sun an integral number of times while Jupiter completes one revolution. The method of attack is similar to that used by Sir George Darwin in England and Professor F. R. Moulton in this country in problems of analogous nature. The results so far secured in this investigation were announced, but Professor Hopkins plans to develop the problem much further and to publish the results upon completion.

W. W. DENTON, *Secretary*

THIRTEENTH ANNUAL MEETING OF THE OHIO SECTION

The thirteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 5, 1928, in connection with the meetings of the Ohio College Association. Chairman C. H. Yeaton presided over the afternoon session. Following the dinner in the evening there was an informal discussion of the question, "What shall we do for our students of superior ability?" Professor E. H.Clarke, chairman of the program committee, presided over this discussion.

Forty-six persons registered attendance, among whom were the following thirty-one members of the Association; J. P. Albert, R. B. Allen, W. E. Anderson, L. C. Bagby, L. Battig, Grace M. Bareis, P. E. Baur, H. M. Beatty, J. B. Brandeberry, C. T. Bumer, W. D. Cairns, V. B. Caris, F. E. Carr, E. H. Clarke, R. Crane, O. L. Dustheimer, E. E. Erickson, H. Hancock, Marie M. Johnson, H. W. Kuhn, C. C. Morris, R. L. Newlin, J. Pierce, S. E. Rasor, C. E. Rhodes, Hortense Rickard, J. H. Weaver, R. B. Wildermuth, F. B. Wiley, B. F. Yanney, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, E. H. CLARKE; Secretary-Treasurer, RUFUS CRANE; Member of Executive Committee, S. A. ROWLAND; Member of Program Committee, LOUIS BRAND. It is expected that the next meeting will be held at the Ohio State University on Thursday, April 4, 1929.

Three papers were presented, titles and abstracts of which follow:

1. "The construction of the tangent to certain curves" by the Chairman, Professor C. H. YEATON, Oberlin College.

2. "The analogy of the theory of Kummer's ideal numbers with chemistry and its prototype in Plato's concept of idea and number" by Professor HARRIS HANCOCK, University of Cincinnati.

3. "Some methods to be used in obtaining rapidly converging power series for certain root approximations" by Professor E. H. CLARKE, Hiram College.

1. To the early mathematician the determination of the tangent at an arbitrary point of a curve was a geometrical problem; general principles and methods of attack were lacking. With the invention of the geometry of Descartes, the synthetic method gave way to the analytic; early definitions were modified and the solution gained in generality. In the present paper Professor Yeaton reviews a geometrical construction of the tangent which has been developed by Jules Poivert, Professor of Architecture, École Polytechnique, Montreal. In many cases the construction is purely geometric; in others, the position of the tangent to some fundamental curve is assumed.

2. Mathematics, occupying in its realm an intermediate position between the realm of the natural sciences and the realm of abstract reason, is both a means of computation and a medium of thought. The mathematical realm so overlaps the two adjacent realms that a transition from the one to the other through it may give rise to confusion both in term and in concept. With Plato there appear two kinds of number, the mathematical numbers consisting of homogeneous unities which can therefore be reckoned together, each with each, and the ideal numbers not identical with ideas but intermediate between them and the things of sense. Accordingly the mathematical numbers express quantitative, the ideal numbers logical determinations. This appears to be the interpretation of Zeller. Prof. Paul Shorey seems in a measure at variance with the Zeller interpretation.

Professor Hancock holds that if the Plato ideal number is rightly interpreted, there is no incorrectness in either the Zeller or the Shorey contention; and he asserts that the theory of algebraic ideal numbers of Kummer is the key for the solution of the difficulty. Its analogy with chemistry offers convincing evidence.

Professor Hancock presents his thesis in a form that can be read by philosophers as well as by mathematicians. Two illustrations of Sommer and Hensel are used to introduce the notion of the ideal number. This notion is generalized for the mathematicians by means of an example taken from an algebraic realm. The Dedekind ideals and modals are introduced and the correlation with the quadratic forms is pointed out. The analogy with chemistry is drawn from the

paper of Kummer in Crelle's Journal, vol. 35, p. 359 (1846) and references are given to the works of Shorey and Zeller.

3. Professor Clarke showed how the binomial series could be made very rapidly converging for the computation of the roots of numbers by the introduction of close rational approximations to the roots, obtained from continued fractions.

RUFUS CRANE, *Secretary*

THE FIFTH ANNUAL MEETING OF THE NEBRASKA SECTION

The fifth annual meeting of the Nebraska section of the Mathematical Association of America was held on April 28, 1928, at Midland College, Fremont, Nebraska. Twenty persons were present including six members of the Association. The following program was presented:

1. "Reducents, or values of the variable for which a polynomial with integer coefficients can not represent a prime" by Mr. O. C. COLLINS, University of Nebraska.

2. "A standard curve for grades in college mathematics" by Professor H. C. FEEMSTER, York College. (By title.)

3. "The borrower's rate in building and loan associations" by Professor C. C. CAMP, University of Nebraska.

4. "A finite projective geometry" by Miss GLADYS BALDWIN, University of Nebraska.

Abstracts of these papers follow:

1. Mr. Collins showed that when certain functions of the coefficients are substituted for the variable in a polynomial with integer coefficients the result is the product of factors which are functions of the coefficients. For integer values of the variable corresponding to such substitutions the polynomial will in general represent a composite number.

3. Professor Camp reviewed various methods for computing rates of interest, among them a new one which is both simple and accurate. A random sample was taken from associations in Illinois, and the rate of the borrower correlated with that of the investor. The average rates were 8.87 percent and 6.87 percent respectively, contrary to popular opinion.

4. Miss Baldwin made a study of the finite projective geometry in which each line consists of eight points. She enumerated the number of involutions on the line, the number of conics in the plane and developed a number of theorems characteristic of this particular geometry.

Officers elected for the year 1928–29 are: H. C. FEEMSTER, York College, Chairman; ELLEN H. FRANKISH, North High School, Omaha, Secretary-

Treasurer; J. M. HOWIE, Nebraska Wesleyan University, Member of Executive Committee.

ELLEN H. FRANKISH, *Secretary*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-third regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the United States Naval Academy, Annapolis, Maryland, on Saturday, May 5, 1928. Sessions were held in the morning and afternoon, Professor J. R Musselman, Chairman of the Section, presiding at both sessions.

About sixty persons attended the meeting, including the following forty-seven members of the Association: O. S. Adams, R. N. Ashmun, Clara L. Bacon, G. A. Bingley, L. M. Blumenthal, C. C. Bramble, J. A. Bullard, Paul Capron, G. R. Clements, A. B. Coble, A. Cohen, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. B. Eppes, P. J. Federico, Michael Goldberg, Harry Gwinner, W. M. Hamilton, P. E. Hemke, L. S. Hulburt, F. E. Johnston, H. P. Kaufman, L. M. Kells, W. D. Lambert, A. E. Landry, Bessie I. Miller, A. K. Mitchell, T. W. Moore, Frank Morley, F. D. Murnaghan, J. R. Musselman, J. W. Peters, E. C. Phillips, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, Marian M. Torrey, John Tyler, W. J. Wallis, Paul Wernicke, Elizabeth W. Wilson, E. W. Woolard.

In persuance of their longstanding tradition for generous hospitality the Annapolis members entertained those attending the meeting at luncheon during the interval between the two sessions. A hearty vote of thanks was passed in appreciation of this hospitality and of the courtesy of the authorities of the Academy for their welcome to the Section within its walls.

At the beginning of the afternoon session a business meeting was held at which the committee on nominations, consisting of Dr. Adams, Professor Cohen, and Professor Root, presented the following list of nominees: For chairman, Professor C. C. BRAMBLE; for secretary, Rev. E. C. PHILLIPS; as members of the executive committee, Professor HARRY GWINNER and Dr. L. M. BLUMENTHAL. No other nominations were offered and the election of the above named candidates was made unanimous.

The following eight papers were presented:

1. "A transformation of the Clifford configuration into itself" by Professor FRANK MORLEY, Johns Hopkins University.

2. "A cubic curve and a reflector" by Dr. BESSIE I. MILLER.

3. "An interesting conformal projection of the sphere depending upon elliptic functions" by Dr. OSCAR S. ADAMS, U. S. Coast and Geodetic Survey.

4. "Two mutual relations possible between a pair of tetrahedra" by Dr. PAUL WERNICKE, U. S. Patent Office.

5. "The first English Euclid" by Professor WALTER F. SHENTON, American University.

6. "Some researches on the triangularly symmetric rational quartic" by Mr. FRANK E. SMITH, Catholic University of America (introduced by Professor A. E. Landry.)

7. "On the genetic equilibrium of a random bred population" by Professor TOBIAS DANTZIG, University of Maryland, and Professor WILLIAM B. KEMP, University of Maryland, (introduced by Professor Dantzig).

8. "An invariant of a space configuration" by Dr. LEONARD M. BLUMENTHAL, Johns Hopkins University.

The papers led to rather extensive discussion in which the following members took part: Messrs. Blumenthal, Capron, Coble, Dantzig, Gwinner, Lambert, Morley, Murnaghan, Wernicke, and Miss Wilson.

Abstracts of these papers follow:

1. A Clifford configuration is the figure of n lines of a plane, say $1, 2, \cdots, n$, completed. That is the points 12, 13, 23 are on a circle 123. The circles 123, 234, 341, 412 are on a point 1234. The points such as 1234, 2345, 3451, $\cdots$, are on a circle 12345. And so on. There are then, including ∞, which is here a point, 2^{n-1} points and 2^{n-1} circles; there are $n-1$ points on each circle and $n-1$ circles on each point. Professor Morley indicated transformations T_n of a C_n into itself in the simpler cases. For C_3, the transformation T_3 is focal pairing or transformation by isogonal conjugates. The transformation T_4 is quadratic involution or polarity, $xy=k$. The transformation T_5 is a contact transformation reducible to $xy=1+\mu t$, where $|t|=1$. The transformation T_6 is a Cremona transformation with four singular points (in the inversive plane). Only the points are sent into themselves in this case. In canonical form T_6 is $\sum^4[\lambda_i/(x-a_i)]=0$, where $\sum\lambda_i=0$ and the values of λ_i are real.

2. A physician, Dr. Wood, of Cleveland, the inventor of one of the forms of reflectors much used in automobile lamps, has made a modification of that reflector. A plane section of the new reflector is a portion of a nodal cubic. Commercially a parabola which approximates this portion of the cubic will be used as the generating curve of the surface of the reflector. This work is covered by a patent.

3. This paper described the method of computation employed in placing the whole sphere within a square with the poles situated on one of the diagonals

and symmetrically placed with respect to the center of the square. The calculations were made for the ten degree intersections and a copy of the constructed projection was shown. The method of attack employed was aimed to simplify the computations as much as possible so that they can be made by one who has a slight knowledge of trigonometry with the use of tables. No further use was made of elliptic functions than simple interpolations in Legendre's table of the first elliptic integral for $k=1/\sqrt{2}$.

4. A quadrilateral in space S_3, or a "tetragram" T, defines a tetrahedron the faces of which, in (reversible) cyclic order, are the planes laid through pairs of consecutive sides of T. Four vectors representing these faces, i.e. perpendicular to them and proportional to their areas, form when placed end to end, a second tetragram T'. Starting with T' we find T, or a tetragram similar to T, by the same process. T and T' are here called *orthogrammatic* to one another. The same term when applied to the tetrahedra, defines a 3 to 1 reciprocal relation.

Steiner has applied the term *orthologiques* to two triangles t, t' if the perpendiculars from the vertices of t to the sides of t' are concurrent. He proves that the perpendiculars from the vertices of t' to the sides of t then also concur. A proof by vector algebra is here given and proves applicable to tetrahedra T, T' or to *simplicia* [$(n+1)$-points] in any S_n. If the perpendiculars from the vertices of T to the faces of T', and vice versa, concur, T and T' may therefore be called *orthologic*.

5. A description, with illustrations, of the two folio volumes of "The Elements of Geometrie of the most auncient Philosopher Euclide of Megara, Faithfully (now first) translated into the Englishe toung, by H. Billingsley, Citizen of London." This interesting work is from the press of John Daye and was printed in London in 1570. A compendious preface by John Dee adds to the interest of the work, as do the figures in Book Eleven, which are made of printed papers, so pasted and folded into the work that they may be unfolded to make actual models for the space theorems. These volumes, together with many others of intense historical interest, are a part of the mathematical library of the late Dr. Artemas Martin, mathematician and collector of Washington, D.C., which he donated to the American University of that city.

6. If the equations of the plane rational quartic are set up so that it has metric as well as projective symmetry, it has parametric value at the double-points: α, α^{-1}; $\omega\alpha$, $\omega\alpha^{-1}$; $\omega^2\alpha$, $\omega^2\alpha^{-1}$, where the parameter $\alpha=e^{i\theta}$ takes all values on the unit circle in the complex domain. The problem is: how many of the 8 possible triangles simultaneously in-and-circumscribed to this curve can be real. It is found that the 8 triangles break up into a set of two equilateral triangles and a set of six non-equilateral triangles. The sets appear either all together or not at all. The set of six is real for a very small range of the para-

meter when $\theta = 77° 45'$, approximately, and for the range $0° < \theta < 60°$. The set of two is real for the range $0° < \theta < 60°$ and for the additional range $2 < \rho < 3$ where $\rho = \alpha + \alpha^{-1}$. The "trioval" shape $(0 < \rho < 1)$ has, then, for a small range 6 real in-and-circumscribed triangles. The "loop" shape $(1 < \rho < 2)$ has always 8 real. The "acnodal" shape $(\rho > 2)$ has two real for the range $2 < \rho < 3$.

For no proper form of this curve does an equilateral triangle have a side in common with a non-equilateral triangle.

7. In this paper the Mendelian law is generalized by assigning general values (u, v, w) to the reproductive ratios of the dominant, heterozygote and recessive types of a species. The mathematical formulation of the law leads to the recurrence formula

$$(1) \qquad r_k \equiv (\alpha r_k + 1) r_k / (r_k + \beta), \text{ where } \alpha = u/v,\ \beta = w/v,\ r_k = a_k/b_k.$$

r_k is the dominance ratio for the kth generation. The question of equilibrium is then reduced to the study of iteration (1) and particularly of the limiting points of the set $r_1, r_2, r_3, \cdots$. It is found that for $\alpha < 1, \beta < 1$ the equilibrium is stable, the dominance ratio tending towards $r = (1-\alpha)/(1-\beta)$.

When $\alpha > 1, \beta > 1$, the equilibrium is unstable and the evolution results in the rapid extermination of either the dominant or of the recessive characteristic. Finally, the classical case $\alpha = \beta = 1$ is that of indifferent equilibrium.

The reproductive capacity is given by the quadratic form

$$P_{k+1} = u a_k^2 + 2v a_k b_k + w b_k^2$$

In the case of stable genetic evolution the function P reaches its maximum at equilibrium. Therefore a selective process based on the adjustment of the dominance ratio will invariably reduce the reproductive capacity of the species.

8. This paper is an extension to Euclidean spaces of higher dimensions of the results recently obtained for the plane. (L. M. Blumenthal, *Lagrange resolvents in Euclidean Geometry*, American Journal of Mathematics, vol. 49, 1927, pp. 511–522). It is shown that attached to an ordered $2n$-point in S_n there is a lineo-linear invariant under translations of either component n-point. This invariant (entirely analogous to the one developed in the paper above) is called the *norm* of the configuration and is expressed in terms of the Lagrange Resolvents of the vertices of the $2n$-point.

As a result of the vanishing of their norms, two interesting theorems are developed for the space 6-point and 8-point.

The norm for ordered points on a sphere is defined and a theorem for the ordered 6-point is deduced.

EDWARD C. PHILLIPS, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The regular spring meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Benedict, St. Joseph, Minnesota, on Saturday, May 19, 1928. At the request of the Chairman Sister Claudette, O.S.B., Professor H. H. Dalaker of the University of Minnesota presided.

The attendance was 65, including the following 23 members of the Association: Sister Alice Irene, W. O. Beal, R. W. Brink, W. H. Bussey, Elizabeth Carlson, H. H. Dalaker, J. M. Earl, Clara L. Hancock, W. L. Hart, H. E. Hartig, D. Jackson, W. H. Kirchner, W. D. MacMillan, M. Nordgaard, W. C. Risselman, R. R. Shumway, A. K. Solum, F. J. Taylor, Ella Thorp, A. L. Underhill, E. D. Wells, H. B. Wilcox, G. Winkelmann.

At the afternoon session the following officers were elected for the coming year: Chairman, Sister ALICE IRENE, College of St. Catherine, St. Paul; Secretary, A. L. UNDERHILL, University of Minnesota; an Executive Committee consisting of the Chairman, the Secretary, and H. E. HARTIG, University of Minnesota, MARIE NESS, University of Minnesota, and CLARA L. HANCOCK, Junior College, Virginia, Minnesota.

A motion was passed expressing the appreciation of the Section for the hospitality of the College of St. Benedict.

The following six papers were read:

1. "Some early applications of calculus to mechanics" by Professor W. D. MACMILLAN, University of Chicago.

2. "On the existence of certain approximating functions" by Professor W. C. Risselman, University of Minnesota.

3. "A course in mathematics for students of statistics" by Professor R. W. BRINK, University of Minnesota.

4. "Analytic and topological transformation of closed surfaces of genus p" by Dr. EDWARD SAIBEL, University of Minnesota.

5. "Prediction of student achievement in freshman mathematics" by Professor W. L. HART, University of Minnesota.

6. "Concerning integration by parts" by Professor H. E. HARTIG, University of Minnesota.

Abstracts of these papers follow:

1. Professor MacMillan gave an account of some of the early applications of the calculus to mechanics, and especially considered problems relating to the gravitational potential of spherical shells.

2. This note dealt with the problem of approximating to a given function

of a complex variable along a curve C by means of a linear combination of unitary bi-orthoganal functions so as to minimize the integral along C of the square of the absolute value of the error. The method of completing squares was used in proving existence and uniqueness. The whole purpose of the note was to call attention to this elementary method of proving an important well-known theorem.

3. In his paper Professor Brink described a course which he has been giving this year at the University of Minnesota. It is designed to give rapid preparation in the mathematics necessary for the study of the mathematical theory of statistics and for reading the literatures of statistics, to persons who have attained some maturity and cannot spare the time from work in their own fields to follow the ordinary courses in mathematics through the calculus.

4. It has been shown by Franklin and Wiener[1] that every continuous one-to-one transformation of a two dimensional region of finite connectivity may be approximated to an arbitrary degree of exactness by an analytic one-to-one transformation. It was the object of Dr. Saibel's paper to show that this theorem can be extended to closed surfaces.

5. The department of mathematics in the College of Science, Literature, and the Arts of the University of Minnesota has experimented with various devices for obtaining, early in each term, a prediction of the future achievement of students in freshman courses in mathematics. Professor Hart described the predictive devices used and discussed the application of the predictions in the placement work of the department of mathematics. This placement of students involves the elimination, at the end of two weeks, of the lowest one-sixth of the students in each course and the sectioning of the remainder on the basis of ability.

6. Professor Hartig showed by examples how one can often simplify integration by parts in Case I, in integrating the expression usually called dv, by throwing in the integration constant, which later is chosen for the purpose of simplification.

On the day preceding this meeting of the Minnesota Section, Professor MacMillan delivered a public lecture at the University of Minnesota on "The evolution and dissolution of matter."

A. L. UNDERHILL, *Secretary*

[1] Philip Franklin and Norbert Wiener, *Analytic approximations to topological transformations*, Transactions of the American Mathematical Society, vol. 28 (1926), pp. 762–785.

A SOURCE BOOK IN THE HISTORY OF MATHEMATICS

By DAVID EUGENE SMITH, Columbia University

Professor Gregory D. Walcott of Hamline University, St. Paul, Minnesota, has for some time been planning a series of source books in the history of the sciences, covering the period 1500–1900. To carry out this project the American Philosophical Society has recently secured from the Carnegie Foundation a grant sufficiently large to permit of a beginning in the publication of the series. It is proposed to use this sum to underwrite the publication of the first one or two volumes, and to use the income from the sales to allow for the publication of others. It is expected that the second volume will relate to mathematics. It will be edited by a committee consisting of Professors R. C. Archibald, Florian Cajori, and David Eugene Smith (chairman).

The general tentative plan for this volume is to publish articles of about 6–20 pages each, giving

(1) The title, author and source of the material selected.

(2) The name and position of the editor of the article.

(3) A brief statement, in a few lines, of the reason why the material has been considered important enough to have place in the publication.

(4) The original text, transcribed and checked with great care.

(5) A translation into English, in case the original is in a foreign language.

(6) A brief explanation, by notes or commentary, of any obscure statements needing elucidation. This feature must necessarily be very limited, the space at the Committee's disposal being devoted principally to the source material itself.

For example, if Cardan's solutions of the cubic and biquadratic equations were given, the Latin text would probably fill about four pages, the translation would take four more, and the explanatory notes with brief references to the work of Del Ferro, Tartaglia, and Ferrari might take one more. In any case, if the articles averaged twelve pages, only about forty could be used.

There are no funds with which to pay for the contributions. As with the members of the committee, a contributor must make the work a "labor of love," the publication of his name with the article which he contributes being the only other reward. There is barely enough money available to assume a beginning of the publication, all returns from sales being devoted to extending the series. No doubt any balance at the close of the project, some years hence, would be equally divided, but in any case it could hardly be worth considering.

It is desirable that all members of the American Mathematical Society and of the Mathematical Association of America should feel the importance of the undertaking and that those who have suggestions as to source material of value to college instructors, to graduate students, and to those high school

teachers who may desire to study the origins of the science, should send them to the committee at an early date. What is much more important, however, is that members should, if possible, cooperate in the work by volunteering to transcribe, translate, and annotate certain classical memoirs or extracts from treatises, these being confined to material of undoubted value as foundations upon which various parts of the mathematical structure stand.

Nearly two hundred topics have already been suggested by a few scholars to whom the matter has been informally mentioned. The list is too long to be published in full, and it is evident that only a small fraction of the items can be finally selected. The Committee would be glad, however, to receive as many further suggestions as possible.

It is desirable that

(1) Any piece of source material chosen should be of general interest to mathematicians, or at any rate to a large enough group to warrant the choice.

(2) It should not be already conveniently accessible in books that can be found in university or large public libraries.

(3) If possible, it should be of a nature to permit of printing as a whole, and should therefore be in concise form. This does not exclude the publication of definite chapters of a book or memoir, but a mere series of brief extracts would be better adapted to a series of historical essays than to a source book.

With these points in view the Committee asks for suggestions as to

(1) A limited number of the most important and epoch-making memoirs or definite sections of treatises, of interest to a sufficiently large number of readers, written during the period 1500–1900, and not generally accessible.

(2) A statement of the precise location of such source material; of the probable number of octavo pages (say of 360 words each) required for (a) the original text and (b) the translation; and of the probable necessity for explanatory notes.

(3) Whether or not the correspondent will volunteer to supply the text, translation, and notes.

The Committee volunteers its services to collate all replies and to follow as far as possible the desires of the majority of those who express an interest in the project. It must, of course, reserve the right to select among several volunteers who choose the same subject, and to make final choice as to the publication of the articles submitted. With such a large range of topics and a possible large number of expressed opinions and offers to collaborate, this selection will naturally take some time, but it is hoped to expedite the work as rapidly as circumstances permit It is probable that articles for which there is no space in the volume will be welcomed by various periodicals.

In case this volume meets with general approval, it is the expectation that two other volumes on mathematics may be published later; one on source

material before 1500, and the other on such material produced since 1900 probably with other important memoirs omitted from the present volume through lack of space.

No selections have as yet been made, either of topics or contributors, and it is hoped that volunteers will notify the Committee as soon as possible of any article or articles that they will be able to prepare.

It is desirable to bring this matter to the attention of mathematical clubs of various kinds in the graduate schools of our universities, thus enlisting the interest of the advanced students in mathematical courses. It is probable that a considerable number of the articles will be written by men and women of this type.

Suggestions of topics and, what is even more important, offers to edit certain ones in which the volunteer is especially interested, may be sent to Professors R. C. Archibald, Brown University, Providence, R. I.; Florian Cajori, Berkeley, Calif.; or David Eugene Smith, 501 West 120th Street, New York City.

A few possible topics are the following: decimal fractions; Delamain and Oughtred on the slide rule; the binomial theorem; mathematical induction; Cardan's solution of the cubic and biquadratic (Del Ferro, Tartaglia, Ferrari); De Moivre's theorem $\cos nx+i \sin nx=(\cos x+i \sin x)^n$ and the relation of Cotes, $ix=\log(\cos x+i \sin x)$; definitions of e; uniform convergence; Determinants; Hill-Poincaré infinite determinants; Kummer's ideals; Descartes, selections from *La Géométrie;* Desargues's theorem on perspective triangles; Gauss's statement on the construction of regular polygons; Bolyai's first essay (1823); Lobachevsky's *Pan-Geometry* (1835); circular points at infinity; Gauss's measure of curvature of surfaces; Bolzano (Weierstrass) nowhere differentiable continuous functions; Bessel's functions and numbers.

THE ANALOGY OF THE THEORY OF KUMMER'S IDEAL NUMBERS WITH CHEMISTRY AND ITS PROTOTYPE IN PLATO'S CONCEPT OF IDEA AND NUMBER[1]

By HARRIS HANCOCK, University of Cincinnati

As this article is intended for those who are interested in philosophic as well as mathematical problems, it may be well to indicate the nature of Kummer's ideal numbers by means of two examples. The first is due to Sommer, the second to Hensel. The presentation of the ideal numbers which is given immediately after these examples is not necessary for a casual survey of the subject under consideration. The mathematician, however, who reads the entire article, will, I believe, get a deeper insight into the Plato concept.

[1] Read before the Ohio Section of the Mathematical Association of America, April 5, 1928.

The following is the example used by Sommer (*Vorlesungen über Zahlentheorie*). Consider as a fixed realm of rationality the realm composed of integers of the form $4n+1$ and permit in the discussion only the operations of multiplication and division in their usual sense. In the series

$$1,5,9,13,17,21,\cdots,\ 45,\cdots,\ 117,\cdots,\ 517,\cdots$$

it is clear that the product of any two integers of the series is an integer of the series since $(4m+1)\ (4n+1)$ is of the form $4g+1$, where m and n are any two positive integers and $g=4mn+m+n$. The numbers 5, 9, 13, 17, 21, 29, $\cdots$, are irreducible, that is, they take the place of prime integers in this realm, in that they are not equal to the product of any other two integers of the realm.

Next observe that the number 10857 may be factored in the following two different ways $10857=141\cdot 77=21\cdot 517$, where 21, 77, 141 and 517 are prime numbers in the fixed realm. Thus it is seen that in this realm the factorization of an integer into its prime factors is not a unique process. This factorization, as is well known, becomes unique, if the fixed realm in question is extended so as to comprise what is known as the realm of *all* rational integers, in which $10857=3\cdot 7\cdot 11\cdot 47$.

Kummer's thought when applied to the above special case, consists in replacing the factors 3, 7, 11, 47, by what may be called "ideal numbers" in the restricted realm. In this realm, observe that the integers 3, 7, 11, 47 as such do *not* exist. To grasp the import of the "ideal numbers," denote the greatest common divisor of two integers a and b by the symbol (a, b) and observe that $(a, b)=(b, a)$. In the extended realm (that is, the realm of all natural integers) 3 is the greatest common divisor of 21 and 141 and may be written $3=(21, 141)$, where 21 and 141 are entities in the restricted realm. Further, we may put

$$7=(21,77)\ ;\qquad 11=(517,77)\ ;\qquad 47=(517,141)\ ;$$

$$(141)=(141,21)(141,517)\ ;\quad 77=(77,21)(77,517)\ ;$$

and

$$(10857)=(141,21)(141,517)(77,21)(77,517)=141\cdot 77.$$

On the other hand,

$$(21)=(21,141)(21,77)\ ;\ (517)=(517,77)(517,141)\ ;$$

and

$$(10857)=(21,141)(21,77)(517,77)(517,141)=21\cdot 517.$$

In both cases the factorization of 10857 through its "ideal" prime factors leads to a unique result.

Similarly, it is seen that $693=21\cdot 33=9\cdot 77$, where 21, 33, 9, 77 are irreducible in the restricted realm as is also $441=(21)^2=9\cdot 49$.

Another illustration due to Hensel[1] is of interest. Let all the natural integers be distributed into two classes. Into the class C_0 let unity and those integers enter which when factored offer an even number of prime factors, while the class C_1 is to include all those integers which when factored present an odd number of prime factors. It is seen that

$$C_0 = [1,4,6,9,10,14,15,16,21,22,24, \cdots]$$
$$C_1 = [2,3,5,7,8,11,12,13,17,18,19,20,23, \cdots].$$

Let only the integers of one of the classes, say C_0, form a fixed realm. We shall confine our attention only to integers of this realm. It is seen that $210 = 6 \cdot 35 = 10 \cdot 21 = 14 \cdot 15$ which are three products of prime integers in C_0. Denote the greatest common divisor of two integers a and b by the symbol (a, b) and observe that we may write

$$\begin{aligned} 210 &= (6,10)(6,21)(35,10)(35,21) = 2 \cdot 3 \cdot 5 \cdot 7, \\ &= (6,14)(6,15)(35,14)(35,15) = 2 \cdot 3 \cdot 7 \cdot 5, \\ &= (10,14)(10,15)(21,14)(21,15) = 2 \cdot 5 \cdot 7 \cdot 3. \end{aligned}$$

Observe further that the ideals in each of the last three lines are equal, for example, $(6, 10) = (6, 14) = (10, 14)$. If then we call the integers of C_0 the real integers and those of C_1 the ideal integers, it is seen that 210 is equal to the unique product of the four ideal (Kummer) integers 2, 3, 5, 7, which do not have a real existence in C_0. Notice also that the elements of the ideals, say 6, 10 of (6, 10) are numbers of the fixed realm C_0.

Consider next the factorization of 21 in the realm $R(\sqrt{-5})$, namely,

$$21 = 3 \cdot 7 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}) = (4 + \sqrt{-5})(4 - \sqrt{-5}).$$

Integers of this realm are of the form $a+b\sqrt{-5}$, where a and b are rational integers. Clearly there is something *common* to 7 and to at least one of the factors $(1+2\sqrt{-5})$, $(1-2\sqrt{-5})$. Take the product of these factors and form the congruence

$$1 - 2^2(-5) \equiv 0 \pmod 7. \tag{1}$$

Note that $-5 \equiv 3^2 \pmod 7$, so that (1) becomes $1 - 2^2 \cdot 3^2 \equiv 0 \pmod 7$.

1°. From this it is seen that $1 + 2 \cdot 3 \equiv 0 \pmod 7$. Compare this congruence with the factor $1+2\sqrt{-5}$. Kummer denoted that which is common to 7 and $1+2\sqrt{-5}$ by what I call the *Kummer factor* $\{7, 3\} = K_1$, while the Kummer factor $K_2 = \{7, -3\}$ denotes what is common to 7 and $1-2\sqrt{-5}$.

[1] Hensel, *Gedächtnissrede auf E. E. Kummer*, Abhandlungen zur Geschichte der Mathematischen Wissenschaften, vol. 29 (1910), No. 22.

Observe in a similar manner the congruence

$$1 - 2^2(-5) \equiv 0 \pmod 3 \tag{2}$$

and note that $-5 \equiv 2^2 \pmod 3$. Writing the congruence (2) in the form $1-2^2 \cdot 2^2 \equiv 0 \pmod 3$, it is seen that $K_3 = \{3, 2\}$ and $K_4 = \{3, -2\}$ are the Kummer factors of 3 and $1+2\sqrt{-5}$ and of 3 and $1-2\sqrt{-5}$, respectively.

2°. Similarly, that which is common to 7 and $4+\sqrt{-5}$ is the Kummer factor $K_1 = \{7, 3\}$, while $K_2 = \{7, -3\}$ is the factor of 7 and $4-\sqrt{-5}$; further, $K_3 = \{3, 2\}$ is the Kummer factor of 3 and $4+\sqrt{-5}$, $K_4 = \{3, -2\}$ being that of 3 and $4-\sqrt{-5}$.

3°. To derive that which is common to each of the pairs $1+2\sqrt{-5}$, $4+\sqrt{-5}$; $1+2\sqrt{-5}$, $4-\sqrt{-5}$; $1-2\sqrt{-5}$, $4+\sqrt{-5}$; and to $1-2\sqrt{-5}$, $4-\sqrt{-5}$, we may proceed as follows:

Denote what is common to two numbers α and β by the symbol (α, β). Thus $(\alpha, \beta) = T$, means "what is common to α and β is T." From above

$$(7,1+2\sqrt{-5}) = K_1 = (7,4+\sqrt{-5}); \quad (7,1-2\sqrt{-5}) = K_2 = (7,4-\sqrt{-5});$$
$$(3,4-\sqrt{-5}) = K_3 = (3,1+2\sqrt{-5}); \quad (3,4+\sqrt{-5}) = K_4 = (3,1-2\sqrt{-5}).$$

Observe that

$$(4+\sqrt{-5},1+2\sqrt{-5}) = (4+\sqrt{-5},1+2\sqrt{-5},2[4+\sqrt{-5}]-[1+2\sqrt{-5}])$$

so that

$$(4+\sqrt{-5},1+2\sqrt{-5}) = (4+\sqrt{-5},1+2\sqrt{-5},7)$$
$$= (4+\sqrt{-5},7) = (1+2\sqrt{-5},7) = K_1.$$

Similarly,

$$(4-\sqrt{-5},1-2\sqrt{-5}) = (4-\sqrt{-5},7) = K_2 = (1-2\sqrt{-5},7).$$

Note that there is nothing save unity in common to $4+\sqrt{-5}$ and $4-\sqrt{-5}$. For, if there were, there would be something common to these two quantities and their sum, which is 8. As there is something in common with either of these quantities, say $4+\sqrt{-5}$ and 21, it would follow that there is something in common with

$$(21,8,4+\sqrt{-5}) = (21,8,8\cdot 8 - 3\cdot 21, 4+\sqrt{-5}) = (1,4+\sqrt{-5}) = (1),$$

contrary to the hypothesis.

Next observe that

$$(4+\sqrt{-5},1-2\sqrt{-5}) = (4+\sqrt{-5},1-2\sqrt{-5},2[4+\sqrt{-5}]$$
$$+[1-2\sqrt{-5}]) = (4+\sqrt{-5},1-2\sqrt{-5},9) = (4+\sqrt{-5},1$$
$$-2\sqrt{-5},9,[4+\sqrt{-5}][4-\sqrt{-5}]) = (4+\sqrt{-5},1-2\sqrt{-5},9,21)$$
$$= (4+\sqrt{-5},1-2\sqrt{-5},3) = (4+\sqrt{-5},3) = K_4 = (1-2\sqrt{-5},3).$$

Similarly,

$$(4 - \sqrt{-5}, 1 + 2\sqrt{-5}) = K_3 = (4 - \sqrt{-5}, 3) = (1 + 2\sqrt{-5}, 3).$$

It follows from the above scheme that associated with the factors

$$(K_1K_2)(K_3K_4) = (K_1K_4)(K_2K_3) = (K_1K_3)(K_2K_4)$$

are the integers

$$7 \cdot 3 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}) = (4 + \sqrt{-5})(4 - \sqrt{-5}).$$

Thus the unique factorization of 21 in $R(\sqrt{-5})$ is $K_1K_2K_3K_4$. In the realm of natural numbers the Kummer factors have no objective reality. Hence the name *ideal*.

In the more general[1] quadratic realm $R(\sqrt{m})$, $m \not\equiv 1 \pmod 4$, we have to do with the factorization of integers $x+y\sqrt{m}$. And as above we are led to the consideration of the congruence $x^2 - my^2 \equiv 0 \pmod p$. If $w^2 \equiv m \pmod p$, or $w^2 - pr = m$, r an integer, we have a Kummer factor $\{p, w\}$ defined through the congruence $x + wy \equiv 0 \pmod p$. This congruence put in the form of an equation is $x = pz - wy$. From this it follows that $x^2 - my^2 = p(pz^2 - 2wzy + ry^2)$. Hence, corresponding to the Kummer factor $\{p, w\}$ of p and $x+y\sqrt{m}$, there is an associated quadratic form (p, w, r), and consequently there exists a class of equivalent forms with determinant m through which p (connected with w as above defined) may be expressed. Then and only then, when the class to which (p, w, r) belongs is a principal class $(1, 0, m)$, can p be expressed through the form $p = x^2 - my^2 = (x+y\sqrt{m})(x-y\sqrt{m})$. In this case and only in this case are the Kummer ideal factors $\{p, w\}$ and $\{p, -w\}$ numbers (algebraic) and have a real existence.

We thus have the condition under which the rules of division that exist in the rational realm are also true in the quadratic realms without the necessity of introducing the Kummer factors, this being evidently the case when the number of non-equivalent classes of quadratic forms with determinant m is unity. When the quadratic form through which the prime ideal p may be expressed is not equivalent to the principal form, it is necessary to introduce the ideal factors to effect uniquely the factorization of the rational prime integers into irreducible factors. Thus it is shown that the theory of quadratic forms with determinant m is exactly correlated with the theory of complex numbers of the realm $R(\sqrt{m})$.

The Kummer theory may with some modification be so changed that the ideal factors of unreal existence may be replaced by ideals of a concrete form. For, if a Kummer ideal prime factor $\{p, w\}$ of p is defined through the congruence $x + wy \equiv 0 \pmod p$ it is seen that the collectivity (complex) of all

[1] See Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, p. 146.

integers of the form $x+y\sqrt{m}$, which are divisible by $\{p, w\}$ with a suitable choice of x, y, may also be expressed through $x=pz-wy$. And that is, the complex of all those algebraic numbers that are divisible by $\{p, w\}$ is of the form $pz+(\sqrt{m}-w)y$. This complex of numbers constitutes what is known as the modul $\mathfrak{a}=[p, \sqrt{m}-w]$. And it is further seen that any number of this modul $pz+(\sqrt{m}-w)y$ when multiplied by any integer of the realm, say $x'+y'\sqrt{m}$ is equal to

$$p(zx' - ryy' + wzy') + (\sqrt{m} - w)(yx' + pzy' - wyy'),$$

if we write $w^2 \cdot m=pr$. Observe that this latter expression is of the form

$$pZ + (\sqrt{m} - w)Y ;$$

and that is, a number of the modul $\mathfrak{a}$ when multiplied by an integer of the realm $R(\sqrt{m})$ is a number of the modul $\mathfrak{a}$. The counterpart of this in the theory of rational integers is: if an integer is divisible by the rational integer a, then the product of the first integer by any other integer is divisible by a. This might in a measure be used to define a rational integer a. It is used by Dedekind to define an ideal $=\mathfrak{i}=[\alpha, \beta]$, where the element p above is replaced by α and where β stands for $\sqrt{m}-w$. Accordingly, the ideal $=\mathfrak{i}$ is defined as the complex of integers $\alpha\lambda+\beta\mu$ where λ and μ run through all the integers of the given realm, and where α and β are definite fixed integers of this realm.

Kummer, in Crelle's Journal, vol. 35 (1846), p. 359, writes as follows regarding the analogy that exists between the theory of complex (algebraic) numbers and chemistry: "Multiplication for the complex numbers corresponds to the chemical compound (Verbindung); to the chemical elements or atomic weights there correspond the prime factors, and the chemical formulae for the decomposition of bodies are precisely the same as the formulae for the factorization of numbers. And the ideal numbers of our theory appear also in chemistry as hypothetical radicals (Radicale) which have not as yet been isolated (dargestellt), but have their reality, as do the ideal numbers in their combination (Zusammensetzung). For example, fluorine[1] which has not yet been isolated (1846) and is still counted as one of the elements, is an analog of an ideal prime factor. Idealism in chemistry however is essentially different from the idealism of complex (algebraic) numbers in that chemical ideal materials when combined with actual materials produce actual compounds: but this is not the case with ideal numbers. Furthermore, in chemistry the materials making up an unknown dissolved (aufgeloesten) body may be tested by means of reagents, which produce precipitates, from which the presence of the various materials may be recognized. Observe that the multiplication of prime ideal numbers produce

[1] Fluorine was isolated in 1886 by Moissant at Paris.

a rational prime integer, so that the reagents of chemistry are the analogues of the prime ideals, these prime ideals being exactly the same as the insoluble precipitate which, after the application of the reagent, settles.

"Also, the conception of equivalence is practically the same in chemistry as in the theory of algebraic numbers. For, as in chemistry, two weights are called equivalent if they can mutually replace each other either for the purpose of neutralization or to bring about the appearance of isomorphism. Similarly, two ideal numbers are equivalent if each of them can make a real rational number out of the same ideal number."

"These analogies which are set forth here are not to be considered as mere play things of the mind' in that chemistry as well as that portion of the number theory which is here treated have the same basic concept, namely, that of combination even though within different spheres of being. And from this it follows that those things which are related to this (principle of combination) and the given concepts which necessarily follow with it (that is, with the principle of combination) must be found in both (fields) by similar methods: The chemistry on the one hand of natural materials and on the other hand the chemistry of ideal numbers may both be regarded as the realizations (Verwirklichungen) of the concept of combination and of the concept-spheres dependent upon this (principle of combination) the former to be regarded as a physical which is bound with the accidents of external existence and therefore richer, the latter a mathematical, which in its inner necessity (nothwendigkeit) is perfectly pure, but therefore also poorer than the former."

While the above is not an exact translation, a few phrases having been added to make the meaning clearer, it gives, I believe, the full and exact import of what Kummer wished to convey.

Professor Edward Zeller, the world's authority on Greek philosophy writes:[1]

"Plato's pure mathematics is primarily a preparatory stage of Dialectic (general logic), the number with which it has to do are not ideal but mathematical numbers: not identical with ideas but intermediate between them and the things of sense. Side by side with numbers, the ideas of numbers are also spoken of, but only in the same sense that ideas generally are opposed to things."

And further:

"The more exact distinction between the two kinds of numbers is this; that the mathematical consists of homogeneous unities which can therefore be reckoned together, each with each, whereas with the ideal numbers this is not the case; consequently the former expresses merely quantitative, the latter logical determinations."

It is seen from this that in the notion handed down from Plato addition is an

[1] *Plato and the Older Academy.* By Eduard Zeller. Translated by Alleyne and Goodwin. Page 256.

operation belonging to the mathematical numbers while with the ideal numbers this is not the case. Even with the Greeks it would appear that mathematics was an apparatus for numerical calculation as well as the philosophy of thought.

From recent articles which have to do with the historical development of chemistry and other physical sciences, which have appeared for example in *Science* I am led to believe that the assertion is justified that the lines of thought as well as the laws of thought in mathematics, the sciences and philosophy are today so closely allied with those of earlier times that they are one and the same. Evolution of the human mind has been exceedingly slow. And it is gratifying to note that progress in the entire intellectual structure has been made and is being made in the right direction and along lines where the truth seems of itself to be the guide.

I cannot refrain however from noting certain difficulties which the philosophers seem to experience regarding the *Ideas* and *Numbers* of Plato:

Jowett, in *The Dialogues of Plato*, vol. 2, p. 13 (third edition), writes; "Plato's doctrine of ideas has attained an imaginary clearness and definiteness which is not to be found in his own writings." Jowett then proceeds to enumerate the ways in which these ideas are interpreted and describes the reasons of their misinterpretations and misunderstanding.

Professor Paul Shorey, in *The Unity of Plato's Thought*, Chicago Decennial Publications, first series, vol. 6, p. 82, writes: "Aristotle's account of Plato's later identification of ideas and numbers has been generally accepted since Trendelenburg's dissertation on the subject. Zeller rightly points out that the doctrine is not found in the extant writings, but adds that for Plato numbers are entities intermediate between ideas and things of sense. In my discussion of the subject I tried to establish two points; first that we need not accept the testimony of Aristotle, who often misunderstood Plato and was himself not clear as to the relation of mathematical and other ideas: second, that the doctrine of numbers as intermediate entities is not to be found in Plato, but that the passages which misled Zeller may well have been the source of the whole tradition about ideas and numbers."

It does not seem to me in view of the above quoted passages from Zeller and from what I have given above regarding the Kummer ideal numbers, that there is any incorrectness either in Zeller's interpretation or in that of the equally eminent Plato scholar. In the quoted passage, Zeller does not use the word "entity." Further Shorey (p. 83) writes; "The 'mathematical' numbers then are plainly the abstract, ideal numbers of the philosopher." Surely Kummer and Dedekind were philosophers as well as mathematicians. And, continuing, Professor Shorey writes; "The numbers of the vulgar are concrete numbered things." And this is precisely what I have indicated above. See also Shorey, *Ideas and Numbers Again*, Classical Philology, vol. 22 (1927). What particularly

impresses me is the following, also from Professor Shorey (Proceedings of the Sixth International Congress of Philosophy, p. 578); "And the more closely and critically one studies Plato, the more apparent it becomes that the one way to misunderstand him is to make condescending allowance for the naïveté, the immaturity, the primitive, unscientific, unsophisticated quality of his thought. Mankind has undergone many vicissitudes of experience since Plato wrote, by which philosophers must endeavor to profit. But philosophy itself, metaphysics, epistemology, speculation about ultimates in cosmogony, religion, ethics and politics, and the subtlety and logic of the art of debate have progressed since Plato just as much as epic has improved since Homer, tragedy since Sophocles, sculpture since Phidias, and eloquence since Demosthenes." Happily Professor Shorey does not include the mathematicians in the category of those whose progress by implication has been in the negative direction.

FORCED VIBRATIONS WITH SMALL RESISTANCE, APPROXIMATING RESONANCE[1]

L. W. BLAU, University of Texas

1. Introduction. The general subject of forced vibrations has been discussed rather fully by different authors[2], but the relations derived do not adapt themselves readily to the investigation of certain especially important cases. Some of these are; 1. Approximating resonance, with small resistance; 2. Resonance, with small resistance; 3. Approximating resonance, with zero resistance. In particular, the relation of these three cases to a fourth case; 4. Resonance with zero resistance, has not been clearly set forth. Rayleigh[3] states, in discussing the equation derived by him, that "the change of phase from complete agreement to complete disagreement, which is gradual when friction acts, here" (that is, when there is resonance and the friction is permitted to approach zero) "takes place abruptly. At the same time the amplitude becomes infinite." This statement is indefinite. "At the same time" does not mean *instantaneously*, for it is well known that on solving the differential equation of case 4 the amplitude, that is the maximum of the absolute value of the displacement, does not increase to infinity monotonically, but the relative maxima of this value increase directly as the first

[1] Read before the Texas section of the Mathematical Association of America, Jan. 28, 1928.

[2] Rayleigh, *Theory of Sound*, 2nd edition. The Macmillan company, 1894. Duffing, *Erzwungene Schwingungen Bei Veränderlicher Eigenfrequenz und Ihre Technische Bedeutung*, F. Vieweg u. Sohn, Braunschweig, 1918. Riemann-Weber, *Die Differential-und Integralgleichungen der Mechanik und Physik*, Zweiter, Physikalischer, Teil. F. Vieweg u. Sohn, Braunschweig, 1927, pp. 86 ff.

[3] *Theory of Sound*, 2nd edition. The Macmillan company, 1894, vol. 1, p. 48.

power of the time. From the theory of differential equations it is also known that the solution of (1), below, is a continuous function of the coefficients, so that cases 1, 2, and 3 should pass over into case 4 smoothly. Physical considerations also lead to a similar conclusion, for the impressed force must impart to the system sufficient energy to cause an infinite amplitude, and it is not very plausible that this should be accomplished within one half of a cycle. It is the purpose of this paper to derive an equation for the amplitude of vibration in terms of the frequency of the impressed force, the difference between this frequency and the natural frequency, and the resistance: this relation is well adapted to the investigation of the cases mentioned above. It is found that in cases 1, 2, 3, the first maximum is a maximum maximorum if the initial displacement and the initial velocity are each equal to zero; there is a "beat" frequency the period of which depends upon the difference between the frequency of the impressed force and the natural frequency of the system as well as on the resistance. In passing over to case 4 the above "beat" frequency becomes zero, and the amplitude increases with the first power of the time.

2. The Differential Equation of Motion. Given a system having unit mass and natural frequency of vibration $n/(2\pi)$: let the oscillations of the system be opposed by a force which is directly proportional to the first power of the velocity and let a periodic force of frequency $p/(2\pi)$ be impressed on the system. The differential equation of motion then becomes

$$\ddot{y} + 2R\dot{y} + n^2 y = E \cos pt, \tag{1}$$

where y is the displacement from the position of rest, $\dot{y}$ the velocity, and $\ddot{y}$ the acceleration, while n, R, and E are constants. Let the initial displacement and the initial velocity be each equal to zero; then we have the conditions

$$y(0) = 0 \text{ and } \dot{y}(0) = 0. \tag{2}$$

The general solution of equation (1) is

$$y(t) = A \cos pt + B \sin pt + Ce^{-Rt} \cos (n^2 - R^2)^{1/2} t + De^{-Rt} \sin (n^2 - R^2)^{1/2} t, \tag{3}$$

where

$$C = \frac{(p^2 - n^2)E}{(n^2 - p^2)^2 + 4R^2p^2} \text{ and } D = \frac{-RE(n^2 + p^2)}{(n^2 - R^2)^{1/2}[(n^2 - p^2)^2 + 4R^2p^2]}.$$

In order to satisfy the conditions (2) the constants A and B must have the following values:

$$A = \frac{(n^2 - p^2)E}{(n^2 - p^2)^2 + 4R^2p^2}; \qquad B = \frac{2RpE}{(n^2 - p^2)^2 + 4R^2p^2}. \tag{4}$$

Let $(n^2-R^2)^{1/2}=n-\Delta n$, and $n=p+s$: then $n-\Delta n=p+s-\Delta n$. Equation (3) then becomes

$$
\begin{aligned}
y(t) &= A\cos pt + B\sin pt \\
&\quad + e^{-Rt}[C\cos(p+s-\Delta n)t + D\sin(p+s-\Delta n)t] \\
&= [A + e^{-Rt}\{C\cos(s-\Delta n)t + D\sin(s-\Delta n)t\}]\cos pt \\
&\quad + [B + e^{-Rt}\{D\cos(s-\Delta n)t - C\sin(s-\Delta n)t\}]\sin pt.
\end{aligned} \tag{5}
$$

This may be written

$$
y(t) = \Big\{A^2 + B^2 + (C^2+D^2)e^{-2Rt} - 2e^{-Rt}[(A^2+B^2)(C^2+D^2)]^{\frac{1}{2}}\sin\Big[(s-\Delta n)t + \tan^{-1}\frac{C}{D} + \tan^{-1}\frac{B}{A}\Big]\Big\}^{\frac{1}{2}}\sin(pt+\beta) \tag{6}
$$

where

$$
\beta = \tan^{-1}\frac{A + e^{-Rt}(C^2+D^2)^{1/2}\sin\Big[(s-\Delta n)t + \tan^{-1}\frac{C}{D}\Big]}{B + e^{-Rt}(C^2+D^2)^{1/2}\cos\Big[(s-\Delta n)t + \tan^{-1}\frac{C}{D}\Big]}.
$$

Equation (6) represents an oscillatory motion. If we let α represent the amplitude, then

$$
\alpha = \text{maximum}\left|\Big(A^2 + B^2 + (C^2+D^2)e^{-2Rt} - 2e^{-Rt}[(A^2+B^2)(C^2+D^2)]^{1/2}\sin(s-\Delta n)t + \tan^{-1}\frac{C}{D} + \tan^{-1}\frac{B}{A}\Big)^{1/2}\right|. \tag{7}
$$

It is clear that the amplitudes form an infinte set, defined for values of t where the velocity is zero.

To find the solution of the differential equation satisfying the boundary conditions (2) for the case when $s=0$, $R=0$, and $\Delta n=0$, we let s, Δn, and R each approach zero in equation (6); then we obtain

$$
y(t) = -(1/2p)Et\sin pt, \tag{8}
$$

which is the equation of case (4).

3. Conclusions: We proceed to the interpretation of equations (6), (7), and (8). If s and Δn are not both equal to zero, equation (6) represents an oscillatory motion of varying amplitude. The maxima occur with a frequency $(s-\Delta n)/(2\pi)$, and according to equation (7) the first maximum is the greatest on account of the exponentials. This is quite apparent on sketching the amplitude as given by equation (7). The accompanying figures 1–5 are graphs of this equation for different values of n, p, R and E. In Fig. 1, n was taken

equal to 25, $p=40$, $R=0.3$ and $E=5000$. Thus s is equal to -15. We note from the graph that the first maximum is the greatest, while the first minimum is the smallest. The same features are evident from Fig. 2; here $n=25$, $p=20$, $R=0.3$ and $E=1000$. The "beat" frequency is smaller than in Fig. 1, and the first minimum is larger. In Fig. 3 n, has the same value as before, $p=24$, R is again 0.3, while $E=200$. There are now only three "beats"; the first

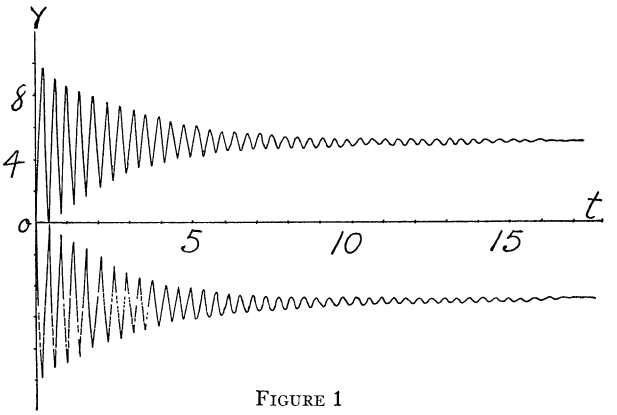


FIGURE 1

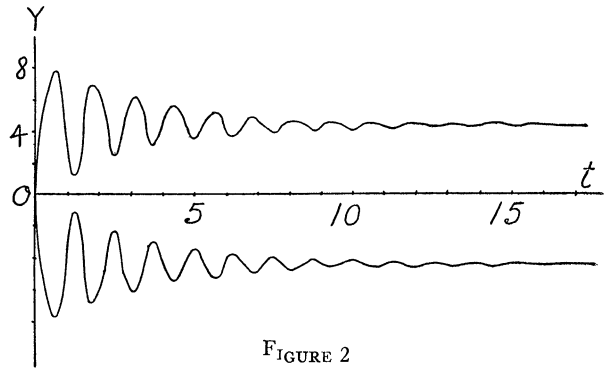


FIGURE 2

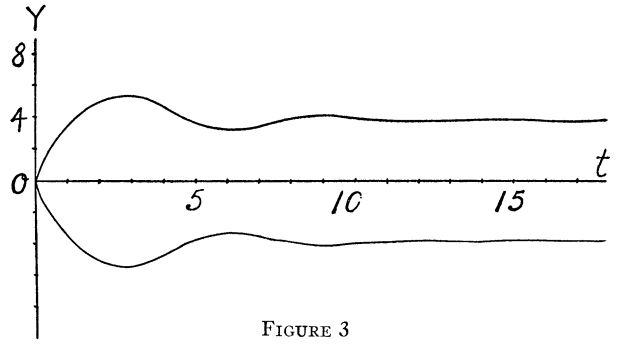


FIGURE 3

minimum is still larger than in Fig. 2. In Fig. 4, $n=p=25$, $R=0.3$ and $E=1000$. Equation (7) shows that even at resonance there is a "beat" frequency of $\Delta n/(2\pi)$. These "beats" are not brought out in this graph, because the frequency n is too large in comparison with R. As R approaches n these "resistance beats" at resonance become more prominent. Fig. 5 shows beats without

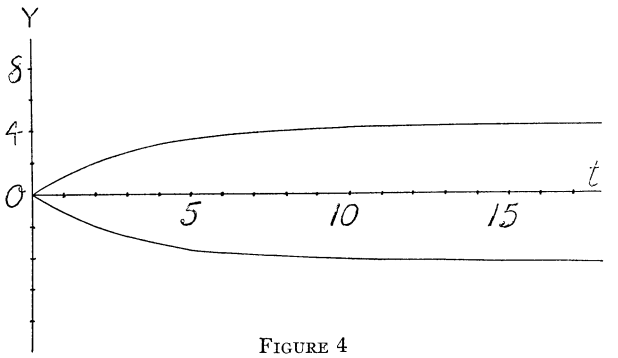


FIGURE 4

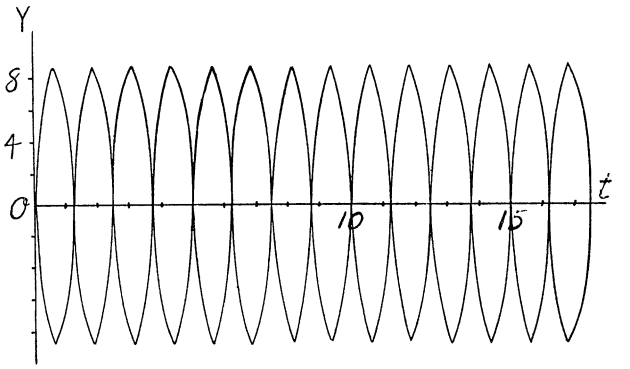


FIGURE 5

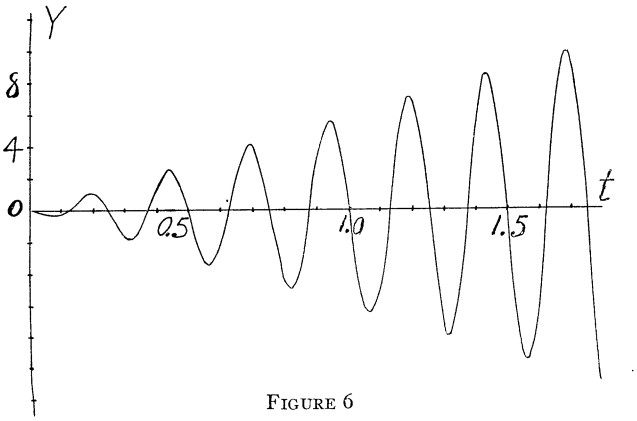


FIGURE 6

damping. In this graph n has the same value as before, $p=20$, and $E=1000$. Fig. 6 is the graph of equation (8).

There is a beat frequency of $(s-\Delta n)/(2\pi)$ for all values of $s \neq \Delta n$. If $s=\Delta n$, that is, if $n^2-p^2=R^2$, the resultant amplitude increases monotonically from zero to the amplitude of the forced vibration in the steady state. This is the only case in which there are no "beats" in a damped system. The difference between the squares of the frequencies may therefore be called the critical frequency difference. An increase in s while Δn remains constant results in an increase in the "beat" frequency. If the system has zero resistance and the impressed force is gradually brought into resonance with the natural frequency of the system, the beat frequency decreases until, when $s=0$ and $\Delta n=0$, the amplitude increases with the first power of the time and becomes infinite with t according to equation (8). In all cases, where the initial displacement and the initial velocity are each equal to zero and where at least one of the quantities s or Δn is different from zero the first relative maximum amplitude is greater than any of the later maxima and greater than the maximum vibrations of the steady motion; for it is apparent from equation (6) that the equation of the steady motion is

$$y(t) = (A^2 + B^2)^{1/2} \sin\left[pt + \tan^{-1}\frac{A}{B}\right]. \tag{9}$$

If the damping is small the steady state is reached much later than if it is large; but in all cases where "beats" are present the first "beat" maximum is greater than any other later maximum vibration.

Equation (6) has been derived from the solution of equation (1) which satisfies the homogeneous boundary conditions. If the boundary conditions are non-homogeneous, the solution of (1) is found by adding to equation (6) the solution of the homogeneous differential equation satisfying the non-homogeneous boundary conditions.

The facts enumerated above show that in any case of forced vibrations the system is subjected to the greatest strain when the first maximum is reached if friction is involved; neglecting fatigue there is little danger of a breakdown after this. This, it may be remarked, confirms our intuitive feeling when standing on a vibrating suspension bridge, that the danger of a collapse has been averted if the bridge has withstood the strain incident to the violent oscillations of the first "beat" maximum.

The writer has made an experimental investigation of these cases; records have been obtained for different values of damping when the frequencies were different and when there was resonance. The results are in excellent agreement with theory.

It is a pleasure to acknowledge my indebtedness to Professor H. J. Ettlinger who suggested the problem.

LINE COMPLEXES IN KINEMATICS

By E. L. REES, University of Kentucky

1. One of the most beautiful applications of the elementary theory of line complexes is found in the theory of the motion of a rigid body in space. Although this subject is treated in several places in the literature it seems to be not very well known and few references are made to it, in spite of the fact that the proofs, both analytic and geometric, are simple and brief.[1]

We shall in this paper develop the theory of linear line complexes and tetrahedral complexes from the point of view of kinematics, using vector methods which are of particular advantage in such work.

2. Central axis. We first show that in the general motion of a rigid body in space the body is instantaneously rotating about and moving in the direction of an axis which is called the *kinematical central axis*.

Let $\boldsymbol{p}$ be the position vector of a point fixed in the body and $\boldsymbol{q}$ the vector from this point to another point also fixed in the body. The position vector of the latter point is $\boldsymbol{r}=\boldsymbol{p}+\boldsymbol{q}$. For points whose velocities have the direction of the angular velocity vector $\boldsymbol{w}$ we have $\boldsymbol{w}\times\dot{\boldsymbol{r}}=\boldsymbol{w}\times\dot{\boldsymbol{p}}+\boldsymbol{w}\times(\boldsymbol{w}\times\boldsymbol{q})=0$, the dots indicating time derivatives. Solving for $\boldsymbol{w}\times\boldsymbol{q}$ we find $\boldsymbol{w}\times\boldsymbol{q}=\boldsymbol{w}^{-1}\times(\boldsymbol{w}\times\dot{\boldsymbol{p}})$, from which $\boldsymbol{q}=\boldsymbol{w}^{-1}\times\dot{\boldsymbol{p}}+u\boldsymbol{w}$, in which u is a scalar parameter and $\boldsymbol{w}^{-1}=\boldsymbol{w}/(\boldsymbol{w}\cdot\boldsymbol{w})$. Hence the locus of points whose velocities have the direction of $\boldsymbol{w}$ is the line $\boldsymbol{r}=\boldsymbol{p}+\boldsymbol{w}^{-1}\times\boldsymbol{p}+u\boldsymbol{w}$, which is therefore the central axis.

3. Definition of linear complex. Before giving the kinematical definition of a linear complex we prove the following theorem:

If a line of a moving body is normal to the velocity of one of its points it is normal to the velocities of all of its points.

PROOF: Let $\boldsymbol{r}=\boldsymbol{p}+u\boldsymbol{q}_1$ be the equation of the line, where $\boldsymbol{p}$ is the position vector of the point whose velocity is normal to the line and $\boldsymbol{q}_1$ is a unit vector having the direction of the line. We have then $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1=0$. Since $\dot{\boldsymbol{r}}\cdot\boldsymbol{q}_1=\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1+u\boldsymbol{q}_1\cdot\dot{\boldsymbol{p}}_1=0$ for all values of u, it follows that the velocity of any other point of the line is also normal to the line.

All lines of the body which are normal to the velocities of their points constitute a *linear complex*. The *central axis* is called the *axis of the complex*. If the motion is simple rotation the complex consists of all lines which intersect the central axis and is called a *special linear complex*. The complex for transla-

[1] This subject is treated geometrically by A. Mannheim in the Journal de Mathématiques, (2), vol. 11 (1866), pp. 273–279; and by Ch. Brisse in the same journal, vol. 15 (1870), pp. 281–314. References are given here to earlier memoirs on this subject by M. Chasles. Also the subject is treated geometrically, but from a different point of view, by A. Schoenflies in *La Géométrie du Mouvement*, French translation by Ch. Speckel, Gauthier-Villars, Paris, 1893.

tion consists of all lines perpendicular to the direction of motion and is a special linear complex with axis at infinity.

4. Polar system. *All lines of the complex through a point of the body,* being perpendicular to the velocity of this point, form *a plane pencil.*

THEOREM: *There is in general one and only one point of each plane of the body whose velocity is normal to the plane.*

To prove this theorem let $\mathbf{a}\cdot\mathbf{q}=k$ be the equation of the plane, the origin being at the tip of $\mathbf{p}$. If the velocity of the point $\mathbf{p}+\mathbf{q}$ is normal to the plane, we have $\mathbf{a}\times(\dot{\mathbf{p}}+w\times\mathbf{q})=0$, and therefore $w\cdot\mathbf{aq}=kw+\mathbf{a}\times\dot{\mathbf{p}}$, which in general defines one and only one point. If the plane is parallel to the central axis the point is at infinity. If the motion is simple rotation and the plane contains the axis then every point of the plane satisfies the condition.

The plane through a point which is perpendicular to the velocity of this point is called the *polar plane* of the point and the point is called the *pole* of the plane. We have as corollaries:

All lines of a plane through its pole belong to the complex, and conversely, all lines of the complex in the plane pass through the pole.

The polar planes of points of the central axis are normal to this axis.

5. Conjugate lines. If the equation of a line l of the body is $\mathbf{r}=\mathbf{p}+u\mathbf{q}_1$, then the equation of the polar plane of any point of l is

$$[\mathbf{r}-(\mathbf{p}+u\mathbf{q}_1)]\cdot(\dot{\mathbf{p}}+u\dot{\mathbf{q}}_1)=0, \text{ or } \mathbf{r}\cdot\dot{\mathbf{p}}-\mathbf{p}\cdot\dot{\mathbf{p}}+u(\mathbf{r}\cdot\dot{\mathbf{q}}_1-\mathbf{p}\cdot\dot{\mathbf{q}}_1-\dot{\mathbf{p}}\cdot\mathbf{q}_1)=0.$$

Therefore all such polar planes pass through the intersection of the planes $\mathbf{r}\cdot\dot{\mathbf{p}}=\mathbf{p}\cdot\dot{\mathbf{p}}$ and $\mathbf{r}\cdot\dot{\mathbf{q}}_1=\mathbf{p}\cdot\dot{\mathbf{q}}_1+\dot{\mathbf{p}}\cdot\mathbf{q}_1$. The equation of this line of intersection, which is the axis of the pencil or polar planes corresponding to the points of l regarded as poles, is $\mathbf{r}=m\dot{\mathbf{p}}+n\dot{\mathbf{q}}_1+v\dot{\mathbf{p}}\times\dot{\mathbf{q}}_1$, where m and n are determined by the equations

$$\dot{\mathbf{p}}^2m+\dot{\mathbf{p}}\cdot\dot{\mathbf{q}}_1n=\mathbf{p}\cdot\dot{\mathbf{p}} \text{ and } \dot{\mathbf{p}}\cdot\dot{\mathbf{q}}_1m+\dot{\mathbf{q}}_1^2n=\mathbf{p}\cdot\dot{\mathbf{q}}_1+\dot{\mathbf{p}}\cdot\mathbf{q}_1.$$

It is easy to show that the polar planes of the points of l', the axis of the pencil, form another pencil with l as axis. The lines l and l' are called *conjugate lines.* Thus the conjugate of a line may be regarded as the axis of the pencil of polar planes of points of the line or as the locus of the poles of the planes through the line.

If l is parallel to the central axis, that is, if $w\times\mathbf{q}_1=0$, and therefore $\dot{\mathbf{q}}_1=0$, the conjugate l' is at infinity, for then one of the planes which determine l' is at infinity. Conversely, the conjugate of a line at infinity is parallel to the central axis. It is the locus of the poles of a system of parallel planes and is called a *diameter of the complex.* Hence *all diameters are parallel to the central axis.*

By simple arguments we may prove the following:

All lines of the complex are self conjugate.

All lines intersecting a pair of conjugate lines belong to the complex, and if a line of the complex intersects one of two conjugates it intersects the other also.

We now prove the theorem:

The common perpendicular of two conjugate lines intersects the central axis perpendicularly.

Let $\boldsymbol{b}$ be a vector perpendicular to l and l', then $\boldsymbol{b}\cdot\boldsymbol{q}_1=0$, and $\boldsymbol{b}\cdot\dot{\boldsymbol{p}}\times\dot{\boldsymbol{q}}_1=0$ or $\boldsymbol{b}\cdot\dot{\boldsymbol{p}}\times(\boldsymbol{w}\times\boldsymbol{q}_1)=0$. Expanding this last equation, we get $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\boldsymbol{w}\cdot\boldsymbol{b}-\dot{\boldsymbol{p}}\cdot\boldsymbol{w}\boldsymbol{b}\cdot\boldsymbol{q}_1=0$, or in view of the first equation, we have $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\boldsymbol{w}\cdot\boldsymbol{b}=0$. Either $\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1=0$ and the line belongs to the complex and is therefore self conjugate, or $\boldsymbol{w}\cdot\boldsymbol{b}=0$ and $\boldsymbol{b}$ is perpendicular to the central axis. This common perpendicular is accordingly a line of the polar plane of a point of this central axis; and being a line of the complex it must intersect the axis in the pole of this plane.

6. Motion of a plane. We next consider certain theorems concerning the motion of a plane of the body.

The locus of points of a plane whose velocities lie in the plane is in general a straight line; and this line is the characteristic of the plane.

The equation of a plane may be written $\boldsymbol{r}=\boldsymbol{p}+u\dot{\boldsymbol{q}}_1+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$, where $\boldsymbol{p}$ is a point of the plane and $\boldsymbol{q}_1$ a vector normal to the plane.[1] For points whose velocities are in the plane, we have $\dot{r}\cdot\boldsymbol{q}_1=\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1+u\boldsymbol{q}_1\cdot\ddot{\boldsymbol{q}}_1=0$. Since $\boldsymbol{q}_1\cdot\ddot{\boldsymbol{q}}_1=-\dot{\boldsymbol{q}}_1^2\neq 0$, this equation determines a unique value of u which, when substituted in the equation of the plane, gives a line of that plane for the locus of points whose velocities are in the plane, as was to be proved. If $\boldsymbol{p}$ is a point of this line, $u=0$ and the equation of the line is $\boldsymbol{r}=\boldsymbol{p}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$.

The equation of the characteristic[2] of this plane is $\boldsymbol{r}=\boldsymbol{p}+\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\dot{\boldsymbol{q}}_1^{-1}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$; and if $\boldsymbol{p}$ is a point with velocity in the plane then $\boldsymbol{r}=\boldsymbol{p}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$. Hence the theorem.

The direction of the conjugate of the characteristic is determined by $\dot{\boldsymbol{p}}\times\mathrm{t}(\boldsymbol{q}_1\times\ddot{\boldsymbol{q}}_1)$, that is, by $\boldsymbol{q}_1$, which shows that it is normal to the plane and therefore to the characteristic.

By a simple argument it may be shown that the point at which the conjugate intersects the plane is the pole of the plane and that *the velocity of the point of the plane which lies at the foot of the perpendicular from the pole to the characteristic is in the direction of the characteristic.*

[1] We assume in this discussion that $\dot{q}_1=\neq 0$. If $\dot{q}_1=0$ then $W\times\boldsymbol{q}_1=0$ and the plane is rotating instantaneously about an axis perpendicular to it or there is no rotation at all. In this case all of the points of the plane would have velocities in the plane or there would be no such points.

[2] We use the term *characteristic of a plane* in the usual sense to mean the ultimate intersection of neighboring planes of a one parameter family. Here the successive positions of a plane of the body constitute a one parameter family. The characteristic is found by the usual method to be the intersection of the planes $(\boldsymbol{r}-\boldsymbol{p})\cdot\boldsymbol{q}_1=0$ and $(\boldsymbol{r}-\boldsymbol{p})\cdot\dot{\boldsymbol{q}}_1-\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1=0$, which is the line $\boldsymbol{r}=\boldsymbol{p}+\dot{\boldsymbol{p}}\cdot\boldsymbol{q}_1\dot{\boldsymbol{q}}_1^{-1}+v\boldsymbol{q}_1\times\dot{\boldsymbol{q}}_1$.

This relation is a reciprocal one, the conjugate line being the characteristic of the plane through it which is perpendicular to the characteristic of the original plane. Indeed either of two perpendicular conjugates is the characteristic of a plane perpendicular to the other and the pole of this plane is the point in which the other conjugate pierces it.

7. The tetrahedral complex. The lines determined by the points of the body, and their velocities at a given instant, form a tetrahedral complex. In view of the preceding article it follows from this definition that *the characteristics of the planes of the body form a tetrahedral complex.*

8. Properties of tetrahedral complexes. Since the direction of the characteristic of a plane is determined by $q_1 \times \dot{q}_1 = w - w \cdot q_1 q_1$, which is the component of w parallel to the plane, it may be shown geometrically that *the lines of a tetrahedral complex which pass through a point form an orthogonal cone*; also it is easy to see that *the lines of the complex which lie in a plane envelop a parabola.*

The axes of the two pencils of orthogonal planes, whose lines of intersection are the elements of the orthogonal cone, are the tangent to the trajectory of the vertex of the cone and a line through the vertex parallel to the central axis.

The point of the characteristic of a plane whose velocity has the direction of the characteristic is the foot of the perpendicular from the pole of the plane to the characteristic. This point is also the vertex of the parabolic envelope of the lines of the complex in the plane.

The focus of the parabolic envelope of lines of the complex in a plane is the pole of the plane with respect to the linear complex. If the intersection of two orthogonal planes is the axis of the parabolic envelope of one of the planes it is also the axis of the envelope of the other plane. The vertex of each parabola lies on the focus of the other. In other words the two envelopes are focal parabolas.

9. Relations between the Linear Complex and the Tetrahedral Complex. Below are given a few of the more important relations between the lines of the linear complex and those of the tetrahedral complex.

The lines of the tetrahedral complex which are determined by the velocities of the points of a given line are perpendicular to the conjugate of this line with respect to the linear complex.

A pair of orthogonal conjugate lines with respect to the linear complex are lines of the tetrahedral complex; and conversely, any line of the tetrahedral complex is one of a pair of orthogonal conjugate lines with respect to the linear complex.

These statements may be proved by simple geometric arguments.

AMERICAN STANDARD MATHEMATICAL SYMBOLS

Prepared by the SECTIONAL COMMITTEE ON SCIENTIFIC AND ENGINEERING SYMBOLS AND ABBREVIATIONS

Foreword

The Sectional Committee on Scientific and Engineering Symbols and Abbreviations was organized under the procedure of the American Engineering Standards Committee as the culmination of a series of efforts on the part of the A.S.C.E., A.S.T.M., A.I.E.E., S.P.E.E., A.S.M.E., and other national organizations. The societies which accepted joint sponsorship are the A.A.A.S., A.S.C.E., A.I.E.E., S.P.E.E., and the A.S.M.E. This Sectional Committee consisting of representatives of thirty-four national societies, associations, and government departments held its organization meeting on January 21, 1926, and Subcommittee No. 6 was appointed to report upon mathematical symbols.

This Subcommittee held its organization meeting May 1, 1926. Its report was submitted to the Sectional Committee under date of January 22, 1927. After approval by the Sectional Committee the report was submitted to the sponsors under date of June 3, 1927. In due course approval was voted by all five sponsors.

After a review of the report in the light of certain criticisms not of a fundamental nature which had been brought out in the course of securing sponsorial approval the Subcommittee reaffirmed the report as originally drafted with the exception of a few minor revisions of an editorial character.

The proposed standard was submitted under date of December 5, 1927, to the American Engineering Standards Committee for approval and on January 11, 1928, the sponsors were notified of its approval as an "American Standard."

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MATHEMATICAL SYMBOLS

Arithmetic and Algebra

1. $=\ \neq\ +\ -\ \pm\ \mp\ <\ >\ \leqq\ \geqq$ () [] % ∞ $\approx$ (for approximately equal to)
2. $a\times b=a\cdot b=ab$; $a\div b=a/b=\frac{a}{b}$ (Influence extends to next + or −). Thus, $a-b/c-d$ should not be used for $(a-b)/(c-d)$. Note that $\frac{a}{b}$ is difficult to print in running text.
3. $a/b=c/d$ for proportion. Discourage $a:b::c:d$.
4. Notation by powers of 10 for very large or very small numbers is recommended: as 3.140×10^{9} and 3.140×10^{-6}. The notation $0.0^{5}314$ is useful in tables, to indicate that there are five zeros after the decimal point.
5. In writing numbers having a larger number of digits, half-spaces may well be used instead of commas to separate groups of digits. In writing decimals, the 0 before the decimal point should not be omitted (except in tables).
6. $|x|=$absolute value of x. $x!=1\cdot2\cdot3\cdots x$. Discourage $\underline{|x}$.
7. $\sqrt{x}=+\sqrt{x}$, not $\pm\sqrt{x}$ (x being real and positive). $a^{1/n}=\sqrt[n]{a}$. $a^{-n}=1/a^{n}$. $\exp x=e^{x}$ is useful when x is a complicated expression. Note that the bar or vinculum after the $\sqrt{}$ is very expensive to print.
8. When $\log x$ is ambiguous, use $\log_{10}x$ or $\log_{e}x$. The notation $\ln x$ may be mentioned as an alternative for $\log_{e}x$.
9. $P(n, r)=n(n-1)(n-2)\cdots(n-r+1)$
 $C(n, r)=[n(n-1)(n-2)\cdots(n-r+1)]/[1\cdot2\cdot3\cdots r)$ = binomial coefficients.

A common alternative for $C(n, r)$ is $\binom{n}{r}$; this however is difficult to print in running text.

10. $a \propto b$ (meaning a varies directly as b).

Elementary Geometry

11. $\angle$ Δ $\parallel$ $\perp$ $\odot$ $\square$ $\therefore$

Analytic Geometry

12. x, y, z; ξ, η, ζ; rectangular coordinates. Right-handed system preferred.
13. ρ, s = intrinsic coordinates. ρ = radius of curvature, s = length of arc.
14. $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, direction cosines.
15. r, θ = polar coordinates. ψ = angle from radius vector to tangent.
16. r, θ, ϕ = spherical coordinates. θ = co-latitude, ϕ = longitude. (Usage general in mathematical physics; other notations are used in astronomy.)
17. r, θ, z = cylindrical coordinates. (Usage diverse.)
18. Conics: e = eccentricity. p = semi-latus rectum (usage general in U.S.).
19. Straight line; $y = mx + b$.

Trigonometric and Hyperbolic Functions

20. $^\circ$ $'$ $''$ $\sin x$, $\cos x$, $\tan x$, $\operatorname{ctn} x$, $\sec x$, $\csc x$.
21. $\sin^{-1}x$ = the principal value of the angle whose sine is x (when x is real). Thus, $-\pi/2 \leqq \sin^{-1}x \leqq \pi/2$, $0 \leqq \cos^{-1}x \leqq \pi$, $-\pi/2 \leqq \tan^{-1}x \leqq \pi/2$. (Discourage arc sin x).
22. $\sin^2 x$ for $(\sin x)^2$ is an exceptional notation, justified by usage. Similarly for $\cos^2 x$, etc.
23. $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{ctnh} x$, $\operatorname{sech} x$, $\operatorname{csch} x$.
24. $\cosh^{-1}x$ = the principal value (when x is real). (Discourage arc sinh x).
25. $\sinh^2 x$ for $(\sinh x)^2$ is an exceptional notation, justified by usage. Similarly, $\cosh^2 x$, etc.
26. In general f^{-1} means the inverse of the function f; and f^2 denotes iteration of the functional operation. But in exceptional cases, f^2 may denote the square of the function f (as in $\sin^2 x$ and $\sinh^2 x$). In general, $[f(x)]^{-1} = 1/f(x)$.

Calculus, etc.

27. If $y = f(x)$, derivative $= y' = f'(x) = \dfrac{dy}{dx} = D_x y$.

Second derivative $= y'' = f''(x) = \dfrac{d(y')}{dx} = D_x^2 y = \dfrac{d^2y}{dx^2}$. Note: $\dfrac{d^2y}{dx^2}$ cannot be regarded as a fraction, except when x is the independent variable; in general, $\dfrac{d^2}{dx^2} = D_x^2$ is a symbol of operation on y.

28. If $u=f(x, y)$, partial derivative $=u_x=f_x(x, y)=D_xu=\dfrac{\partial u}{\partial x}$. Similarly, $u_{xy}=f_{xy}(x, y)=D_y(D_xu)=\dfrac{\partial^2 u}{\partial y\partial x}$. Note: $\dfrac{\partial^2 u}{\partial y\partial x}$ and $\dfrac{\partial u}{\partial x}$ are not fractions; $\dfrac{\partial}{\partial x}=D_x$ and $\dfrac{\partial^2}{\partial y\partial x}=D_yD_x$ are symbols of operation.

29. $\Delta y=$ increment, $dy=$ differential. $\delta y=$ variation. $\sum_{i=a}^{b}=$ summation.

30. $\dot{x}=dx/dt=v$; $\ddot{x}=dv/dt$ (used only for differentiation with respect to the time t, and difficult to print).

31. $\lim_{x\to a}(y)=b$; $y\to b$ as $x\to a$. (Discourage $\doteq$.)

32. $\int_a^b f(x)dx$. $F(x)\Big|_a^b=F(b)-F(a)$. $\int\int f(x,y)dxdy=\int[\int f(x,y)dx]dy$.

33. $\pi=3.1416\ldots e=2.718\ldots i$ (or j) $=\sqrt{-1}$.

34. If $z=x+iy$, then $|z|=$ absolute value, or magnitude, $<z=$ angle, $R(z)$ and $I(z)=$ real and imaginary parts, $\bar{z}=$ conjugate of z. (Where $\bar{z}$ is difficult to print, use conj z.)

Special Functions

35. Bessel Functions. The notation used in G.N. Watson's Treatise, 1922, as endorsed by E.P. Adams in the Smithsonian Tables, 1922, is recommended.

36. Bernoulli numbers. Of the five or six different notations in use, the notation $B_1, B_3, B_5, \cdots$ has historical priority and many practical advantages, but the notation $B_1, B_2, B_3, \cdots$ is the one most used in recent years. To indicate what usage is being followed, authors will do well to state explicitly the value of the first few numbers, as $B_1=1/6$, $B_2=1/30$, $B_3=1/42, \cdots$

37. $\gamma=0.5772\ldots$ (Euler's constant.)

Vector Analysis[1]

38. Vectors to be indicated in printed matter by letters in bold face type, and in written manuscript by letters modified by a bar above (or by the doubling of some part of the character). The magnitude of a vector to be indicated in print by the corresponding italic letter, and in manuscript (when necessary) by the use of the absolute value signs, $|\ |$.

39. The scalar product, or dot product, $=\boldsymbol{a}\cdot\boldsymbol{b}$, the dot being centered. (Other notations are $S\boldsymbol{ab}$, or $(\boldsymbol{ab})$ in round parentheses.)

40. The vector product, or cross product, $=\boldsymbol{a}\times\boldsymbol{b}$, the cross being small. (Other notations are $V\boldsymbol{ab}$, or $[\boldsymbol{ab}]$ in square brackets.)

41. $i, j, k=$ unit vectors along the axes (right-handed system).

[1] NOTE: As to further questions of notation in Vector Analysis (including Tensor Analysis), the desirability of a thorough-going attempt to bring uniformity out of the present diversity of usage is recognized, and the appointment of a special committee to take up this subject has been recommended.

Abbreviations

42. It is desirable to distinguish between (1) a "symbol," that is, a single letter or single letter affected with subscripts, etc., which is to be used to represent a numerical value in a formula; and (2) an "abbreviation," which may consist of several letters, but is not intended to be substituted for a numerical quantity in a formula.
43. Abbreviations such as ft/sec^2, ft-lb/min, etc., should not be further condensed, lest clearness be sacrificed to brevity.

NOTE: The recommendations concerning terms and symbols in elementary mathematics contained in Chapter 8 of the report on the Re-organization of Mathematics in Secondary Education, made in 1923 by the National Committee on Mathematical Requirements (under the auspices of the Mathematical Association of America and reprinted by Houghton Mifflin Company in 1927), are, with one or two important exceptions, endorsed.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. INTEGRAL BASES

By E. T. BELL, California Institute of Technology

Under the above title a review of W. E. H. Berwick's recent Cambridge Tract on this topic appeared in the Monthly for March, 1928, pp. 142–3. The following remarks are not intended in any way as a criticism either of the tract or of the review, both of which are admirable from the respective points of view from which they were written. It is important, however, to call attention to a significant, highly practical paper on this subject which is not mentioned either by the author of the tract or by the reviewer. (The author's tract went to press before the paper appeared). Students who are interested in these subjects will find it profitable to master the paper by Professor N. R. Wilson in the Transactions of the American Mathematical Society, vol. 29 (1927), pp. 111–126.

The minimum of space in which Wilson disposes of Woronoi's cubic problem will commend itself to all who have ploughed through the Russian original or Sommer's abstract of the same. Further, Wilson's methods apply at once to the quintic field, and, with but slight extensions, settle the points in question for such fields and for all higher fields.

There is another item of importance, not considered by the reviewer. In his tract, Berwick bases his account of Zolotareff's theory (apparently) on the

German abstract (including Bachmann's, Zahlentheorie, vol. 5). On reading Berwick's remarks, one gets the impression, perhaps not intended by the author, that Zolotareff's theory is a rank failure. This is unfortunate, as the facts are otherwise. Zolotareff's theory should be rescued from oblivion in the form in which Zolotareff himself put it forth. The theory is sound. I wo uld hesitate to make this assertion were it not reinforced by an opinion which all arithmeticians must receive with respect—namely that of Professor J. V. Ouspensky. The fact, independent of any mere opinion, seems to be thus: the accounts in Bachmann and the Fortschritte are based on insufficient knowledge of the Russian original. On other grounds (postulational formulations) also I believe that the current estimate of Zolotareff's theory is erroneous. At worst, a slight refinement of statement is all that is required to make it unassailable.

It is time that Zolotareff's work, with an adequate critical commentary, was made available to those who do not read Russian.

II. Beginning Geometry and College Entrance

By H. W. Tyler, Massachusetts Institute of Technology

In a recent interesting letter to the Monthly,[1] Professor Beatley has cogently urged the appropriateness of including a certain moderate amount of three-dimensional work in connection with the present requirement in plane geometry, in view of the fact that a larger proportion of the students concerned will have no other systematic study of solid geometry. Sympathizing decidedly with this idea, I have been from the standpoint of a scientific institution much concerned with the probable reaction on the solid geometry in case a substantial portion of it should have been anticipated as a part of the more elementary subject. The logical remedy for such duplication would naturally be the division of elementary geometry as a whole on a more fundamental basis than that traditionally expressed by the titles Plane and Solid. The division ought, it seems to me, to be based largely on relative difficulty and relative importance. Geometry A would, accordingly, include those parts of present plane geometry which are deemed important for boys and girls who expect to attend arts colleges or who are taking the school course without reference to more advanced study. It would contain also the simpler and more frequently used theorems of solid geometry, with special attention to mensuration. Geometry B, would then consist of the principal matters left over from the present plane and solid geometry so far as they are of value for the future student of science, with a large proportion of original exercises in both two and three dimensions. This course

[1] Ralph Beatley, *Beginning Geometry and College Entrance*, this Monthly, vol. 35 (1928), pp. 80–83.

would be taken by boys and girls of superior mathematical capacity and ordinarily required of candidates for admission to scientific institutions.

Such a redistribution of material would naturally carry with it the probability of a serious examination of the whole field of elementary geometry and would signify something more than a mere permutation of present elements and textbook pages. It would need the careful study of a commission, representing both schools and colleges, and would lead presumably to such a progressive modification of College Board examinations as Professor Beatley has suggested.

III. Remark on the Note "Some Geometric Illustrations for the Elementary Course in Differential Equations[1]

By J. D. Tamarkin, Brown University

It was not until recently that I became aware of the fact that property (A) (p. 27) had been stated and used for graphical solution of linear differential equations of the first order by E. Czuber, *Beitrag zum graphischen Integration der linearen Differentialgleichungen erster Ordnung*, Zeitschrift für Mathematik und Physik, 44 (1899), pp. 41–49. The proof is different from ours.

Czuber's method has been extended to some non-linear equations in an interesting paper by T. Kojima, *On graphical solutions of some differential equations of the first order*, The Science Reports of the Tôhoku Imperial University, (1), vol. 3, (1914), pp. 289–302.

IV. A Note on the Linear Differential Equation of the First Order

By E. G. Olds, Carnegie Institute of Technology

In the January, 1928, Monthly[2], Mr. J. D. Tamarkin gives *Some Geometric Illustrations for the Elementary Course in Differential Equations*. In the case of $dy/dx+p(x)y+q(x)=0$ he shows that its family of integral curves possess the property: (*A*) *The tangents at the points of intersection of the family of integral curves with any line parallel to the y-axis are either concurrent or parallel.*

Another proof of the above property is as follows: Let the line, $x=a$, cut the integral curves C_1, C_2, C_3, in points, the ordinates of which are y_1, y_2, y_3, respectively. Also set $q(a)=Q$, and $p(a)=P$.

Then the slopes of the tangents to the integral curves at $x=a$ are, respectively, $y_i' = -Qy_i - P$ $(i=1, 2, 3)$ and, for $Q=0$, the slopes become equal to $-P$ and the lines are parallel.

In general, the equations of the tangents are:

$$y - y_i = y_i'(x - a) \text{ or, in another form, } y_i' x - y + (y_i - ay_i') = 0.$$

[1] This Monthly, vol. 35 (1928), pp. 27–29.

[2] This Monthly, vol. 35 (1928), pp. 27-29.

The condition that these lines be concurrent is given by the formula

$$\begin{vmatrix} y_1' & 1 & y_1 - ay_1' \\ y_2' & 1 & y_2 - ay_2' \\ y_3' & 1 & y_3 - ay_3' \end{vmatrix} = 0,$$

which is readily verified.

V. An Extension of the Gauss Problem of Eight Queens

By Leonard M. Blumenthal, Johns Hopkins University

1. Introduction. The problem of ascertaining the number of ways in which eight queens may be placed on a chess board so that no queen can capture any other was proposed to Gauss by Nauck.[1] In a correspondence with his pupil, Schumacher, Gauss, "after finding the number to be 76 and then 72, ultimately arrived at 92, which has since been recognized as the correct solution.[2]" The purpose of this paper is to consider an extension of the problem to the case of n queens on a board of n^2 squares. It is shown that whenever n is not a multiple of two or three, many solutions may readily be obtained. Further, the problem is seen to be equivalent to one in combinatory analysis[3].

2. If the determinant $|a_{ij}|$ $(i, j=1, 2, \cdots, n)$ be expanded and all terms rejected in which two elements have either the sum or the difference of their subscripts identical, then the remaining terms will give all the solutions of the problem.[4] A moment's inspection justifies the rule, since each term involves but one element from each row and column (thus preventing castle-wise capture) and the rejected terms exclude the cases in which two or more queens lie in the same diagonal.

This method of solution is of practical value only for $n=3, 4, 5$; for the board of 64 squares, for example, it would involve calculation of 20160 terms. The method does give rise, however, to the following very suggestive statement of the problem:

[1] For a history of the problem see S. Günther, *Zur mathematischen Theorie des Schachbretts*, Grunert's Archiv der Mathematik und Physik, vol. 56, part 3, pp. 281–292. See also W. W. R. Ball, *Mathematical Recreations and Essays*, 7th edition pp. 113–118.

[2] J. W. L. Glaisher, *On the Problem of Eight Queens*, Edin. Phil. Mag., vol. 48, pp. 457–467.

[3] Quite another method of approach is given by W. H. Bussey, *A Note on the Problem of the Eight Queens*, this Monthly, vol. 29 (1922), p. 252. This interesting note shows "that in the special case in which n is a prime number p there is a connection between the problem of the queens and the lines of the finite plane geometry of p points to the line." Prof. Bussey obtains by use of finite geometry methods p^2-3p of the total number of solutions. Only for $p=5$ does this method give all of the solutions of the problem.

[4] This method, due to Günther and developed somewhat by Glaisher in the paper referred to, is rendered simpler of application and more suggestive with the notation adopted here.

Given two sets of numbers $(i, j = 1, 2, 3, \cdots, n)$, *to select n pairs, exhausting the two sets, so that no two pairs will have identical sums or differences.*

Using each pair as the subscripts of an element, the n pairs thus selected will furnish a solution, and the total number of ways in which the n pairs may be chosen will be the totality of solutions.

3. Consider the pairing $(n \mp 2k)$

$$(1) \qquad \begin{pmatrix} 1,2,3, \cdots, & \frac{1}{2}(n+1), & \frac{1}{2}(n+1)+1, \cdots, & n \\ 1,3,5, \cdots, & n, & 2, \cdots, & n-1 \end{pmatrix}$$

where a number i in the first row has its j directly beneath it. We obtain for $i+j$ the two sequences

$$(a) \qquad 2,5,8, \cdots, \tfrac{1}{2}(3n+1)$$

$$(b) \qquad \tfrac{1}{2}(n+1)+3, \tfrac{1}{2}(n+1)+6, \cdots, 2n-1$$

Evidently no number of sequence (b) will be contained in sequence (a) if the first term of (b) is not contained in (a). Now the numbers of (a) are all congruent to 2 (mod 3), whence if $\frac{1}{2}(n+1)+3$ is in (a) we must have

$$\tfrac{1}{2}(n+1)+3 \equiv 2 \pmod 3 \text{ or } n = 3k.$$

It is easily shown[1] that if $n=3k$, then $\frac{1}{2}(n+1)+3$ will be contained in (a).

The differences $i-j$ form the two sequences

$$0, -1, -2, \cdots, -\tfrac{1}{2}(n-1) \; ; \; 1,2, \cdots, \tfrac{1}{2}(n-1)$$

and hence no two of these are identical. Thus we have that (1) is a *solution for every n not a multiple of two or three*, and for *only* such n's.

4. Solutions may be obtained from a given solution by the following substitutions:

(1) $\binom{i,\ j}{j,\ i}$, reflection in principal diagonal.
(2) $\binom{i,\ j}{n-j+1,\ n-i+1}$, reflection in secondary diagonal.
(3) $\binom{i,\ j}{i,\ n-j+1}$, reflection in middle column.
(4) $\binom{i,\ j}{n-i+1,\ j}$, reflection in middle row.
(5) $\binom{i,\ j}{j,\ n-i+1}$, rotation 90 degrees clockwise.

Carrying out the square and the cube of (5) rotates a solution 180 and 270 degrees respectively. The solutions obtained by the above substitutions are not, of course, all distinct: e.g., the product of (3) and (4) is the square of (5).

A method of obtaining $(n-1)$ *distinct* solutions from a given one is to apply the substitution $\binom{i,\ j}{i,\ j+1}$ $(n-1)$ times to the given solution, the subscripts to be

[1] Let $n=3k$ and represent by

$$3j-1 \quad (j=1, 2, \cdots, \tfrac{1}{2}(3k+1))$$

the numbers of sequence (a). Then $3j-1=\frac{1}{2}(3k+1)+3$ for $j=\frac{1}{2}(k+3)$ which is among the values j assumes; in fact, the number $\frac{1}{2}(3k+1)+3$ will be the $\frac{1}{2}(k+3)$rd number of sequence (a).

reduced, mod n. The solutions will be distinct whenever the method furnishes them. To secure criteria to determine when the substitution yields solutions let

$$(A) \qquad \begin{pmatrix} 1,2,3, \cdots, j, \cdots, n \\ i_1, i_2, i_3, \cdots, n, \cdots, i_n \end{pmatrix}$$

be a solution, and consider the result of the substitution, i.e.,

$$(B) \qquad \begin{pmatrix} 1, & 2, & 3, \cdots, j, \cdots, & n \\ i_1+1, & i_2+1, & i_3+1, \cdots, 1, \cdots, & i_n+1 \end{pmatrix}.$$

Applying the test developed in section 2, we have that (B) will be a solution if and only if

$$j \neq k + i_k; \quad j \neq k - i_k \qquad (k = 1,2, \cdots, n).$$

This furnishes a very practical device in testing for a solution when n is large.

In conclusion, we remark that the above methods yield all of the 10 solutions for the case $n=5$ with a minimum of labor; that 28 of the 40 solutions for the case $n=7$ are obtained with the same ease (10 more solutions being immediately forthcoming upon securing two solutions that are not of the type treated here). More than 88 solutions for the case $n=11$ may be written down at once.

RECENT PUBLICATIONS

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REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Wave Mechanics; An Introductory Sketch. By H. F. BIGGS. Oxford University Press, American Branch, New York, 1927; 77 pages. Price $1.50.

In the words of the author, "This sketch is written for physicists of ordinary mathematical attainments, especially those to whom the language difficulty would make the reading of Schrödinger's original papers too laborious for a first plunge."

Wave-mechanics had its origin in the attempt by de Broglie to reconcile the undulatory theory of light with the apparent necessity of ascribing atomic properties to a field of radiation in order to account for the phenomena associated with the photoelectric effect. The hypothesis which de Broglie makes at the outset is that every mass particle, in the older sense, is the centre of a pulsation in space; to be consistent with the requirements of the restricted theory of relativity, the phase velocity, u, must be connected with the speed of the

particle, v, by the relation $uv = c^2$, where c is the velocity of light. One at once asks what it is that is pulsating, or through what are the pulsations transmitted? No answer can be given to these questions; our customary concepts are inadequate to picture a model that works in accordance with the requirements of such a theory. Schrödinger, however, showed that it is possible to relate the amplitude of these hypothetical pulsations to the electrical density in such a way as to lead to satisfactory results in a purely formal manner.

In the first chapter of this book, the author gives an excellent description of de Broglie's theory, and shows how it leads to Schrödinger's differential equation. This equation is then applied to the hydrogen atom; and the appropriate solution is stated without giving the calculation in detail. It may be noted that Schrödinger's original solution of this problem can be very much simplified by using the Laguerre polynomials. This solution is then shown to lead to the same spectral series as are given by Bohr's theory. *Wave-mechanics*, however, goes further in that it determines the intensities of spectral lines, as well as derives the selection principles that were found empirically.

In the second chapter, the author describes the method used by Schrödinger, in his second paper, to obtain the equation that goes by his name. This method is based on the Hamiltonian dynamics. When it is remembered that the starting-point of Hamilton's theory was the attempt to connect the undulatory and the corpuscular theories of light, it will be evident that this furnishes a natural foundation for the new wave-mechanics. Briefly, the essential feature of Schrödinger's theory is the interpretation of Hamilton's "action," when divided by Planck's constant, h, as the phase of a wave-function. Division by h is of course an arbitrary hypothesis, but a plausible one, since its dimensions are the same as those of "action."

The author of this small volume has carried out his purpose admirably. His insistence upon the purely formal nature of the theory is particularly to be commended; while the terminology of the older physics is used, we must be careful not to carry over too far the concepts to which we have become accustomed. One who reads this book will undoubtedly wish to pursue the subject further by going to the original papers. E. P. ADAMS

The Pythagorean Proposition. By ELISHA SCOTT LOOMIS. Cleveland, Ohio. Published by the Masters and Wardens Association of the 22nd Masonic District of the Most Worshipful Grand Lodge of Free and Accepted Masons of Ohio. 214 pages. Price $2.00.

Professor Loomis states, in the foreword of his book, *The Pythagorean Proposition*, that "the object of this work is to present to the future investigator, under one cover, simply and concisely, what is known relative to the

Pythagorean proposition, and to set forth certain established facts concerning the proofs and the geometric figures pertaining thereto."

There are discussed 58 algebraic proofs, 167 geometric proofs, four quaternionic proofs and one dynamic proof. Many of these proofs admit of variations so numerous that the author states that the number of algebraic proofs, and also the number of geometric proofs, is limitless. Technically, the author is unquestionably correct in this statement. Since, however, most of the variations are produced by the introduction of unnecessary complications into an otherwise simple proof, the number of proofs worthy of serious consideration is, for most of us, certainly finite. Nevertheless, even the most critical must admit that the number of essentially distinct interesting proofs is surprisingly large, and that the realization of this fact should be a source of inspiration for both student and teacher of geometry.

Professor Loomis has produced a valuable and interesting work. Merely as an up-to-date book of reference, it is useful a contribution to mathematical literature. But it is more than a book of reference, more than a cold-blooded compilation of facts. It reflects clearly the happy individuality and the unbounded enthusiasm of its author.

The necessity of preparing for college entrance examinations, and other exigiencies of the modern secondary school program, place serious restrictions on the enthusiastic teacher. The publication of Professor Loomis's book will be amply justified if it serves no other purpose than to encourage teachers to break away from a deadly routine of formal instruction and to reclaim more of the romantic and inspirational values which are the just heritage of geometry.

RALPH D. BEETLE

College Algebra. By ARTHUR M. HARDING and GEORGE W. MULLINS. New York, The Macmillan Co., 1928. vii+324 pages.

A natural reaction to the appearance of an addition to an already large group of text books is one of curiosity concerning the purpose of and need for such addition. The authors of this *College Algebra* make no plea for the book other than that of the differing needs of colleges, the flexibility of their product, and the incorporation of some important features presumably somewhat lacking in existing college algebras.

The content, general style, and arrangement of material are much the same as those of most good elementary books on mathematics. The chapter headings include the usual ones and others not so universal, such as partial fractions, graphical representation of numbers, interest and annuities, and scales of notation. The topics are presented in an easy conversational way, for the most part by using the scheme of statement followed by illustration; and the arrangement of material on the page is excellent. Flexibility is maintained through separa-

tion of the subject into discrete divisions from which interdependence has been eliminated as far as possible without making the book seem choppy. Short tables adequate for all the numerical computations are provided, and answers to the exercises are appended.

Exclusive of the introduction, the book contains over a thousand exercises. Of these the proportion of verbal problems to purely formal drill problems is roughly one to four. They, also, are of the usual type to which some might object on the grounds of lack of imaginative stimulation for the student.

Noteworthy features of the book include those mentioned by the authors and a few others. Straightforward graphical methods are frequently employed to advantage in locating and discussing the nature of roots of polynomials, in the study of quadratic functions, and in the solutions of simultaneous equations. As an aid to graphing quadratic functions, simple derivatives are introduced; and they are further utilized in Newton's method for approximating irrational roots of polynomials. Review material is amply provided for in the introduction. A chapter on "Scales of Notation" is unusual but of doubtful value.

Among the probable objections to the book none seem to be extraordinary. The rules for finding the characteristic of the common logarithm are the customary clumsy ones and there is the usual ambiguity of statement of problems in permutations and combinations. Teachers of algebra object to the use of the term "ways" as a substitute for both "arrangements" and "groups"; they also take exception to omitting the geometrical scheme usually understood to accompany the idea of permutations in a given connection and the point of view to be assumed in counting them. Although the treatment of permutations and combinations in this book is up to par, it is evident that the authors overlooked the opportunity of relieving teachers of some of the avoidable burdens connected with what is intrinsically difficult exposition for the classroom. Problems on the change of base in logarithms and discussion of homogeneous simultaneous equations are omitted. The definition of an infinite progression is faulty, and the statement that "r^n may be made as small as one pleases by taking n larger and larger" might well be made more rigorous, even for a beginner. Furthermore, "*the* quadratic equation which has two roots" may lead beginners to believe in the uniqueness of such an equation. The student will be amused by "an airplane 900 feet high."

Aside from these details the book possesses directness and simplicity of statement. The development of the quadratic formula without the use of fractions is impressively neat; exercises concerning the roots point out several interesting relations. The idea of logarithm is clearly presented; synthetic division is clearly explained and extensively used. The outstanding feature, perhaps, is the direct use of graphs in the theory of equations and in the discussion of the quadratic function.

The conclusion arrived at by the reviewer is that although the book is not a material improvement over current algebra books and although some features of it are not wholly commendable, teachers will find it a good basis for a beginning course in the subject.

EARL L. MICKELSON

Plane and Solid Analytic Geometry. By JAMES MCGIFFERT. Ginn and Company. xiv+335 pages. Price $2.48.

This text is the result of many years of experience in teaching by the author in the Rensselaer Polytechnic Institute. It is meant as a first course in the subject and is especially suitable for freshmen.

Polar coördinates are put off until the latter part of the course, after the student has taken up in rectangular coordinates the parabola, ellipse, and hyperbola. This is not in accord with the practice in many of our best texts, where polar coordinates are taken up early and carried along concurrently with the rectangular coordinates.

Most teachers will agree with the author in not emphasizing the general equation of the second degree in the usual first course. The plan of arranging the text so that teachers may easily omit the articles on tangents and normals is also to be commended. In giving answers to all the problems, the author is doing something to help the busy teacher. Most of us do not want to take the time to determine answers when students are in doubt.

The introduction of transcendental functions in the early part of the text will give the student something of a jolt; but it has the advantage of giving the opportunity for continued application throughout the course. Difficult topics relegated to the last chapter of a text are seldom well understood.

In the opinion of the reviewer insufficient attention is given to the graphing of illustrative solved problems. This is apt to give the student the impression that graphs are not of great importance in the solution of problems. Also in discussing theory, the graph should in general be kept continually before the student's eye. The instructor should see to it that proper diagrams are always made.

The introduction of the general equation of the second degree in the chapter on the circle is to be commended. The treatment here merely shows how the general equation includes the circle as a special case. In this chapter the introduction of the derivative before the discussion of the tangent to a circle is a step in the right direction. However, the instructor must exercise patience in thus introducing some of the elements of differential calculus at this point.

This text differs from the average in having a chapter of eighteen pages in the latter part of the plane geometry part of the book entirely devoted to oblique

coördinates. Many teachers will not care to lay much stress upon this division of the subject. Freshmen usually find this type of discipline rather irksome.

The general emphasis upon theory throughout the text should be looked upon as one of the strong features of the book. In many texts there is too much cheap pedagogy with emphasis largely upon numerical illustrations.

M. O. TRIPP

Calculus of Variations. By A. R. FORSYTH. Cambridge, England, University Press, 1927. xxii+656 pages. Price $16.00.

After a ten-page "Introduction," in which the historical development of the calculus of variations from its beginnings through the time of Weierstrass is admirably sketched, the author proceeds to a consideration of integrals involving derivatives of the first order. In Chapter I are discussed integrals of the form $\int f(x, y, y')dx$, $y'=dy/dx$. The Euler, Legendre, and Jacobi tests for maxima and minima are derived by applying special weak variations, viz., those that affect only the dependent variable. Here and elsewhere throughout the book extremals are called "characteristic curves" or merely "characteristics." The word *extremal* was nowhere discovered. The theory of the first chapter is well illustrated by examples, including the Bernoulli brachistochrone and the minimum surface of revolution, which are interestingly discussed in some detail. To be noted is the case in the minimum surface problem in which the third variation has to be considered to show that an extremum does not exist. In one problem it is shown that although a minimum is provided for weak variations of the type considered, it is possible to find variations which will give the integral a still smaller value; thus the necessity of a later consideration of strong variations is shown.

Chapter II takes up the parametric form introduced by Weierstrass as applied to integrals of first order (those in which the integrand contains no derivatives of higher order than the first). Forsyth here uses the notation x_1, y_1 for dx/dt, dy/dt, respectively, and prefers to regard x as the independent, y as the dependent, variable, rather than to regard them both as dependent upon the parameter t. The minimum surface of revolution is again used for illustrative purposes. Geodesics are also employed. In this chapter the so-called corner conditions are derived.

In the third chapter, the substance of which has previously been given in a memoir by the author in the Proceedings of the Royal Society of Edinburgh, vol. 46, part 2 (1926), pp. 149–193, integrals involving second derivatives are treated. Both the simple form and the parametric form are considered. Some propositions are stated for integrals involving derivatives of order up to and including n. Terminal conditions for the case of variable end-points are fully

treated. An interesting discussion of Euler's evolute problem illustrates the general theory.

Chapters IV, V, VI are devoted to integrals of the forms $\int f(x, y, y', z, z')dx$, $\int F(x, x_1, y, y_1, z, z_1)dt$, $\int f(x, y, y', y'', z, z', z'')dx$, $\int F(x, x_1, x_2, y, y_1, y_2, z, z_1, z_2)dt$, in which accents denote differentiation with respect to x, and subscripts indicate differentiation with respect to t, of an order equal to the subscript. Results are stated, without proof, for integrals involving $n-1$ independent variables and their first derivatives with respect to the independent variable x. As illustrative examples, the problem of joining two points not in one plane with the x-axis by a curve whose first moment about that axis is a minimum, and a similar problem for moment of inertia are used.

Strong variations are introduced in Chapter VII. The Weierstrass E-function is expressed in a variety of forms, and is developed for integrals in which second order derivatives occur, for skew curves, and, in later chapters, for the isoperimetric problem and double and triple integrals. Of note is the author's proof that neither a maximum nor a minimum can exist when the parametric integrand is a rational function of the derivatives. Application is made to the problem, originally propounded by Newton, of the solid of revolution of minimum resistance (for various laws of resistance), as well as to the least action problem, the brachistochrone, geodesics, etc.

Chapter VIII is largely concerned with isoperimetric problems, including the case involving second derivatives, even in the space problem. The classic problems of finding a curve of given length which shall enclose a maximum area, and of finding a curve which when rotated will give a maximum volume with a fixed area are among those used for illustration. For exemplifying the space theory the author employs the problem of finding a curve of given length which joins two points not in one plane with a given straight line, and which has a maximum or a minimum moment about that line. The question of relative extrema restricted by equations of non-integral type is very briefly considered. An interesting illustration of this theory is his solution of the problem of finding the shortest curve of given constant circular curvature joining two points in space. He also treats the problem of geodesics on a surface $S(x, y, z)=0$ by minimizing the length integral subject to the relation $S=0$.

The next three chapters deal with double integrals. In Chapter IX a theorem on minimal surfaces, due to Schwarz, is extended so as to obtain an analytic expression of the Jacobi test regarding the conjugate of the initial boundary curve. Chapter X shows how to construct the E-function for double integrals and proves that the Weierstrass test is satisfied by minimal surfaces. It also considers the simplest case of isoperimetric problems for double integrals. These two chapters center upon the problem of minimal surfaces. The eleventh chapter treats of double integrals involving partial derivatives of second order; but,

as the author states in the preface, "there is no attempt at a full discussion, mainly because, after the application of even the simpler tests, the analysis becomes unwieldy and the developments demand the differential geometry of the curvature of surfaces."

The final chapter, number XII, deals with triple integrals. Its chief point of interest is perhaps the proof, by application of the Weierstrass E-function test, that Dirichlet's principle is not valid.

In the "Conclusion," the author mentions four questions, the investigation of which he deems desirable. The first of these has to do with conjugate points for integrals of the second order. The second is the influence of the parametric multipliers in the isoperimetric problem upon the so-called normal form of the second variation. It is often stated that an extremal surface for a double integral of first order is determined by the condition that it pass through two curves in space. The Cauchy primitive of the corresponding characteristic (Euler) equation is determinate under the condition of being tangent to a given developable along an assigned curve. Professor Forsyth's third question is whether these two determinations are equivalent and if so whether there is any limitation on the form or on the position of the final curve. Finally, he urges a study of strong variations for single integrals of second order, and of general weak variations and also strong variations for double integrals of second order.

The book seems comparatively free from typographical and other errors. The following, however, were noted:

On page 12, line 7, "increase" should be "decrease."

On page 12, line 8, "increase" should be "decrease."

On page 12, line 12, "decrease" should be "increase."

On page 12, line 13, "decrease" should be "increase."

On page 41, *Ex.* 1, y_1 should be y'. Also it is not clear that the two integrals are "effectively the same" analytically.

On page 125, the last equation should have the factor n in the last term.

On page 326, in line 5 from bottom, the letter "l" has been left out of "let."

On page 342, in line 3 from bottom, the letter "f" has been left out of "function."

On page 343, in line 2 from bottom, the letter "l" has been left out of "value."

On page 455, in the last displayed equation, the subscript r should be n.

The Weierstrass E-function is first derived by considering two neighboring characteristic curves (extremals) AP and AQ and a *straight line* PQ. It becomes necessary later to make an extension to the case in which PQ is a curvilinear arc. Circular and parabolic arcs are considered as special cases. The advantage of this method is not clear to the reviewer. It would seem more desirable to introduce the idea of a "field" of extremals, which Forsyth does not do, and to employ Hilbert's invariant integral, which he barely mentioned in an early

chapter. This same criticism applies to his development of the E-function for other problems, e.g., double integrals. The reviewer would take issue with the statement regarding the arc PQ mentioned above, that "the investigations of Weierstrass, so far as they have been made known, were limited to the case when the arc is a straight line." Both Kneser and Hancock seem to indicate that Weierstrass used a curvilinear arc in a field of extremals.

In treating double integrals some use could well have been made of papers later than those of Kobb (Acta Mathematica, vol. 16, pp. 65–140; vol. 17, pp. 321–343), notably a paper by Radon (Monatshefte für Mathematik und Physik, vol. 22, pp. 63–63). But the name of Radon does not occur in the book, nor do the names of Bliss, Carathéodory, and others who have played important rôles in advancing the theory of the calculus of variations.

Much space is devoted to ingenious but tedious transformations of the second variation in the proofs of the Jacobi condition and its extensions. This could have been avoided by employing a method developed by Bliss and described in the papers referred to in the next paragraph.

The book seems unduly long for the topics it treats, for, after all, it covers little but the elements of the subject, with the possible exception of the problem involved in integrals containing second derivatives. Practically as complete a story of the calculus of variations can be obtained by reading Bliss's delightful little book which recently appeared as number 1 of the Carus Monographs. It is the opinion of the reviewer that a book so voluminous as the one under review should be more encyclopedic in character and that it should contain something of the recent advances in the subject. Professor Forsyth states that sources which were of particular use to him are the books of Moigno-Lindelöf and Hancock. He would have done well to make more use of such books as those of Kneser, Bolza, and Hadamard. Also, two articles on recent developments in the subject (G. A. Bliss, Bulletin of the American Mathematical Society, vol. 26, pp. 343–361; and Arnold Dresden, *ibid.*, vol. 32, pp. 475–521) should have been consulted. Reference is made to the articles on the calculus of variations in the German *Encyklopädie*, but the later article in the French *Encyclopédie* is not mentioned. Practically no recent developments are included in the book, and the paucity of references would make it difficult for the inquirer to discover what is at present taking place in the domain of the calculus of variations. Several problems for research are suggested in the "Conclusion," as stated above, but aside from this the volume could hardly be called inspiring. On the other hand, it is lucid and interesting; and the author is to be commended on his treatment of many topics. In particular should be mentioned his treatment of integrals involving second derivatives, his discussion of terminal conditions, the clear manner in which he emphasizes the importance of distinguishing between extrema under weak and under strong variations, the summaries at

the ends of the chapters, and above all the excellent way in which he illustrates the theory by applications to special problems, a number of which have been specifically referred to in this review.

PAUL R. RIDER

An Introduction to the Theory of Infinite Series. By T. J. I'A BROMWICH. Second Edition, revised with the assistance of T. M. MACROBERT. London, The Macmillan Co., 1926. xv+535 pages.

The first edition of this book published in 1908, is familiar to readers of the Monthly. This second edition is largely a reproduction of the first with additional theorems and examples. Chapters I–VII have had only slight alterations. To chapter VIII there has been added a discussion of the solution of linear differential equations of the second order. Chapter X of the first edition, on "Complex series and products," becomes Chapter X on "The general theory of complex series and products" and Chapter XI on "Special series and functions."

Chapter XI of the first edition, on "Non-convergent and asymptotic series" becomes Chapter XII. The entire discussion of the theory of summable series, apart from the historical introduction, has been omitted. The discussion of asymptotic series has been enlarged. Among the additions are the asymptotic expansions of Bessel's functions, trigonometric series, Stokes's transformation, and Gibbs's phenomena.

Only slight changes have been made in the Appendices. Appendix I remains practically unchanged; appendix II now includes an account of Napier's invention of logarithms.

For those unacquainted with the first edition we add that this volume contains a mass of material some of which is not otherwise easily available to English readers. Free use is made of the calculus. Proofs are not always given. The truth of some theorems is frankly allowed to rest on the reader's intuition. Valuable as a book of reference, this volume does not meet the requirements of a class room text.

The style in the revised edition, as well as in the first edition, at no point approaches the simplicity and clarity characteristic of many of the best French mathematical works.

W. V. LOVITT

Investigations on the Theory of the Brownian Movement. By ALBERT EINSTEIN. Edited with notes by R. FÜRTH: translated by A. D. COWPER. New York, E. P. Dutton and Co., 1927. viii+124 pages.

The practice of collecting, in book form, the original papers which have played a conspicuous part in the development of an important field of scientific research is to be highly commended. Such papers are of very great value in many ways. While it is, of course, best to read them in the original language,

nevertheless a translation makes them readily available to a larger circle and hence fills a real need. Einstein's five papers on the "Brownian Movement" included in the present volume are of unusual interest and significance; they were contributed to the Annalen der Physik and the Zeitschrift für Elektrochemie during 1905–1908, and opened up an extensive field of investigation, the cultivation of which has since led to results of wide application and fundamental importance in many branches of mathematical physics and physical chemistry.

The "Brownian Movement" is the continual, irregular, zigzag motion shown by microscopic particles suspended in liquids or gases. First observed in 1827 by the English botanist Robert Brown, it remained unexplained for fifty years, when finally several investigators suggested that it might be due to the continual bombardment of the particles by the molecules of the fluid in which they were suspended. A precise test of this hypothesis first became possible as a result of Einstein's brilliant theoretical work; in the first paper here reprinted, he deduced that "bodies of microscopically-visible size suspended in a liquid will perform movements of such magnitude that they can be easily observed in a microscope, on account of the molecular motions of heat," and suggested that "it is possible that the movements discussed here are identical with the so-called 'Brownian molecular motion.' " In this and the following papers, Einstein established the now famous equations which the phenomenon should obey; but he remarked, "I will not attempt here a comparison of the slender experimental material at my disposal with the results of the theory, but will leave this comparison to those who may be handling the experimental side." The classical experiments that soon were carried out by Perrin and others completely confirmed Einstein's theoretical deductions, and incidentally provided a most direct and convincing experimental proof of the atomic hypothesis.

The final paper of the volume, "The Elementary Theory of the Brownian Movement," might well be read first in order to get a general idea of Einstein's method of procedure. His deductions are mainly thermodynamical, and it has been said that the most universal generalization that can be made about mathematical physics is that everyone finds thermodynamics a very difficult subject; furthermore, Einstein makes extensive use of the concept of "osmotic pressure," a term rather loosely employed to denote three different things. Einstein apparently uses it in the sense of diffusion pressure. His principal equations, however, have since been obtained in many alternative ways, which will be found in current treatises on physics and physical chemistry.

It is of especial interest and significance that the "Brownian Movement" introduces us to a realm in which classical thermodynamics, particularly the "Second Law," is not obeyed.

The volume opens with a brief biographical sketch; and closes with an ex-

tensive collection of helpful notes, which amplify and criticise Einstein's text and give references to much of the relevant literature.

EDGAR W. WOOLARD

Collected Papers of Srinivasa Ramanujan. Edited by G. H. HARDY, P. V. SESHU AIYAR, and B. M. WILSON. Cambridge, University Press, 1927. xxxvi+355 pages.

The papers are preceded by two sketches of the life and work of Srinivasa Ramanujan (1887–1920), the first by P. V. Seshu Aiyar and R. Ramachandra Rao and the second by G. H. Hardy. Ramanujan's career was unlike that of any modern mathematician. Until he arrived in England in 1914 he had had little access to modern works on mathematics. To quote Hardy: "Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equation of the zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intitution, and induction, of which he was entirely unable to give any coherent account." It is therefore not surprising that a large part of his earlier work had been discovered by others. That he should have discovered them at all is in itself quite remarkable, for, as Hardy says in connection with one of Landau's theorems which Ramanujan discovered independently: "Ramanujan had none of Landau's weapons at his command; he had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvelous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day."

There are thirty-seven papers in all, of which seven are in collaboration with G. H. Hardy. Most of the papers deal with the theory of numbers. Ramanujan had a most amazing faculty for conjecturing theorems from empirical data. For example, from a table giving the number of unrestricted partitions $p(n)$ of n, he conjectured that if $\delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1$, mod δ, then

$$p(\lambda),\ p(\lambda+\delta),\ p(\lambda+2\delta),\ \cdots \equiv 0,\ \text{mod}\ \delta;$$

and he proved the theorem for a few particular cases.

The reader will be grateful to the editors for the numerous notes on Ramanujan's papers in Appendix I. Errors are pointed out and references are given to writers who commented on or made additions to Ramanujan's work.

LOUIS WEISNER

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3326. Proposed by J. J. Ginsburg, Student, Cooper Union.

Prove that, if R, r, $2s$ are the circumradius, inradius, and perimeter of a triangle, its altitudes satisfy the cubic

$$2Rx^3 - (s^2 + r^2 + 4Rr)x^2 + 4rs^2x - 4r^2s^2 = 0.$$

3327. Proposed by J. H. Neelley, Carnegie Institute of Technology.

Bisect a given line segment by means of compasses only.

3328. Proposed by Paul Wernicke, Washington, D. C.

Let A, B, C, D be four points in a plane no three of which are collinear. If AC, BD, produced, meet in F, and AD, BC meet in G, prove that $(AF/FC)/(AG/GD) = (BG/GC)/(BF/FD)$.

3329. Proposed by Lester S. Hill, Hunter College of the City of New York.

It is desired to make the following construction: N objects to be arranged in N groups, all of the same size, in such a manner that (1) every two groups have one and only one object in common, (2) corresponding to every two objects there is one and only one group which contains both. It is required to determine whether the following statement is correct or false: A necessary and sufficient condition for the existence of the required construction is that $N = 1+q+q^2$, where $q = p^r$, p denoting a prime positive integer, and r denoting a positive integer or zero.

3330. Proposed by J. B. Reynolds, Lehigh University.

A uniform elastic thread of natural length a, weight W and coefficient of elasticity e lies within and is attached to one end of a smooth tube. The tube is rigidly attached at this end to a thin vertical rod with which it makes an angle θ. Find the length of the elastic thread in steady motion under a constant angular velocity n about the vertical rod and locate the center of gravity of the thread. θ is measured clockwise from the tube to the vertical.

3331. Proposed by Otto Dunkel, Washington University.

In a spherical triangle ABC such that $AC < BC$ two points A', B', are taken on the side AB so that $\angle A'CA = \angle BCB' \leqq C/2$. Prove that $AA' \gtreqless B'B$ according as $AC + CB \gtreqless 180°$.

SOLUTIONS

263 [1917, 177]. Proposed by J. L. Riley.

To find values, positive integral, which verify the equation $x^3+2=y^2$.

Partial Solution by Lawrence Hampton, University of Oklahoma

By making use of the known facts that x^3 will be congruent to zero, one, or eight, modulo nine, and that y^2 will be congruent to zero, one, or four, modulo eight, we find that x must have the form $24\,n-1$. The equation is not satisfied by any positive numbers less than one hundred.

Note by Otto Dunkel, Washington University

By an extension of this process and the use of a table of squares and cubes it appears that no solution exists for a positive value of x less than 1500. It would be desirable to have a proof of the existence, or non existence, of a solution in positive integers.

274 [1917, 467]. Proposed by J. L. Riley.

Solve in positive integers the equation

$$x^4+x^3+x^2+x+1=y^2.$$

Solution by A. A. Bennett, Brown University

The complete solution of this equation in integers is given in the solution [1926, 281] of problem 2784 [1919, 366].

Also solved by Lawrence Hampton

436 [1917, 388]. Proposed by Artemas Martin, LL. D., Washington, D. C.

A circle of radius a is drawn at random on a circular slate of radius r. If another circle be drawn on the slate, what is the probability that the second circle will intersect the first?

Solution by R. E. Gaines, University of Richmond, Va.

Let O be the center of the slate with radius r, let C_4 denote the concentric circle with radius $r-a$, and C_5 the concentric circle with radius $r-3a$. Also let C_1 and C_2 denote the two circles of radius a, and C_3 the circle of radius $2a$, concentric with C_1. It will be assumed that the problem means that C_1 and C_2 shall lie wholly on the slate, and hence that the centers of C_1 and C_2 must lie within or on the boundary of C_4. First suppose that the center of C_1 falls within C_5. Then C_3 lies wholly within C_4, and if the center of C_2 falls anywhere within C_3, the circles C_1 and C_2 will intersect. The chance that the center of C_1 will fall within C_5 is $(r-3a)^2/(r-a)^2$, and the probability that the center of C_2 will fall within C_3 is $4a^2/(r-a)^2$. The probability of intersection is therefore $4a^2(r-3a)^2/(r-a)^4$.

Next suppose that the center of C_1 falls outside of C_5 but within C_4; then the center of C_2 cannot fall in that portion of C_3 which projects over C_4 but only in the area common to C_3 and C_4. Let A be the center of C_1 and C_3, and let B be a point of intersection of C_3 and C_4. Set $OA=x$, $\angle OAB=\theta$ and $\angle AOB=\phi$. The other two sides of the triangle OAB are $AB=2a$, $OB=r-a$. Then, if we take a narrow circular path of width dx, with the center at O and radius x, the chance that A will fall in this path is $2\pi x dx\div\pi(r-a)^2$; and for any particular position of A the chance that the center of C_2 will fall within C_3 is $K\div\pi(r-a)^2$, where K is the area common to C_3 and C_4. Hence the probability of intersection in this case is $2Kxdx\div\pi(r-a)^4$, where $K=(r-a)^2\phi+4a^2\theta-2S$ and $S=$ the area of OAB. To get the sum of the probabilities for all the circular paths for values of x from $r-3a$ to $r-a$, we have the following integral:

$$\frac{1}{\pi(r-a)^4}\int_{x=r-3a}^{x=r-a}[(r-a)^2\phi+4a^2\theta-2S]2xdx,$$

where $\phi=\cos^{-1}\dfrac{x^2+(r-a)^2-4a^2}{2x(r-a)}$, $\theta=\cos^{-1}\dfrac{x^2+4a^2-(r-a)^2}{4ax}$, and

S, the area of $OAB=\frac{1}{4}[\{x^2-(r-3a)^2\}\{(r+a)^2-x^2\}]^{1/2}$.

The first term of the integral gives

$$(r-a)^2x^2\phi - 2(r-a)^2S - 4a^2(r-a)^2\sin^{-1}\frac{4a^2+(r-a)^2-x^2}{4a(r-a)}.$$

The second term may be obtained from this result by replacing ϕ by θ and interchanging $r-a$ and $2a$.

The third term is

$$[4a^2+(r-a)^2-x^2]S + 4a^2(r-a)^2\sin^{-1}\frac{4a^2+(r-a)^2-x^2}{4a(r-a)}.$$

Combining these results and applying the required limits, after which certain terms may be further simplified and combined, we have as the probability of intersection in the case where A falls outside of C_3 but within C_4,

$$\frac{1}{\pi(r-a)^4}\left[2(r-a)^4\sin^{-1}\frac{a}{r-a} + 8a^2(r-a)^2\cos^{-1}\frac{a}{r-a} - 2a(r^2-2ra+3a^2)(r^2-2ra)^{1/2} - 4a^2(r-3a)^2\pi\right].$$

To this expression must be added the result previously obtained for the probability of intersection when A falls within C_5, namely $4a^2(r-3a)^2/(r-a)^4$. This will simply cancel the last term and give as the final result,

$$\frac{1}{\pi(r-a)^4}\left[2(r-a)^4\sin^{-1}\frac{a}{r-a} + 8a^2(r-a)^2\cos^{-1}\frac{a}{r-a} - 2a(r^2-2ra+3a^2)(r^2-2ra)^{1/2}\right].$$

It will be noticed that if $r=2a$ this result reduces to unity, as it should, since under this condition the circles C_1 and C_2 always intersect. If $r=3a$ the probability is $1-3^{3/2}\cdot 4^{-1}\cdot\pi^{-1}=.587$.

510 [1917, 124, 231]. Proposed by Joseph E. Rowe.

Show how to find the equation of a line perpendicular to a side of the triangle of reference and passing through a given point, in a system of homogeneous coördinates, using the condition that two lines be parallel but not the condition that two lines be perpendicular. Illustrate the method by using it to find the trilinear coördinates of the points of contact of an escribed circle of the triangle.

Solution by the Editors

Let the homogeneous coördinates (x, y, z) of a point be proportional to the distances of the point from the sides of the reference triangle ABC, and let (1, 1,1) be the center of the inscribed circle. The orthocenter has the coördinates $(\sec A, \sec B, \sec C)$, and the line at infinity is $L\equiv x\sin A+y\sin B+z\sin C=0$. A line perpendicular to $z=0$ has the equation

$$\lambda(x\cos A - y\cos B) + \mu L = 0, \tag{1}$$

and it cuts the side $z=0$ in a point determined by setting $z=0$, in (1). It will also pass through the center of an escribed circle $(-1, 1,1)$, if we set $\mu=\cos A+\cos B$, $\lambda=\sin B+\sin C-\sin A$. We thus find the point of contact in this case to be $(\cos B-1)$, $(\cos A+1)$, 0. The other points of contact may be found in a similar manner.

2764 [2740; 1919, 171]. Proposed by E. H. Clarke, Hiram College.

If the coefficients of $(a-b)^k$, where k is a positive integer, be multiplied term by term by the nth powers (n being zero or a positive integer) of the terms of any arithmetic progression with common difference $d\neq 0$, the sum of the products will vanish if $n<k$; will be $(-d)^k(k!)$ if $n=k$; and, if $n=k+1$, will be the product of this last result and the sum of the terms of the arithmetic progression.

Solution by O. J. Ramler, The Catholic University of America.

The results will be obtained by expanding $(e^{ax}-e^{bx})^k$ in two ways. Expanding first the expression inside the parenthesis to terms of the second degree in x and then raising this result to the kth power, we obtain

$$(a-b)^kx^k + \tfrac{1}{2}k(a-b)^k(a+b)x^{k+1} + \cdots.$$

Now carrying out the expansion in the reverse order, we shall find for the coefficient of x^n, after setting $ka=u$ and $b-a=d$,

$$\frac{1}{n!}\sum_{i=0}^{i=k}(-1)^i{}_kC_i(u+id)^n, \qquad {}_kC_i=\frac{k!}{i!(k-i)!}.$$

Hence

$$\sum_{i=0}^{i=k}(-1)^i{}_kC_i(u+id)^n \begin{cases} =0, & n<k,\\ =k!(-d)^k, & n=k,\\ =(k+1)!(-d)^k\dfrac{2u+kd}{2}, & n=k+1. \end{cases}$$

2807 [1920, 80]. Proposed by S. A. Corey, Des Moines, Iowa.

Establish the identity

$$\begin{vmatrix} -x & bcy & -acu & -abv\\ y & x & -av & au\\ u & bv & x & by\\ -v & cu & cy & -x \end{vmatrix}^2 + ac\begin{vmatrix} x & bcy & -x & -abv\\ -y & x & y & au\\ u & bv & u & by\\ -v & cu & -v & -x \end{vmatrix}^2$$

$$+ ab\begin{vmatrix} x & bcy & -acu & -x\\ -y & x & -av & y\\ u & bv & x & u\\ -v & cu & cy & -v \end{vmatrix}^2 = \begin{vmatrix} x & bcy & -acu & -abv\\ -y & x & -av & au\\ u & bv & x & by\\ -v & cu & cy & -x \end{vmatrix}^2$$

Solution by the Proposer

Lagrange's like-producing quadrinomial is

$$(x^2+bcy^2+acu^2+abv^2)(p^2+bcq^2+acr^2+abs^2)=(P^2+bcQ^2+acR^2+abS^2)$$

where

$$P=xp-bcyq-acur-abvs, \quad Q=-yp-xq-avr+aus,$$
$$R=up-bvq+xr+bys \qquad S=-vp-cuq+cyr-xs.$$

Now assume for p, q, r, and s such values as to render $P=-x$, $Q=y$, $R=u$, and $S=-v$. Substitute these values of p, q, r, and s in the like-producing quadrinomial, divide by the first factor, clear of fractions, and obtain the given identity, since $q\equiv 0$.

3119 [1925, 95]. Proposed by N. A. Court, University of Oklahoma.

Through a given point to draw a line meeting the sides BC, CA, AB of a given triangle ABC in the points P, Q, R such that $PQ=kQR$, where k is a given constant.

Solution by Otto Dunkel, Washington University

Let O be the given point and OT, a variable straight line cutting the sides CB and CA in P and Q, respectively. Let the point T be taken on this variable line so that $PQ=kQT$. If we draw the ray CT and then the ray $C\infty$ parallel to OT, the cross ratio of the pencil $C(A, B, T, \infty)$ will be constant. Hence the pencils $C(T)$ and $C(\infty)$ are projective; but the latter is also projective with the pencil $O(T)$. Therefore the pencils $O(T)$ and $C(T)$ cut the side AB in two projective ranges of points. The self-corresponding points of the two ranges give the desired points R on AB. The points R may be determined by aid of an auxilliary circle as explained in 3240 Note, 1928, 46 from three pairs of corresponding points. If OT is parallel to CB, T is at infinity on CB, and this gives one pair of corresponding points on AB. A second pair may be constructed similarly by making OT parallel to CA. The third pair is constructed by taking any other position for OT. There may be two solutions, one solution, or no solution.

It is apparent from the above discussion that the locus of T is an hyperbola through O and C with asymptotes parallel to CA and CB.

3268[1927, 335]. Proposed by Tobias Dantzig, University of Maryland.

A and B are two fixed points on a conic and M and N, two variable points on the same conic, such that the cross ratio $(A, B; M, N)$ with regard to the conic remains constant. Find the envelope of the line MN.

SOLUTION BY THEODORE BENNETT, University of Illinois.

By making a suitable choice of trilinear coordinates the points A and B may be taken as (0, 0, 1) and (1, 0, 0), while the conic has the equation $x_2^2 - x_1x_3 = 0$.

This conic may be written parametrically $x_1 : x_2 : x_3 = t^2 : t : 1$. The points A and B are given by $t = 0$ and $t = \infty$.

Let M and N be given by the parametric values t and s; then the equation of MN is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ t^2 & t & 1 \\ s^2 & s & 1 \end{vmatrix} = 0, \text{ or } x_1 - (t + s)x_2 + stx_3 = 0,$$

But $(A, B; M, N) = (0, \infty; t, s) = t/s = k$. Substituting $t = ks$, the equation of MN becomes

$$x_1 - s(1 + k)x_2 + s^2kx_3 = 0.$$

The envelope of this system of lines has the equation

$$(1 + k)^2x_2^2 - 4kx_1x_3 = 0.$$

The envelope is therefore a conic which touches the given conic at A and B.

NOTE BY THE EDITORS.

Let $[A, B, M, N]$ denote the given cross ratio, where A, B, M, N are points on the conic (C). Consider any other point M' on (C) and let $M'N$ cut AB in K. Draw MK cutting (C) again in N'. Then the pencils $M'(A, B, M, N)$ and $M(A, B, M', N')$ are projective, and hence $[A, B, M', N'] = [A, B, M, N]$. Thus if A, B, M, N remain fixed and K varies on AB, the problem requires the envelope of $M'N'$. This is problem 3156[3152; 1925, 481] a solution of which is given in [1926, 387].

Also solved by N. A. COURT AND PAUL WERNICKE.

3270[1927, 335]. Proposed by N. A. Court, University of Oklahoma.

If a triangle is self-polar with respect to a circle, the center of the circle is the orthocenter of the triangle.

State this proposition in projective form, and give a direct proof of the proposition so stated.

SOLUTION BY THEODORE BENNETT, University of Illinois.

There are several ways in which the proposition may be stated, all of which amount to the sam thing. The following is a fairly concise way of putting it.

Given a point C, a conic K, and a triangle $P_1P_2P_3$ which is self-polar as to K. The lines which join C to the points in which the polar line of C cuts the sides of the triangle, and the lines which join C to the vertices of the triangle form three pairs in an involution whose double rays are the tangents to K from C.

PROOF: Let P_2P_3 cut the polar line of C at Q_1; then the polar line of Q_1 passes through both C and P_1, whence CP_1 and CQ_1 are harmonic to the tangents to K drawn from C. Similarly, for the other two pairs, which proves the proposition.

Also solved by PAUL WERNICKE.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The twelfth annual summer meeting of the Mathematical Association of America will be held at Amherst College, Amherst, Massachusetts, on September 3 and 4, 1928. Professor B. H. Camp, of Wesleyan University, chairman of the program committee, announces that the following papers will be read at the meeting:

1. "What we mean by postulates," by Professor ALBERT A. BENNETT, Brown University.
2. "Convergence andcontinuity," by Professor ARNOLD DRESDEN, Swarthemore College.
3. "Higher forms of the icosahedron," illustrated by models, by Mr. A. HARRY WHEELER, Lecturer, Wellesley College.
4. "The past decade in cosmology," by Professor WILLIAM D. MACMILLAN, University of Chicago.
5. "Some geometrical aspects of physics," by Professor LUTHER P. EISENHART, Princeton University.
6. "The development of the Russian mathematical school," by Professor JACOB D. TAMARKIN, Brown University.

(Note: All the papers are mainly expository in character.)

A complete announcement of the meeting will be sent to members of the Association by the secretary in due time.

At their last meeting in Boston on February 24 and 25, 1928, The National Council of Teachers of Mathematics decided to publish a register of all members of the Council as a supplement to one of the forthcoming issues of "The Mathematics Teacher." This register will serve as a sort of "Whos Who?" in secondary mathematics and all such teachers of mathematics will wish to be included. The register will be sent free to all who are members at the time of publication. Any teachers of mathematics who are not now members should send in the annual dues of $2 at once to "The Mathematics Teacher," 525 West 120th Street, New York City. The payment of these dues entitles each member to a year's subscription to "The Mathematics Teacher." The address and position of each subscriber should be given.

The National Council also has published three yearbooks. The first one on "A General Survey of Progress in the Last Twenty-five Years" may be secured for $1.10 from C. M. Austin, Oak Park High School, Oak Park, Ill. The second on "Curriculum Problems in Teaching Mathematics" may be secured for $1.25 from the Bureau of Publications, Teachers College, 525 West 120th Street, New

York City. The third yearbook on "Selected Topics in Teaching Mathematics," which is just off the press, may also be obtained for $1.75 (bound volume) from the Bureau of Publications, Teachers College.

The University of Arkansas has conferred the degree of LL.D. on Professor H. E. BUCHANAN of Tulane University.

The Messel medal of the British Society of Chemical Industry has been awarded to Dr. R. A. MILLIKAN, of the California Institute of Technology, in recognition of his achievement in measuring the electrostatic charge of the electron.

On the occasion of its semi-centennial celebration, the University of Colorado conferred a doctorate of laws on Professor R. A. MILLIKAN, of the California Institute of Technology.

Dr. GEORGE SAUTÉ, of Harvard University, has been awarded one of the fellowships of the Commission for Relief in Belgium Educational Foundation, for the study of mathematics in Belgium.

It is announced that Guggenheim fellowships in astronomy, mathematics, and physics have been awarded to the following persons: to Professor PERRY BYERLY, of the University of California, for the study of mathematical geophysics; to Professor OLIVE C. HAZLETT, of the University of Illinois, for the study of the arithmetics of linear associative algebras together with their application and interpretation in other lines of mathematics, especially the theory of numbers; to Professor J. J. HOPFIELD, of the University of California, to study the Zeemann effect of the infra-red spectrum of oxygen and nitrogen; to Dr. R. J. KENNEDY, of the California Institute of Technology, for research in the theory of radiation; to Professor N. C. LITTLE, of Bowdoin College, for the study of thermo-magnetic properties of gaseous molecules; to Professor F. W. LOOMIS, of New York University, for the study of quantum mechanics; to Professor L. E. REUKEMA, of the University of California, for the theoretical and experimental study of electric discharge in gases at high frequencies; to Professor OTTO STRUVE, of the University of Chicago, for a theoretical study of the distribution and physical properties of diffuse matter in interstellar space; to Professor W. W. WATSON, of the University of Chicago, for the study of molecular spectra.

Professor PAUL ALEXANDROFF of the University of Moscow, who has been on the staff of Princeton University during the present academic year, has recently made a lecture tour of the Middle West. His tour included the Ohio State University, the University of Michigan, and the University of Iowa. At

the Ohio State University, Professor Alexandroff lectured on "Some phases of analysis situs."

The papers read at the bicentenary Newton celebration held in New York under the auspices of the History of Science Society have been published in book form with the title, *Sir Isaac Newton*, 1727-1927, *A Bicentenary Evaluation of his Work*. The book was prepared for publication by F. E. Brasch, and has an introduction by D. E. Smith; it is published by Williams and Wilkins, of Baltimore.

The Louisiana Academy of Sciences held its first annual meeting at Louisiana College, Pineville, May 5, 1928. Professor I. MAIZLISH, of Centenary College, is president, and Professor H. L. SMITH, of Louisiana State University, is vice-president. Professor H. A. WILSON, of Rice Institute, delivered the principal address, entitled *Recent theories of atomic structure*. Mathematical papers were read by Professors J. A. HARDIN, C. D. SMITH, H. L. SMITH, and Mrs. YETTA V. MAIZLISH. The next annual meeting will be held at the Southwestern Louisiana Institute, Lafayette.

An article by Professor C. A. Garabedian of the University of Cincinnati was recently published in the Journal de l'Ecole Polytechnique (2eS., Cahier 26, 1927). The title is: *Correction de certains résultats sur la flexion d'une plaque circulaire épaisse donnér par de Saint-Venant dans la célèbre "Note finale du paragraphe 45" de la traduction de Clebsch.*

The following graduate courses in mathematics are announced for the academic year 1928–29:

Harvard University—By Professor W. F. OSGOOD: Advanecd calculus, II; Theory of functions, II. By Professor J. L. COOLIDGE: Higher geometry. By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics. By Professor O. D. KELLOGG: Theory of potential functions. By Professor G. D. BIRKHOFF: Partial differential equations of mathematical physics; Advanced dynamics and the quantum theory. By Professor MARSTON MORSE: Theory of functions, I; Calculus of variations. By Professor J. L. WALSH: Introduction to modern geometry; Dynamics, II. By Professor H. W. BRINKMANN: Advanced calculus, I; Theory of groups; Differential geometry, II. By Professor M. H. STONE: Functions of real variables, II. By Dr. M. M. SLOTNICK: Differential geometry, I. Professor Walsh will conduct a fortnightly seminar on the theory of functions. Courses of research are offered by Professor Osgood in the theory of functions, by Professor Coolidge in geometry, by Professor Huntington in postulate-theory, by Professor Kellogg in the theory of potential functions, by Professor Birkhoff in the theory of differential equations, by Professor Morse in analysis situs, by Professor Walsh in analysis, by Professor Brinkmann in the theory of groups, and by Professor Stone in expansion problems.

University of Pennsylvania—By Professor E. S. CRAWLEY: Higher algebraic geometry. By Professor G. H. HALLETT: Infinite series and products, first term. Functions of a complex variable. By Professor H. B. EVANS: Quaternions and vector methods, second term. By Professor F. H. SAFFORD: Partial differential equations. By Professor H. H. MITCHELL: Analytic theory of numbers, first term. Algebraic numbers, second term. By Professor M. J. BABB: History of mathematics. By Professor F. W. BEAL: Differential equations, first term. Differential geometry, first term. By Professor J. R. KLINE: Functions of a real variable. By Professor J. D. ESHLEMAN: Synthetic projective geometry. By Professor P. A. CARIS: Higher algebra. By Professor J. M. THOMAS: Pfaff's problem, first term. Continuous groups, second term.

Professor J. D. TAMARKIN, of Brown University, has been elected a member of the American Academy of Arts and Sciences.

Professor HERMAN WEYL, of the Zurich Technical School, has been appointed to the Thomas D. Jones research professorship of mathematical physics at Princeton University.

Associate Professor J. W. ALEXANDER, of Princeton University, has been promoted to a full professorship of mathematics.

Dr. R. G. ARCHIBALD, of Columbia University, has been promoted to an assistant professorship of mathematics.

Professor DANIEL BUCHANAN, head of the department of mathematics in the University of British Columbia, has been appointed dean of the faculty of arts and sciences at that University.

Assistant Professor WILLIAM B. CAMPBELL, of Colgate University, has been appointed by the Baptist Mission Board as lecturer in mathematics at Judson College, Rangoon, Burma.

Assistant Professor CHARLES DUNCAN GREGORY, of William and Mary College, has been promoted to an associate professorship of mathematics.

Dr. LULU HOFMANN has been promoted to a lecturership in mathematics in Columbia University.

Dr. B. O. KOOPMAN, of Columbia University, has been promoted to a full professorship of mathematics.

Associate Professor S. LEFSCHETZ, of Princeton University, has been promoted to a full professorship of mathematics.

At Yale University, Professor W. R. LONGLEY has been appointed to the Colgate professorship, as successor to Professor Luquiens.

Associate Professor C. R. MACINNES, of Princeton University, has been promoted to a full professorship of mathematics.

Dr. MAX MASON, president of the University of Chicago and formerly professor of mathematical physics at the University of Wisconsin, has resigned to accept an appointment as director of the newly created division of natural sciences of the Rockefeller Foundation.

Dr. L. T. MOORE has been promoted to an assistant professorship of mathematics at Yale University.

Associate Professor G. W. MULLINS, of Barnard College, Columbia University, has been promoted to a full professorship of mathematics.

Assistant Professor OYSTEIN ORE, of Yale University, has been promoted to an associate professorship of mathematics.

Dr. D. J. STRUIK has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Assistant Professor J. D. TAMARKIN, of Brown University, has been promoted to a full professorship of mathematics.

Professor JOHN HASBROUCK VAN VLECK, of the department of physics of the University of Minnesota, has been appointed professor of physics at the University of Wisconsin. His work will be in theoretical physics. He is a son of Professor E. B. VAN VLECK of the department of mathematics of the University of Wisconsin.

Associate Professor J. H. M. WEDDERBURN, of Princeton University, has been promoted to a full professorship of mathematics.

The following appointments to instructorships in mathematics are announced:

Columbia University, Dr. M. S. DEMOS;

Harvard University, Dr. M. M. SLOTNICK (National Research Fellow), A. C. BERRY, A. B. BROWN, A. H. FOX, G. A. HEDLUND, J. J. HINRICHSEN, J. K. PETERSON, G. B. PRICE, G. SAUTÉ, and CHARLES WEXLER.

Dr. J. B. CHITTENDEN, professor of mathematics at the Brooklyn Polytechnic Institute, died March 20, 1928, at the age of sixty-four.

Professor GAETANO LANZA, emeritus professor of theoretical and applied mechanics at the Massachusetts Institute of Technology, died March 21, 1928, at the age of seventy-nine.

Dr. F. S. LUTHER, emeritus president and formerly professor of mathematics at Trinity College, Hartford, died January 4, 1928, at the age of seventy-seven.

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DIRECTORY

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BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 31.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, April 28.

OHIO, Columbus, Ohio, April 5.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Dunham, N. C., March.

SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

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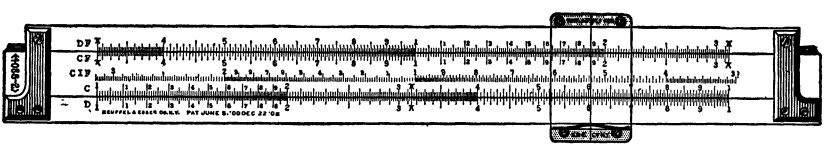
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THE TWELFTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twelfth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado School of Mines, Golden, Colorado, on April 20–21, 1928. There were fifty-five present including the following twenty-five members of the association: Ralph Beatley, A. G. Clark, E. A. Cummings, I. M. DeLong, J. R. Everett, G. W. Finley, Philip Fitch, J. C. Fitterer, G. W. Gorrell, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, G. H. Light, E. P. Martinson, S. L. Macdonald, Margaret McGinley, J. Q. McNatt, W. K. Nelson, O. H. Rechard, W. J. Risley, Mary S. Sabin, C. H. Sisam, E. B. Stouffer, C. W. Wray.

The section voted to hold its next annual meeting at the Colorado Teachers' College, Greeley, Colorado. The following officers were elected: G. W. FINLEY, chairman; G. W. GORRELL, vice-chairman; PHILIP FITCH, secretary, G. H. LIGHT, treasurer.

At a complimentary dinner given by the School of Mines on Friday, President Coolbaugh delivered an address of welcome to which Professor G. W. Finley made the response. The section was favored Friday evening by an address, "Mathematics in Italian Universities" by Professor E. B. Stouffer of the University of Kansas. Professors DeLong and Gorrell gave brief talks on the life and work of the late Professor H. E. Russell.

The following fourteen papers were read:

1. "A problem" by Mr. J. Q. MCNATT.

2. "The intersection of two special ruled cubics" by Professor SAUL POLLOCK (by invitation).

3. (a) "A method of solving quadratics" and (b) "A model for trigonometric functions" by Mr. J. W. HAZARD (by invitation).

4. "Teaching mathematics by subjects or topics" by Professor G. W. GORrell.

5. "Note on the convergence of an infinite series" by Professor C. A. HUTCHINSON.

6. "Teaching engineering mathematics to the poor student" by Professor J. R. EVERETT.

7. "An interpretation of the imaginary resulting from an application of the principle of continuity" by Mr. PHILIP FITCH.

8. "The use of the discriminant in solving a certain type of differential equation" by Professor G. H. LIGHT.

9. "Mathematics and the Society for the Promotion of Engineering Education" by Professor J. W. RISLEY.

10. "Remarks on a question raised by Hessenberg" by Professor A. J. KEMPNER.

11. "On a special quartic surface with a double line" by Professor C. H. SISAM.

12. "Cocoanuts and congruence" by Professor RALPH BEATLEY.

13. "Some fundamental concepts in differential projective geometry" by Professor E. B. STOUFFER.

14. "Civic values in the study of mathematics" by Professor A. S. ADAMS (by invitation).

At the close of the Friday afternoon session, Dr. C. A. Heiland gave an explanation of the apparatus used in geophysics for the purpose of locating bodies of ore and oil.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

In his address on "Mathematics in Italian universities," Professor Stouffer told of various experiences in connection with his study of mathematics in Italy during the year 1926–27. Special mention was made of the mathematics curricula in Italian universities and of the character of the lectures delivered by some of the distinguished Italian geometers.

1. Mr. McNatt gave a method of calculating the volume of liquid contained in a right circular cylindrical tank, when the tank is inclined at any given angle.

2. Professor Saul Pollock exhibited a series of six models illustrating the special curves of intersection of two ruled cubic surfaces each having two pinch points. These curves of the ninth degree were composites and pointed out the results of matching pinch points, rulings, common conics, etc. In addition, a device was demonstrated by means of which it is possible to study space curves of any degree. The principle consisted of intersecting a surface with another surface consisting of light. The resulting curve of intersection is at once visible, and by moving one of the surfaces it is possible to study the variation in the curve.

3. (a) Mr. Hazard described a method of solving the quadratic equation in one unknown by means of a table of quarter-squares. Such tables have been computed for the purpose of finding the product of numbers by means of the relation: the quarter-square of the sum of two numbers minus the quarter-square of their difference equals their product. In the quadratic we have the sum and the product of the roots given, hence we can use the above relation to solve for the difference. The roots are then found from their sum and difference. This method has marked advantages over solving the quadratic equation by logarithms when the coefficients are large.

3. (b) Mr. Hazard demonstrated a rotating model for use in class instruction, in which the signs and values of sines and cosines are shown with their continuous variations throughout the entire circle.

5. Mr. Hutchinson's note dealt with real infinite series in a variable, x, which may be converted into power series in y by a substitution, $y=f(x)$. The interval of convergence of the power series is projected orthogonally onto the curve $y=f(x)$, thence orthogonally onto the x-axis, giving the regions of convergence, and divergence, of the original series.

8. There is always an extraneous factor when a differential equation of the type $f(x, y, p)=0$ has a singular solution. This paper by Professor Light shows how that factor can be found before formally solving the differential equation.

10. In a long article, Grundbegriffe der Mengenlehre, Neue Abhandlungen der Fries' schen Schule, 1906, G. Hessenberg raised the question as to what would happen if it could be proved, for example, that it is impossible to decide whether the number 2^{μ}, where $\mu=\sqrt{2}$ is an algebraic or a transcendental number. In his opinion mathematics would in this case be confronted with a difficulty more serious than any which has so far been encountered. The writer has not had access to the paper for many years and does not recall that any qualifying statements were made concerning the nature of the system of axioms used or concerning the laws of formal logic. In the absence of any such restrictions, it is easy to show that the difficulty would be solved by the introduction of two algebras, in one of which it would be an axiom that 2^{μ} is algebraic, in the other of which 2^{μ} would be transcendental.

11. In this paper, the author discusses some properties of those quartic surfaces with a double line which have the property that the conics on the surface, in pairs, constitute degenerate quartic curves of the first kind.

13. The projective differential properties of many figures may be studied by means of canonical expansions. Professor Stouffer derived such a canonical expansion for the equation of the plane curve, starting from a general expansion for the equation. The method required the proper choice of the triangle of reference. In a similar manner a canonical expansion suitable for the study of curved surfaces was also derived.

14. An answer to the question "What can the average student get out of the study of Mathematics" was given by Professor Adams. The values to be derived from the study were considered from the point of view of the student's social, economic, and individual life. His social life was shown to have been helped in giving him the valuable mental habits of neatness, orderliness, accuracy, persistence, intellectual honesty, and attention. His economic life has been benefited by direct application of mathematical principles to his business or profession, by fundamental training for further study in the sciences, and by the development of his ability to think clearly, rapidly, and accurately. His indi-

vidual life has profited by basic training in appreciation of beauty and by a realization of the exactness and order of the Universe.

PHILIP FITCH, *Secretary*

THE FIFTH MEETING OF THE INDIANA SECTION

The fifth meeting of the Indiana Section of the Mathematical Association of America was held May 11, 12, 1928 at Butler University, Indianapolis, Indiana.

There were sixty-one present at the meeting including the following thirty-one members of the Association: R. J. Aley, W. C. Arnold, Gladys L. Banes, Stanley Bolks, G. E. Carscallen, P. T. Copp, H. T. Davis, S. C. Davisson, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, H. E. H. Greenleaf, Laurence Hadley, Cora B. Hennel, F. H. Hodge, E. N. Johnson, Kathryn M. Kennedy, Florence Long, Juna M. Lutz, T. E. Mason, H. R. Mathias, R. E. Peterson, C. K. Robbins, D. A. Rothrock, J. R. K. Stauffer, C. E. Stout, K. P. Williams, H. E. Wolfe, W. A. Zehring, and H. A. Zinszer.

On Friday evening at 6:30 a banquet was held at the Claypool Hotel which was attended by members of the Association and their guests. President Aley of Butler University presided.

At eight o'clock a public lecture under the auspices of Butler University was given by President D. W. MOOREHOUSE of Drake University, Des Moines, Iowa, on the subject: "The Milky Way." President Moorehouse, by means of lantern slides, traced the ever interesting history of man's expanding knowledge of the universe. The speaker called special attention to the perplexing problem presented by the presence of black patches in the midst of brilliant star clouds and showed how the evidence points to the existence of great dark nebulous masses in the milky way.

At the session on Saturday morning on the Butler campus, presided over by Professor J. E. Dotterer, Manchester college, chairman, the following officers were elected: Professor H. E. H. GREENLEAF, De Pauw University, chairman; Professor H. A. ZINSZER, Hanover College, vice-chairman; Professor H. T. DAVIS, Indiana University, secretary-treasurer.

A chairman's address was made by Professor J. E. DOTTERER on the subject: "The Mathematician as a Salesman." Professor Dotterer pointed out the duty incumbent upon the teacher, in addition to his actual instruction, of showing the fundamental connection between mathematics and actual living. He urged the need of exhibiting mathematics not only as a tool used in solving and explaining the universe in which we live, but also as a discipline, a cultural subject and an art.

Professor R. H. COON of the Latin Department of Indiana University fur-

nished a pleasing variation to the program with a translation of Tycho Brahe's: "The Science of Mathematics and Astronomy," a public lecture given in the year 1574. The translation was prefaced by a summary of the life and achievements of the famous astronomer.

The morning program concluded with a pageant: "Pictures from the Pages of Mathematical History," which was presented by the department of mathematics of Butler University under the direction of Miss Gladys Banes, Professor E. N. Johnson, and Miss Juna Lutz. The words were written by Miss Lutz. The pageant comprised a survey of mathematical history from the time of Fuh-hi, 2852 B.C. to Henri Poincaré, and was especially noteworthy in the accuracy with which the various mathematicians had been reconstructed from their pictures.

The afternoon program consisted of the following papers:

1. "A generalization of the construction of the ellipse," by Professor F. H. HODGE, Purdue University.

2. "Some remarks concerning the exponential function," by Professor K. P. WILLIAMS, Indiana University.

3. "The teaching of analytic geometry," by Professor T. E. MASON, Purdue University.

4. "Function theoretic determination of the law of sines and the law of cosines for any plane triangle," by Mr. L. R. KELLAM, Indiana University. (Introduced by Professor Williams.)

5. "Some new formulas in curve fitting by least squares," by Mr. V. V. LATSHAW, Indiana University. (Introduced by Professor S. C. Davisson.)

6. "The place of statistics in the mathematics curriculum," by Professor H. T. DAVIS, Indiana University.

7. "On Fresnel's theory of diffraction," by Professor H. A. ZINSZER, Hanover college.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. This paper generalized the method of constructing an ellipse by means of two concentric circles and the eccentric angle. One circle is taken with center at the origin and the second with center at $(h, 0)$. The locus of the point whose x and y intercepts are determined by the respective intersections of a straight line drawn from the origin through the two circles is found to be a cubic, some of whose properties were studied.

2. It is customary in books on the calculus to give only an incomplete discussion of $\lim_{n=\infty}(1+zn^{-1})^n$, leaving the problem to the theory of functions. On the other hand, books on function theory are likely to regard the problem as disposed of in the calculus. Professor Williams spoke on some of the details

necessary for a systematic treatment. Attention was also directed to certain generalizations that throw some light upon the exponential function.

3. Analytic geometry, according to Professor Mason, should be taught in such a manner as to give three things to the student: (1) A certain facility in drawing approximate graphs of functions; (2) ability to put into the form of an equation a description of a locus; (3) ability to interpret results. The learning of specific facts about particular curves is only incidental.

4. This paper showed how the law of sines and the law of cosines for any plane triangle may be determined from function theoretic properties. A simple list of these fundamental properties with specific application to one or two special cases for the determination of constants was used in obtaining the desired expressions.

5. Karl Pearson has indicated the desirability of computing the coefficients of polynomials fitted to data by the method of least squares where the values of the abscissa are assumed to be the integers from 1 to p. This paper gave these coefficients for the straight line, the parabola and the cubic. Coefficients were also computed for the quartic in the case where the number of points is odd.

6. It was pointed out in this paper that there seems to be a new movement toward the emphasis of statistics as a mathematical course. The theory of probability, which deals primarily with *a priori* probabilities, is admittedly a mathematical subject. Statistics which deals, on the other hand, with *a posteriori* probabilities obtained from empirical data, must not be approached with an aversion inherited from mathematical training in more subjective theories. The progress of this work in the mathematics department at Indiana University was explained.

7. The speaker gave a resumé of the Fresnel theory of diffraction with special reference to diffraction through a narrow slit bounded by parallel edges. The application of Cornu's spiral was explained.

At the conclusion of the meeting a vote of appreciation was extended to the department of mathematics of Butler University. The time and place of the next meeting were left in the hands of the executive committee.

H. T. DAVIS, *Secretary*

OBSERVATIONS ON SIMULTANEOUS QUADRATIC EQUATIONS[1]

By ARCHIBALD HENDERSON, University of North Carolina

1. Observations upon esthetic technic. It is the common practice of teachers of mathematics to consider their subject from a purely scientific, and often, alas! from an exclusively utilitarian point of view. In this paper the attempt is made to approach the problems treated from the esthetic standpoint. The mathematician, one finds, tends to make fetiches of brevity and uniformity, as other names for efficiency. Yet it is often the case that the sterotyped method, while reaching the sought for result in the briefest space of time, veils or ignores deeper, subtler, and more elegant properties associated with the problem treated. A straight line may be what is (inaccurately) designated the "shortest distance" between two points. It is, however, not necessarily the path which affords the widest horizon or the greatest beauties of the landscape.

It is not uncommon for mathematicians to comment admiringly upon the beauty or the elegance of a certain demonstration. How many of them would be able to describe lucidly the qualities which constitute the meaning of words of such rich content as beauty and elegance? The literature of mathematics is filled with examples of beauty and elegance in demonstration. The works of the classic French mathematicians in especial are characterized by these qualities. It is almost always the case that the supreme works of the greatest mathematicians are conspicuous for beauty and elegance, for an exceptional sense of balance and proportion, for symmetry and richness of content. Esthetic qualities tend to emerge most prominently in the greatest scientists. Nietzche goes so far as to assert that "the scientific man is the finest development of the artistic man." The classic exemplar is Leonardo da Vinci, in whose case no line of demarkation is to be observed between the artist and the scientist. An arresting contemporary illustration is the Theory of Relativity which, for all its scientific complexities, is essentially an artistic construct, marked by rare beauty, elegance, richness and variety. Professor Einstein once remarked to me that, after a certain high level of scientific technic is attained, science and art tend to coalesce in esthetic, plasticity and form. A vast field lies open to the investigator who would make a philosophical study of the esthetic qualities and characteristics of scientific technic, especially in the field of mathematics.

2. A particular case. The attempt is made here to treat, by known methods but with the esthetic aspect in mind, a familiar problem in simultaneous quadratic equations. I do not find this attractive treatment in the text

[1] Read at the twelfth annual meeting of the Mathematical Association of America, Nashville, December 30, 1927. I am indebted to my assistant, Mr. L. E. Bush, for aid in detail work and numerical computation, and for executing the figures.

books. The mathematician need not always go in sombre garb; he may sport a malacca cane and wear a gardenia in his buttonhole.

Consider the solution of the simultaneous system;

$$A:\begin{cases} x^2 + y = a^2 + b, & (1) \\ x + y^2 = a + b^2. & (2) \end{cases} \qquad (a \text{ and } b \text{ real})$$

If we proceed according to the usual stereotyped methods, we shall do one or the other of two things; eliminate one, or the other, of the variables from the two equations, solve the resulting quartic, and from these four solutions find by substitution the four values of the other variable: or plot the two parabolas (1) and (2), the co-ordinates of the intersections furnishing the plotted values to scale of the solutions common to equations (1) and (2).

Such a solution is skeleton, failing to embrace alternative algebraic systems of richer content and auxiliary associated configurations. There is lost to view the fundamental fact that the solution of a quartic may be made to depend upon the solution of a cubic equation: and that curves other than parabolas lend themselves more readily to quick and accurate plotting.

Making use of the two equations, which are readily derived from equations (1) and (2), namely

$$(x^2 - a^2)(y^2 - b^2) - (x - a)(y - b) = 0, \tag{3}$$

$$x^2 + y^2 + x + y - a(a + 1) - b(b + 1) = 0, \tag{4}$$

it is clear that the system A may be replaced by the point

$$x = a,\ y = b, \tag{5}$$

and any one of the following systems:

$$B:\begin{cases} (x + a)(y + b) - 1 = 0, & (6) \\ x^2 + y = a^2 + b, & (1) \end{cases}$$

$$C:\begin{cases} (x + a)(y + b) - 1 = 0, & (6) \\ x + y^2 = a + b^2, & (2) \end{cases}$$

$$D:\begin{cases} (x + a)(y + b) - 1 = 0, & (6) \\ x^2 + y^2 + x + y - a(a + 1) - b(b + 1) = 0. & (4) \end{cases}$$

It is to be noted that if any extraneous solution be introduced in any of these systems, it is rejected by the test of not satisfying equations (1) and (2). Thus from system D enters the extraneous solution $(-a-1,\ -b-1)$ which satisfies neither (1) nor (2). It may be noted in passing that the circle represented by

equation (4) is a member of the one-parameter family of conics passing through the points of intersection of the parabolas given by equations (1) and (2).

System B shows on its face that the solution of the problem, through the happy isolation of the single point (a, b), is reduced to the solution of a cubic, to wit

$$x^3 + ax^2 - (a^2 + 2b)x - (a^3 + 2ab - 1) = 0. \tag{7}$$

Obviously the ordinates of the three points are from symmetry or from system C given by the cubic equation

$$y^3 + by^2 - (b^2 + 2a)x - (b^3 + 2ba - 1) = 0. \tag{8}$$

From the practical standpoint, the graphical solution will not here be made to consist in the laborious plotting, by points, of the two parabolas given by the equations (1) and (2). The system D is chosen as supplying the two conics easiest to construct; the circle and the rectangular hyperbola referred to the asymptotes. For the circle, only the centre and radius are required: and these are given in this case by $(-1/2, -1/2)$ and $\frac{1}{2}[4a(a+1)+4b(b+1)+2]^{1/2}$. For the rectangular hyperbola, freehand plotting is quick and surprisingly accurate when only three points are plotted and the asymptotes, given here by $x+a=0$, $y+b=0$ are known.

The two parabolas given by the equations (1) and (2) cut the axes with respect to which they are severally symmetrical at (a^2+b) and $(a+b^2)$ respectively. Since (a, b) is always a solution, there must always be at least two real intersections. The problem resolves itself into four cases according to the nature of the intersections;

1. Four real and distinct.
2. Two real and coincident and two real and distinct.
3. Two real and distinct and two complex.
4. Two real and coincident and two complex.

As a numerical illustration, consider the case when $a=2$, $b=3$ (Fig. 1). Solving (7) by the usual trigonometric method for the solution of a cubic, and using (1) we obtain

$$x = \quad 3.1310, \quad -3.2829, \quad -1.8482,$$
$$y = -2.8030, \quad -3.7765, \quad \quad 3.5920.$$

Therefore the solutions of (1) and (2) are

$$A:(2,3)\ ;B:(-1.8482,3.5920)\ ;C:(-3.2829,-3.7765)\ ;D:(3.1310,-2.8030).$$

The extraneous solution introduced by system D, namely the point E: $(-3, -4)$, satisfies neither of equations (1) and (2).

3. The general case. The particular case treated above is a special type

which lends itself to the elegant method of solution adopted. This, however, is not the method employed in the treatment of the general case, which exhibits even greater elegance. Needless to say, the particular case may equally be solved according to the method now to be given: and this is done further on in this paper, in order to afford a comparison of methods.

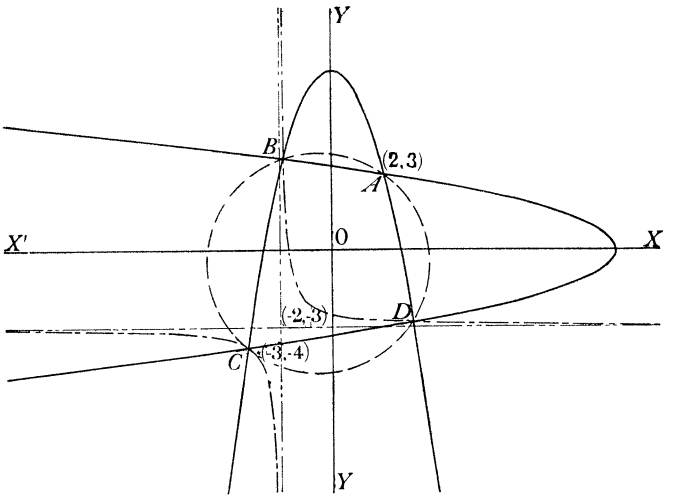


FIG. 1

Klein[1] has outlined a general program of dealing with the elementary mathematics from the higher standpoint. The solution here submitted for the solution of two simultaneous quadratic equations in two variables is, from the pedagogical point of view, a happy and forceful illustration of the Klein program.

Consider the equations

$$C_1: a_0x'^2 + a_1y'^2 + 2a_2x'y' + 2a_3x' + 2a_4y' + a_5 = 0, \tag{9}$$

$$C_2: b_0x'^2 + b_1y'^2 + 2b_2x'y' + 2b_3x' + 2b_4y' + b_5 = 0. \tag{10}$$

These equations represent in rectangular coordinates two conics which intersect in four points of some description, real or complex, rational or irrational, finite or ideal. Since some of the solutions may be infinite, we make use of homogeneous Cartesian coordinates, using the substitution

$$x' = x/z, \quad y' = y/z, \tag{11}$$

giving

[1] Felix Klein, *Elementarmathematik vom höheren Standpunkte aus*, 3te Auflage, Berlin, Springer, 1925.

$$C_1: a_0x^2 + a_1y^2 + 2a_2xy + 2a_3xz + 2a_4yz + a_5z^2 = 0, \tag{12}$$

$$C_2: b_0x^2 + b_1y^2 + 2b_2xy + 2b_3xz + 2b_4yz + b_5z^2 = 0. \tag{13}$$

On the four points of intersection of (12) and (13) there is a pencil (one-parameter family) of conics having the equation

$$C_1 + \lambda C_2 = 0 \tag{14}$$

or, written in full,

$$(a_0 + \lambda b_0)x^2 + (a_1 + \lambda b_1)y^2 + 2(a_2 + \lambda b_2)xy + 2(a_3 + \lambda b_3)xz + 2(a_4 + \lambda b_4)yz + (a_5 + \lambda b_5) = 0,$$

where λ is a parameter.

In this pencil of conics there will occur types of degenerate conics, namely certain pairs of straight lines. The condition that (14) be resolved into linear factors is

$$\begin{vmatrix} a_0 + \lambda b_0, & a_2 + \lambda b_2, & a_3 + \lambda b_3 \\ a_2 + \lambda b_2, & a_1 + \lambda b_1, & a_4 + \lambda b_4 \\ a_3 + \lambda b_3, & a_4 + \lambda b_4, & a_5 + \lambda b_5 \end{vmatrix} = 0. \tag{15}$$

This is a cubic equation in λ; and therefore in general we obtain three values of λ, i.e. $\lambda_1, \lambda_2, \lambda_3$. Each of these three values will cause equation (14) to break up into a pair of linear factors, representing geometrically a pair of straight lines. Denote by p_1 and p_2 the pair of lines arising from λ_1; by q_1 and q_2 the pair arising from λ_2; and by r_1 and r_2 the pair arising from λ_3.

Each of these pairs of lines lies on all four points of intersection of (12) and (13); and since it is impossible for a straight line to cut a conic in more than two points, each line separately passes through two of the points. Therefore on each point we have exactly three lines, one from the pair denoted by p, one from that denoted by q, and one from that denoted by r. Each of these three lines on any one point passes through one other point of the set. Therefore the lines constitute the pairs of opposite sides of a complete quadrangle on the four points, the p-lines constituting one pair of opposite sides, the q-lines another pair, and the r-lines the third (Fig. 2).

If the equation of any one of these lines be solved simultaneously with that of any other of the lines, with the single exception of that line with which it is paired, we obtain one of the required points of intersection of (12) and (13). The following table shows how all of the intersections are obtained, each in three different ways, from the equations of the six lines. Denote by $p_1 = 0$, the equation of p_1, etc. Then we have

$$
\begin{array}{lllll}
A: & \left\{\begin{array}{l} p_1 = 0 \\ q_1 = 0 \end{array}\right. & \left\{\begin{array}{l} p_1 = 0 \\ r_1 = 0 \end{array}\right. & \left\{\begin{array}{l} r_1 = 0 \\ q_1 = 0 \end{array}\right. & p_1,\ q_1,\ r_1, \\
B: & \left\{\begin{array}{l} p_1 = 0 \\ q_2 = 0 \end{array}\right. & \left\{\begin{array}{l} p_1 = 0 \\ r_2 = 0 \end{array}\right. & \left\{\begin{array}{l} q_2 = 0 \\ r_2 = 0 \end{array}\right. & p_1,\ q_2,\ r_2, \\
C & \left\{\begin{array}{l} p_2 = 0 \\ q_1 = 0 \end{array}\right. & \left\{\begin{array}{l} p_2 = 0 \\ r_2 = 0 \end{array}\right. & \left\{\begin{array}{l} r_2 = 0 \\ q_1 = 0 \end{array}\right. & p_2,\ q_1,\ r_2, \\
D: & \left\{\begin{array}{l} p_2 = 0 \\ q_2 = 0 \end{array}\right. & \left\{\begin{array}{l} p_2 = 0 \\ r_1 = 0 \end{array}\right. & \left\{\begin{array}{l} r_1 = 0 \\ q_2 = 0 \end{array}\right. & p_2,\ q_2,\ r_1.
\end{array}
$$

$A, B, C,$ and D above are the four points, and the last column shows the three lines on each point.

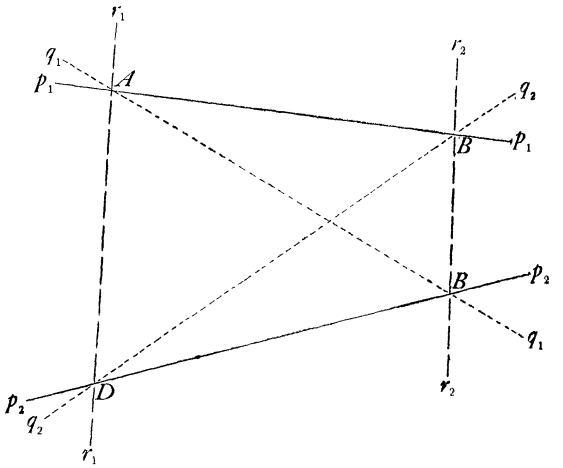


FIG. 2

If any solution gives the coordinate z the value zero, the point in question is infinite. Otherwise we may obtain the non-homogeneous coordinates by dividing x and y by the value of z, thus obtaining the solutions of (9) and (10).

A number of special cases arise according to the collocation of the four points of intersection. The case of three or four distinct points being collinear does not arise, since a straight line cannot cut a conic in more than two points.

CASE I. *Two intersections coincide.* Suppose, for example, that the intersections A and B coincide (Fig. 2). Then p_1 becomes tangent to the two given conics; and the two coincident degenerate conics, the pairs of straight lines $r_1r_2=0$ and $q_1q_2=0$ must eventuate from the same value of λ. Hence equation (15) must have a double root, which however does not make the left side of equation (14) a perfect square. Here the intersections are found by making $p_1=0$ and $p_2=0$ each simultaneous with either $q_1=0$ $(r_2=0)$ or $q_2=0$ $(r_1=0)$.

CASE II. *The intersections coincide in pairs.* Suppose, for example, that A and B coincide, and also that C and D coincide. Then p_1 and p_2 are both tangent to both conics; the former the common tangent at A, B: the latter the common tangent at C, D. Here the conics $r_1r_2=0$, $q_1q_2=0$ are coincident with each other, and in addition each is a pair of coincident lines. Hence equation (15) must have a double root which makes equation (14) a perfect square. In this case, the intersections may be found by making $q_1=0$ ($q_2=0$, $r_1=0$, $r_2=0$) simultaneous with $p_1=0$ and $p_2=0$.

CASE III. *Three intersections coincide.* Suppose, for example, that A, B, and C coincide. Then one line out of each pair becomes the common tangent to the two conics at A, B, C; while the other line in each pair coincides with the line joining A, B, C with D. Since the three degenerate conics are identical, equation (15) has a triple root which does not make equation (14) a perfect square. In this case, the intersections are found by making both $p_1=0$ ($q_1=0, r_2=0$) and $p_2=0$($q_2=0$, $r_1=0$) simultaneous with either one of the original conics given by the equations (12) and (13).

CASE IV. *Four intersections coincide.* In this case all six of the lines coincide with the common tangent to the two conics at the intersection. Hence equation (15) has a triple root, which makes equation (14) a perfect square. In this case, the single intersection (point of tangency) is found by making the equation $p_1=0$($p_2=0$, $q_1=0$, $q_2=0$, $r_1=0$, $r_2=0$) simultaneous with either one of the original conics given by the equations (12) and (13).

CASE V. *One of the original conics proper, the other degenerate.* If C_1 is proper and C_2 degenerate, then equation (15) reduces to a quadratic equation, since one value of λ must give rise to the equation $C_2=0$ when substituted in (14). If C_1 is degenerate and C_2 is proper, then equation (15) has one root zero, since one value of λ must give rise to the equation $C_1=0$ when substituted in equation (14). In this case we have only to make each of the lines of the degenerate conic simultaneous with the other conic, to find the intersections.

CASE VI. *Both original conics degenerate, with non-coincident meets.* Denote the two conics by $l_1l_2=0$, $m_1m_2=0$, and note that their meets (l_1, l_2), (m_1, m_2) are not coincident. Then since equation (15) must give rise to both C_1 and C_2, it must be a quadratic having one root zero. In this case the four intersections are found by making $l_1=0$ and $l_2=0$ simultaneous with $m_1=0$ and $m_2=0$.

CASE VII. *Both original conics degenerate, with coincident meets.* In this case equation (15) vanishes identically. To find the meet of the four lines make the equation of any of the four lines simultaneous with that of any of the other three.

Illustrations. To find the points of intersection of the two conics

$$C_1: 3x^2 + 5y^2 - 2xy + x - 5y - 10 = 0, \tag{16}$$

$$C_2: x^2 - y^2 - 6xy + 3x + y + 2 = 0. \tag{17}$$

The roots of (15) are $\lambda = 5, 1, -7/5$: and the corresponding line-pairs are

$$x(x - 4y + 2) = 0, \quad (x - y - 1)(x - y + 2) = 0,$$
$$(x + 2y - 4)(x + 2y + 2) = 0.$$

Solving these six linear equations in pairs simultaneously, never solving the members of any one pair with each other, we find the solutions of the original pair of equations to be $(0, -1)$, $(0, 2)$, $(2, 1)$, $(-2, 0)$ (Fig. 3).

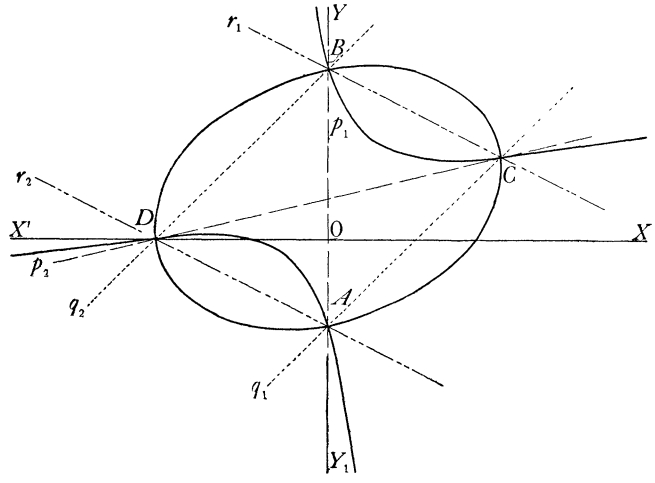


FIG. 3

As a second illustration, let us take the example of section 2:

$$C_1: x^2 + y = 7, \quad C_2: x + y^2 = 11.$$

Equation (15) is $\lambda^3+44\lambda^2+28\lambda+1=0$, the roots of which, by either Newton's or Horner's method, after isolating the roots by the use of Sturm's functions, are found to be $-.038$, $-.608$, -43.355. Corresponding to these three values are the line-pairs

$$\left.\begin{aligned} p_1: x - .195y + 2.547 &= 0 \\ p_2: x + .195y - 2.585 &= 0 \end{aligned}\right\};$$

$$\left.\begin{aligned} q_1: x - .780y + .332 &= 0 \\ q_2: x + .780y - .940 &= 0 \end{aligned}\right\}; \qquad \begin{aligned} r_1: x - 6.584y - 21.583 &= 0 \\ r_2: x - 6.584y - 21.772 &= 0. \end{aligned}$$

Solving these lines in pairs, as indicated in the table below, we find the intersections of the two conics:

A	2	(p_2q_1)	(p_1r_2)	(q_1r_2)
		2.00	2.02	2.02
	3	2.99	3.00	3.00
		(p_1q_2)	(p_1r_2)	(q_2r_2)
B	−1.8482	−1.85	−1.82	−1.82
	3.5920	3.58	3.58	3.59
		(p_1q_1)	(p_1r_1)	(q_1r_1)
C	−3.2829	−3.28	−3.24	−3.24
	−3.7765	−3.79	−3.77	−3.77
		(p_2q_2)	(p_2r_1)	(q_2r_1)
D	3.1310	3.134	3.16	3.16
	−2.8030	−2.815	−2.80	−2.80

4. A method for the solution of the quartic equation, by adopting the device of causing a one-parameter family of conics to degenerate, has been given by Heilermann.[1] Consider the quartic equation

$$a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4 = 0. \tag{18}$$

Writing this in the form

$$a_0t^4 + 4a_1t^3 + 4a_2t^2 + 2a_2t^2 + 4a_3t + a_4 - 4a_0\theta t^2 + 4a_0\theta t^2 = 0, \tag{19}$$

and effecting the transformation

$$x = t^2, \quad y = 2t, \quad z = 1, \tag{20}$$

we have

$$(a_0x^2 + a_2y^2 + a_4z^2 + 2a_1xy + 2a_3yz + 2a_2zx) - a_0\theta(y^2 - 4zx) = 0. \tag{21}$$

Now the condition that the conic (21) shall degenerate into two straight lines is

$$\begin{vmatrix} a_0 & a_1 & a_2 + 2a_0\theta \\ a_1 & a_2 - a_0\theta & a_3 \\ a_2 + 2a_0\theta & a_3 & a_4 \end{vmatrix} = 0, \tag{22}$$

which is the familiar reducing cubic[2]

$$4a_0^3\theta^3 - a_0I\theta + J = 0, \tag{23}$$

where

$$I = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_4a_1^2 - a_2^3.$$

[1] Zeitschrift für Mathematik und Physik, vol. 44 (1898).

[2] Burnside and Panton, *Theory of Equations*, p. 122.

By the use of this equation, it is not difficult to show that the equation (18), multiplied by a_0, breaks up into two factors;

$$[a_0t^2 + 2a_1t + a_2 + 2a_0\theta] \pm$$
$$[2t\{a_1^2 - a_0a_2 + a_0^2\theta\}^{1/2} - \{(a_2 + 2a_0\theta)^2 - a_0a_4\}^{1/2}] = 0. \tag{24}$$

Solving (22) we obtain three values of θ, either of which leads to two quadratic equations (24), giving the roots paired in the three ways

$$\left.\begin{matrix}\alpha,\beta\\ \gamma,\delta\end{matrix}\right\}\ ;\quad \left.\begin{matrix}\alpha,\gamma\\ \beta,\delta\end{matrix}\right\}\ ;\quad \left.\begin{matrix}\alpha,\delta\\ \beta,\gamma\end{matrix}\right\}.$$

An application; Let us apply this method to the solution of the first illustration given in article 3, namely to find the coordinates of the points of intersection of the two conics

$$3x^2 + 5y^2 - 2xy + x - 5y - 10 = 0, \tag{16}$$

$$x^2 - y^2 - 6xy + 3x + y + 2 = 0. \tag{17}$$

Using Sylvester's dialytic method of elimination, the ordinates of the four intersections of equation (16) and (17) are given by

$$\begin{vmatrix} 3 & 1-2y & 5(y^2-y-2) & 0 \\ 0 & 3 & 1-2y & 5(y^2-y-2) \\ 1 & 3(1-2y) & -(y^2-y-2) & 0 \\ 0 & 1 & 3(1-2y) & -(y^2-y-2) \end{vmatrix} = 0, \tag{25}$$

which reduces to the equation

$$y^4 - 2y^3 - y^2 + 2y = 0. \tag{26}$$

Employing the method of Heilermann outlined above, equation (22) furnishes three values of θ: $^7/_{12}$, $-^1/_6$, $-^5/_{12}$. The value $^7/_{12}$, for example, set in equation (24) gives the two equations (y for t):

$$y^2 + y = 0, \quad y^2 - 3y + 2 = 0,$$

furnishing the values $0, -1, 1, 2$. Solving for x from either (16) or (17) we obtain as the values of the coordinates, finally

$$A:(0,-1)\ ; \quad B:(0,2)\ ; \quad C:(2,1)\ ; \quad D:(-2,0)$$

This simple example is given to illustrate the method, although obviously equation (26) may be solved by inspection.

In conclusion, it is worthy of note that the present paper puts constantly in evidence the solution of the quartic equation as dependent upon the solution of a cubic equation.

ON THE EARLY HISTORY OF THE DECIMAL POINT

By JEKUTHIEL GINSBURG, Teachers College, Columbia University

The early history of the decimal fraction has been considered so extensively that it would seem that little could remain unsaid concerning its inventors. Even in the less important matter of symbolism, the ground would seem to have been covered, thanks to the investigations of men like Braunmühl, Cajori, Eneström, Glaisher, Karpinski, Smith, and Tropfke. As to the notion of the fraction itself, the facsimiles in Smith's *History of Mathematics* (Vol. 2 pp. 236–243, with the accompanying text) set forth the case with considerable completeness, showing that such fractions were used some time before the decimal point was suggested. The latter is merely one of several symbols used to separate the fractional and integral parts, and the question as to what device shall be selected is merely one relating to convenience.

The purpose of this article is to call attention to a fact which seems to have been overlooked by all historians, namely, that the credit of being the first to use a dot (point) as a separatrix, with a full knowledge of its significance in this connection, is due to the Jesuit father Christopher Clavius (1537–1612). In his work on the astrolabe, published in Rome in 1593 he gives (pp. 195–270) a "Tabula Sinuum," where the proportional parts are separated from the integers by periods (points, dots). A portion of the table is here shown. So far as now known, this is the first appearance of the decimal point in a work in which its full significance is given. The table antedates the Pitiscus edition of 1608, which has been so carefully studied by Professor Cajori.[1] It also antedates Napier's *Rhabdologia* (1617), Wright's translation of the *Mirifici logarithmorum canonis descriptio* (1616), and Kepler's *Ausszug auss der uralten Messe-Kunst Archimedis* (1616), in each of which the decimal point or comma is used. As to Bürgi's use of the symbol in a manuscript of 1592, the claim is no longer seriously considered.[2] The name of Stevin, who wrote the first book upon decimal fractions,[3] need not be considered in this connection since he made no use of the symbolism under discussion.

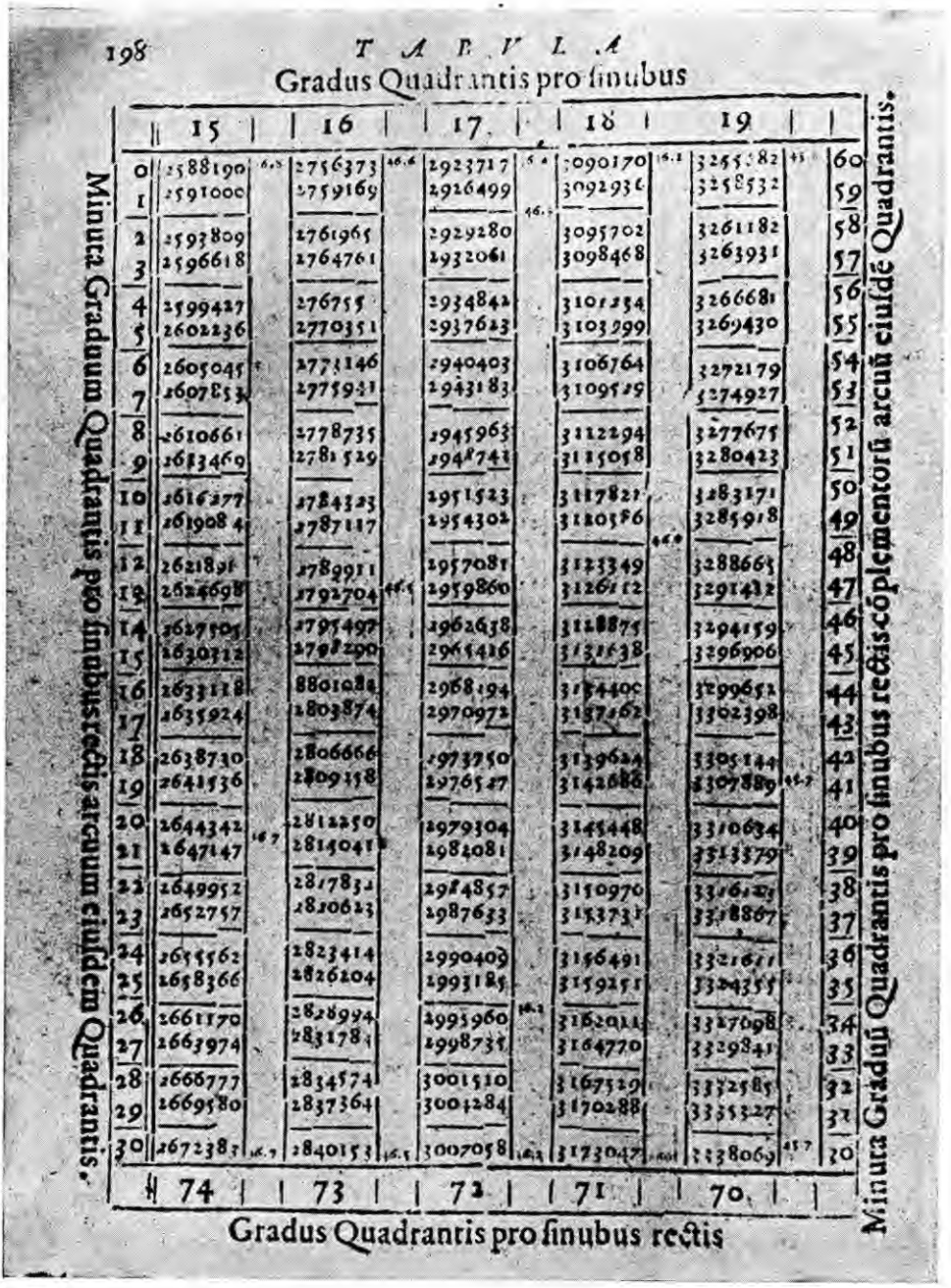
It remains to consider whether Clavius understood the significance of the symbol, for otherwise he would be entitled to but little more credit than Pellos who, in 1492, wrote 538694.3 as the quotient of 53836943 by 10, but showed no further appreciation of decimals. That Clavius actually did understand its significance appears by the following statement on page 228:

——quoniam inter duos sinus grad. 16 min. 12 grad. 16 min. 13 positi sunt duo binumeri 46.5 colligemus uni secundo inter minutum 12 13 gradus 16 congrue particulas $46\frac{5}{10}$, ex differentio 2793 inter duos sinus

[1] *Did Pitiscus use the decimal point?* In Archivio di storia della scienza, vol. 4 (1923), pp. 313–28.

[2] See L. C. Karpinski, *On the decimal point*, Science, (2), vol. 45 (1912), pp. 663–665.

[3] *Le Thiende*, Antwerp, 1535, with a French translation (*La disme*) in the same year.


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TABVLA

Gradus Quadrantis pro ſinubus

	15		16		17		18		19		
0	2588190	6.5	2756373	16.6	2923717	16.4	3090170	16.2	3255782	16	60
1	2591000		2759169		2926499	16.3	3092936		3258532		59
2	2593809		2761965		2929280		3095702		3261182		58
3	2596618		2764761		2932061		3098468		3263931		57
4	2599427		276755		2934842		3101234		3266681		56
5	2602236		2770351		2937623		3103999		3269430		55
6	2605045		2773146		2940403		3106764		3272179		54
7	2607853		2775941		2943183		3109529		3274927		53
8	2610661		2778735		2945963		3112294		3277675		52
9	2613469		2781529		2948743		3115058		3280423		51
10	2616277		2784323		2951523		3117821		3283171		50
11	2619084		2787117		2954302		3120586	16.0	3285918		49
12	2621891		2789911		2957081		3123349		3288665		48
13	2624698		2792704	16.1	2959860		3126112		3291412		47
14	2627505		2795497		2962638		3128875		3294159		46
15	2630312		2798290		2965416		3131638		3296906		45
16	2633118		2801082		2968194		3134400		3299651		44
17	2635924		2803874		2970972		3137162		3302398		43
18	2638730		2806666		2973750		3139924		3305144		42
19	2641536		2809458		2976527		3142686		3307889	16.2	41
20	2644342		2812250		2979304		3145448		3310634		40
21	2647147	16.7	2815041		2982081		3148209		3313379		39
22	2649952		2817832		2984857		3150970		3316123		38
23	2652757		2820623		2987633		3153731		3318867		37
24	2655562		2823414		2990409		3156491		3321611		36
25	2658366		2826204		2993185		3159251		3324355		35
26	2661170		2828994		2995960	16.3	3162011		3327098		34
27	2663974		2831784		2998735		3164770		3329841		33
28	2666777		2834574		3001510		3167529		3332585		32
29	2669580		2837364		3004284		3170288		3335327		31
30	2672383	16.7	2840153	16.5	3007058	16.3	3173047	16.1	3338069	16.7	30
	74		73		72		71		70		

Gradus Quadrantis pro ſinubus rectis

Minuta Graduum Quadrantis pro ſinubus rectis arcuum eiuſdem Quadrantis.

Minuta Graduũ Quadrantis pro ſinubus rectis complementorũ arcuũ eiuſdẽ Quadrantis.

Facsimile of p. 198 of the "Astrolabe" of Clavius (1593)

2789911. 2792704praedictorum arcuum grad. 16 min. 12 et grad 16min. 13 qua to ta differentia secundia 60, Hoc Est.

It amounts to the following:

In the table on p. 198, the number 46.5 is put between sin 16°12′ and 16°13′. Hence to every second (of the difference) between 16°12′ and 16°13′ we will apportion $46\frac{5}{10}$ particles of the difference (2793) between 2789911 (sin 16°12′) and 2792704 (sin 16°13′) etc.

It would seem, therefore, that up to the present time the evidence is conclusive that Clavius was the first to use the decimal point with a clear understanding of its significance. This statement does not, of course, invalidate the positive evidence that the decimal fraction was used with other symbols long before his time.

PROPERTIES OF TWO POINTS ASSOCIATED WITH A TRIANGLE

By J. H. WEAVER, Ohio State University

The Brocard points of a triangle are defined as the intersections of two triads of circles described on BC, CA, AB as chords in such a way that the external segments of the circles contain the angles (1) C, A, B, and (2) B, C, A respectively.[1]

If the triads of circles are drawn in such a way that the internal segments contain the angles (1) C, A, B and (2) B, C, A, respectively, two other points P_2 and P_4 will be determined. It is the purpose of the present paper to discuss some of the properties of the points P_2 and P_4.

THEOREM I: *The pedal triangles of P_2 and P_4 are similar.*

PROOF: Since in the figure P_2 lies within the angle BCA and $D_2E_2F_2$ is the pedal triangle of P_2 we have $\angle BP_2C = \angle(A+D_2)$.[2] Therefore $\angle D_2 = \angle(C-A)$. Similarly $\angle E_2 = \angle(A-B)$; $\angle F_2 = \angle(\pi+B-C)$. Also if, $D_4E_4F_4$ is the pedal triangle of P_4 it may be shown in the same way that

$$\angle D_4 = \angle(A-B), \quad \angle E_4 = (\pi+B-C), \quad \angle F_4 = \angle(C-A).$$

Hence the pedal triangles of P_2 and P_4 are similar.

[1] For a discussion of the various definitions of Brocard points and their properties see McClelland, *Geometry of the Circle*, London, 1891, pp. 60–70; Court, *College Geometry*, Richmond, 1925, pp. 24–38; Gallatley, *Modern Geometry of The Triangle*, London, pp. 94–108; Coolidge, *A Treatise on The Circle and the Sphere*, Oxford, 1916, pp. 60–84; *Encyclopädie der Mathematischen Wissenschaften*, Band III_1, Heft 7.

[2] See Gallatley, *Modern Geometry of the Triangle*, p. 37.

THEOREM II: *The three pairs of lines BP_2 and CP_4, CP_2 and AP_4, AP_2 and BP_4 are parallel.*

PROOF: It is shown in Gallatley's, *Modern Geometry of the Triangle*, p. 42, that the trilinear coordinates of the points P_2 and P_4 may be written as follows: for P_2,

$$\alpha:\beta:\gamma = \frac{\sin C}{\sin(C-A)}:\frac{\sin A}{\sin(A-B)}:\frac{\sin B}{\sin(B-C)};$$

for P_4,

$$\alpha:\beta:\gamma = \frac{\sin B}{\sin(A-B)}:\frac{\sin C}{\sin(B-C)}:\frac{\sin A}{\sin(C-A)};$$

where AB is the line $\gamma=0$, BC is the line $\alpha=0$ and CA is the line $\beta=0$. The equation of AP_2 is

$$\frac{\beta \sin B}{\sin(B-C)} - \frac{\gamma \sin A}{\sin(A-B)} = 0;$$

and the equation of BP_4 is

$$\frac{\alpha \sin A}{\sin(C-A)} - \frac{\gamma \sin B}{\sin(A-B)} = 0.$$

Now the angle which any line $l\alpha+m\beta+n\gamma=0$ makes with the line $\gamma=0$ is given by the relation[1]

$$\tan\psi = \frac{l\sin B - m\sin A}{l\cos B + m\cos A - n}.$$

For the line AP_2 this relation becomes

$$\tan\psi_2 = \frac{\sin A \sin B \sin(A-B)}{\sin B\cos A\sin(A-B) + \sin A\sin(B-C)};$$

and for BP_4 it becomes

$$\tan\psi_4 = \frac{\sin A \sin B \sin(A-B)}{\sin A\cos B\sin(A-B) + \sin B\sin(C-A)}.$$

The values for $\tan\psi_2$ and $\tan\psi_4$ are equal; hence the lines AP_2 and BP_4 are parallel. Similarly it may be shown that CP_2 is parallel to AP_4 and that BP_2 is parallel to CP_4.

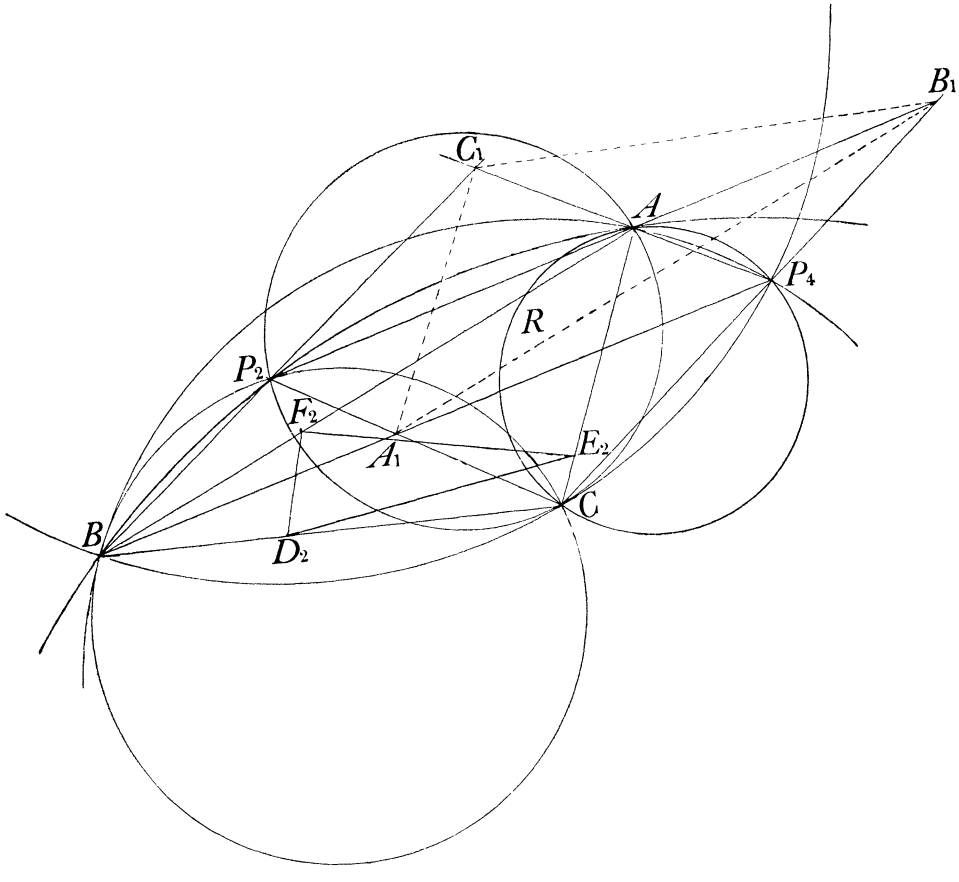
[1] See Whitworth, *Modern Analytic Geometry*, Cambridge, 1866, p. 49.

Let BP_4 and CP_2 intersect in A_1, AP_2 and CP_4 intersect in B_1, and BP_2 and AP_4 intersect in C_1. We may then prove

THEOREM III: *The triangles ABC and $A_1B_1C_1$ are congruent and have the sides AB, BC, CA respectively parallel to A_1B_1, B_1C_1, C_1A_1.*

PROOF: Without much difficulty it may be shown that the equation of A_1C_1 is

$$\begin{aligned}
&\alpha[\sin^2 A \sin^2 B \sin^3 (C - A) \sin (B - C) \\
&\quad - \sin A \sin B \sin^2 C \sin^2 (C - A) \sin^2 (A - B)] \\
&\quad + \beta[\sin^2 A \sin^2 C \sin^2 (A - B) \sin^2 (B - C) \\
&\quad - \sin A \sin^2 B \sin C \sin (A - B) \sin^2 (C - A) \sin (B - C)] \\
&\quad + \gamma[\sin^2 B \sin^2 C \sin (A - B) \sin^3 (C - A) \\
&\quad - \sin^2 A \sin B \sin C \sin^2 (B - C) \sin^2 (C - A)] = 0.
\end{aligned}$$



This line will be parallel to the line $\beta = 0$[1] if

$$\sin^2 A \sin^2 B \sin C \sin^2 (C - A) \sin (B - C)$$
$$- \sin A \sin B \sin^3 C \sin^2 (C - A) \sin^2 (A - B)$$
$$- \sin A \sin^2 B \sin^2 C \sin (A - B) \sin^3 (C - A)$$
$$- \sin^3 A \sin B \sin C \sin^2 (B - C) \sin^2 (C - A) = 0.$$

This is true. Therefore the line A_1C_1 is parallel to AC, and since A_1C_1 and AC are parallels between parallels, we have $A_1C_1 = AC$. A similar relationship holds for the pairs of lines B_1C_1 and BC and A_1B_1 and AB. Hence the theorem.

The following theorem will be evident from a glance at the figure.

THEOREM IV: *The triangles ABC and $A_1B_1C_1$ are homothetic, the homothetic center being R, the mid-point of P_2P_4.*

THEOREM V: *The two points P_2 and P_4 bear the same relationship to each of the triangles ABC and $A_1B_1C_1$.*

PROOF: This follows from the fact that since R is the homothetic center of the two triangles, P_2 referred to the triangle ABC corresponds to P_4 referred to the triangle $A_1B_1C_1$.

THEOREM VI: *The two triangles ABC and $A_1B_1C_1$ are triply perspective, the centers of perspective being the three collinear points P_2, R, P_4, and the three axes being the line at infinity and two parallel lines symmetrically situated with respect to R.*

PROOF: A glance at the figure will show the first part of the theorem; and since when two triangles are perspective from three collinear points they will be perspective from three concurrent lines, and since one of these lines is the line at infinity, the other two will be parallel.

A PROJECTIVE THEOREM ON THE PLANE PENTAGON

By EDWARD KASNER, Columbia University

This paper is concerned with a projective theorem relating to the plane pentagon. In order to be able to state it, we first define two projective constructions, the diagonal construction D and the inscribed construction I, by means of which two new pentagons are derived from the original one.

Let A, B, C, D, E be the corners of the given pentagon P. The construction D then consists in drawing the five diagonals BD, CE, DA, EB, AC of P, intersecting every pair of consecutive ones in the indicated order and taking

[1] See Whitworth, *Modern Analytic Geometry*, p. 49.

these points of intersection (in the same order) for the corners of a new pentagon.[1] The construction I consists in finding the points of contact of the conic inscribed to P with the sides CD, DE, EA, AB, BC of P and taking these points in the order of the sides of P which they lie on for the corners of a new pentagon.[2]

We term the two pentagons derived in this manner from the original one, P, the *diagonal* and the *inscribed pentagon* of P and abbreviate them by $\mathsf{D}P$ and $\mathsf{I}P$.

Our theorem now reads:

For any plane pentagon P the inscribed pentagon of its diagonal pentagon and the diagonal pentagon of its inscribed pentagon are identical. In formulas,

$$\mathsf{IDI}P = \mathsf{DII}P.$$

Differently expressed: *the diagonal construction* D *and the inscribed construction* I *are commutative.*

Let the corners of $\mathsf{D}P$ be called A', B', $\cdots$ so that

$$(1) \qquad A' = (BD)(CE)^3, \qquad B' = (CE)(DA), \cdots,$$

and the corners of $\mathsf{I}P$ be called A_1, B_1, $\cdots$ so that they lie on the sides CD, DE, $\cdots$. Then according to Brianchon's theorem

$$(2) \qquad A_1 = (AA')(CD)^4, \qquad B_1 = (BB')(DE), \cdots.$$

With the same method of notation, superscripts and subscripts being used to denote the diagonal and the inscribed pentagon of a given pentagon, the corners $(A')_1$, $(B')_1$, $\cdots$ of $\mathsf{IDI}P$ and the corners $(A_1)'$, $(B_1)'$, $\cdots$ of $\mathsf{DII}P$ are determined as follows:[5]

$$(3) \qquad \begin{aligned} &(a) \quad (A')_1 = (A'A'')(C'D')^6, \quad (B')_1 = (B'B'')(D'E'), \cdots, \\ &(b) \quad (A_1)' = (B_1D_1)(C_1E_1), \quad (B_1)' = (C_1E_1)(D_1A_1), \cdots, \end{aligned}$$

where

$$(3)' \qquad A'' = (B'D')(C'E'), \quad B'' = (C'E')(D'A'), \cdots.$$

To prove our theorem, we must show that

$$(A')_1 = (A_1)'; \qquad (B')_1 = (B_1)'; \cdots.$$

[1] A', B', C', D', E' in the accompanying drawing.

[2] A_1, B_1, C_1, D_1, E_1, in the accompanying drawing.

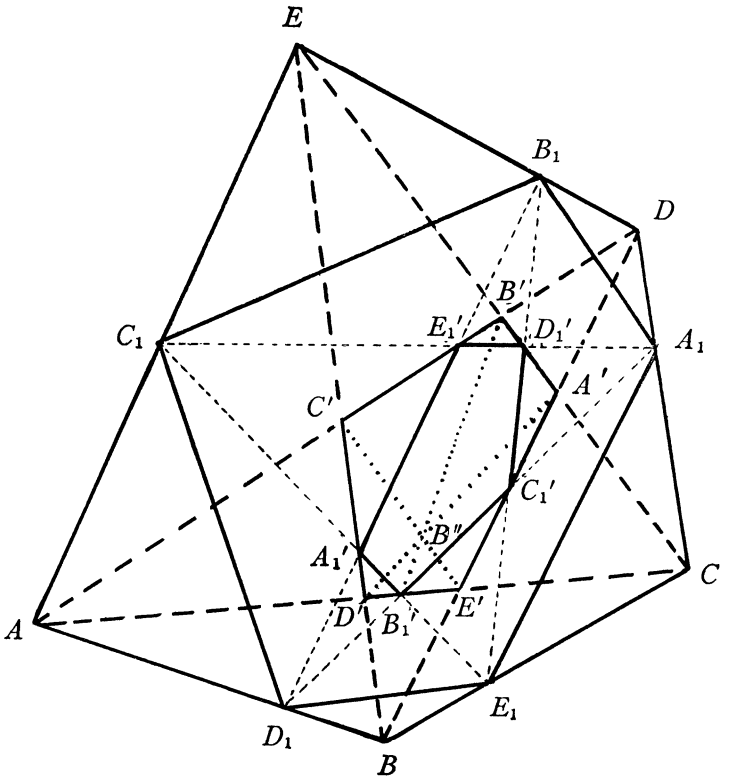
[3] We use this notation to indicate that the point on the left hand side is the point of intersection of the two lines in brackets on the right hand side.

[4] The lines AA', BB', $\cdots$ have not been put into the drawing, to avoid confusing the reader by too many lines.

[5] In the drawing we have indicated the coinciding corners of $\mathsf{IDI}P$ and $\mathsf{DII}P$ by A_1', B_1', $\cdots$.

[6] In the drawing we have carried out the complete construction only for $(B')_1$.

On account of the absolute symmetry of construction for all five pairs of points, it will be sufficient to show the identity for any one pair, say $(B')_1$ and $(B_1)'$.



We give the proof that $(B')_1$ and $(B_1)'$ coincide analytically, and for this purpose introduce the special homogeneous coordinate system which has its fundamental points at the points, A, B and C and its point of unity at the point D. The coordinates of E in this system shall be denoted by a, b, 1.

In this system we find for the coordinates of the corners and sides of P,

$$
\begin{array}{llllll}
A: & 1 \;\; 0 \;\; 0 & & CD: & 1 \quad -1 \quad 0 & \\
B: & 0 \;\; 1 \;\; 0 & & DE: & b-1 \quad 1-a \quad a-b & \\
C: & 0 \;\; 0 \;\; 1 & ; & EA: & 0 \quad -1 \quad b & ; \\
D: & 1 \;\; 1 \;\; 1 & & AB: & 0 \quad 0 \quad 1 & \\
E: & a \;\; b \;\; 1 & & BC: & 1 \quad 0 \quad 0 &
\end{array} \tag{4}
$$

the sides and corners of DP, according to (1),

$$\begin{array}{llllllllll} A'D' = EB: & -1 & 0 & a & & A': & a & b & a & \\ D'E' = AC: & 0 & 1 & 0 & & B': & a & b & b & \\ E'A' = BD: & 1 & 0 & -1 & ; & C': & a & 1 & 1 & ; \\ A'B' = CE: & -b & a & 0 & & D': & a & 0 & 1 & \\ B'C' = DA: & 0 & -1 & 1 & & E': & 1 & 0 & 1 & \end{array} \tag{5}$$

the lines AA', BB', $\cdots$, according to (4) and (5),

$$\begin{array}{lccc} AA': & 0 & -a & b \\ BB': & b & 0 & -a \\ CC': & -1 & a & 0 \\ DD': & -1 & 1-a & a \\ EE': & b & 1-a & -b \end{array} \quad ; \tag{6}$$

the corners of IP, according to (2), (4) and (6),

$$\begin{array}{lccc} A_1: & b & b & a \\ B_1: & a(1-a) & b^2 - 2ab + a & b(1-a) \\ C_1: & ab & b & 1 \\ D_1: & 1-a & 1 & 0 \\ E_1: & 0 & b & 1-a \end{array} \tag{7}$$

To find $(B')_1$ we first calculate the coordinates of B'' according to (3)′ and (5).

$$B'': \quad a(b+1) \quad b \quad a+b.$$

Then, according to (3a) and (5), the coordinates of $(B')_1$ are

$$(B')_1: \quad b \quad 0 \quad 1.$$

The coordinates of $(B_1)'$ are immediately obtained, according to (3b), from (7) as

$$(B_1)' : b \quad 0 \quad 1$$

and are thus equal to those of $(B')_1$.

In this manner we have shown that the points $(B_1)'$ and $(B')_1$ coincide. As stated in the beginning of the proof, it follows by virtue of symmetry that also the remaining points $(A_1)'$, $(C_1)'$, $(D_1)'$, $(E_1)'$ coincide with the points $(A')_1$, $(B')_1$, $(D')_1$, $(E')_1$ respectively. Thus our theorem is proved:

The pentagon obtained by first applying to the given pentagon P the diagonal construction D *and then to the pentagon* DP *the inscribed construction* I, *is the*

same as the pentagon obtained by first applying to P the inscribed construction I *and then to* I*P the diagonal construction* D.

This theorem was first stated by the author in 1896 at a students' colloquium in Columbia University. It was published (among other results) in an abstract in the Bulletin of the American Mathematical Society, vol. 9 (1903), pp. 397, 398. Ten years later Professor Bateman, to whom I communicated the theorem, indicated a synthetic metrical proof. This year, three members of my seminar, Miss Hofmann, Mr. Donahue, and Mr. Feld, worked out synthetic projective proofs.

Among the corollaries stated in the Bulletin abstract, the most interesting is this (valid for at least convex pentagons):

The limit point of the sequence of successive inscribed pentagons coincides with the limit point of the sequence of successive diagonal pentagons.

SYNTHETIC PROOF OF PROFESSOR KASNER'S PENTAGON THEOREM

By LULU HOFMANN, Columbia University

This paper gives a synthetic proof of the pentagon theorem stated and proved analytically by Professor Kasner in the preceding article. We will not re-explain the notions of the diagonal and inscribed pentagon, DP and IP of a given pentagon P nor the notation used by Professor Kasner. We merely repeat the theorem:[1]

For any plane pentagon P the inscribed pentagon IDP *of its diagonal pentagon* DP *and the diagonal pentagon* DIP *of its inscribed pentagon* IP *are identical*:

$$\mathsf{ID}P = \mathsf{DI}P.$$

We conduct our proof in two steps; the first showing the points $(A_1)'$, $(B_1)'$, $\cdots$, the corners of DIP, to lie on the sides $C'D'$, $D'E'$, $\cdots$ of DP; the second showing them to be identical with the points $(A')_1$, $(B')_1$, $\cdots$ which by construction as the corners of IDP, the pentagon inscribed to DP, lie on these same sides $C'D'$, $D'E'$ $\cdots$ of DP.

The first part of the proof consists in an application of the following well known projective theorem:

If of two simple quadrilaterals the first is circumscribed and the second is inscribed to the same conic in such a manner that the corners of the second lie in the points of contact with the conic of the first, then the four diagonals of these two quadrilaterals are concurrent.

[1] A figure illustrating the theorem is also found in Professor Kasner's article.

To show that, for example, the point $(E_1)'$ which by construction is the point of intersection of the lines A_1C_1 and B_1D_1 lies on the line $B'C'=AD$, we construct the point of intersection E^* of the lines AB and CD and consider the two quadrilaterals A, E^*, D, E and D_1, A_1, B_1, C_1. By construction these two quadrilaterals have the mutual position described in the theorem just quoted, D_1, A_1, B_1, C_1 being the points of contact with the lines AB, CD, DE, EA of the conic inscribed to the pentagon P. Therefore the four diagonals AD, EE^*, A_1C_1, B_1D_1 are concurrent, or in a wording better suited for our proof, the point of intersection $(E_1)'$ of A_1C_1 and B_1D_1 lies on the line $AD=B'C'$.

In the same manner it follows that the points $(A_1)'$, $(B_1)'$, $(C_1)'$, $(D_1)'$ lie on the lines $C'D'$, $D'E'$, $E'A'$, $A'B'$, respectively.

It remains to be proved that the points $(A_1)'$, $(B_1)'$, $\cdots$ are identical with the points $(A')_1$, $(B')_1$, $\cdots$. For this purpose we introduce two mutually inverse plane correlative transformations defined with respect to a given pentagon with the corners O_1, O_2, O_3, O_4, O_5. These correlations shall be denoted by $\Omega(O_1O_2O_3O_4O_5)$ and $\Omega^{-1}(O_1O_2O_3O_4O_5)$; they are determined as follows:

(1)

$\Omega(O_1O_2O_3O_4O_5)$	$\Omega^{-1}(O_1O_2O_3O_4O_5)$
$O_1 \rightarrow O_3O_4$	$O_1O_2 \rightarrow O_4$
$O_2 \rightarrow O_4O_5$	$O_2Q_3 \rightarrow O_5$
$O_3 \rightarrow O_5O_1$	$O_3O_4 \rightarrow O_1$
$O_4 \rightarrow O_1O_2$	$O_4O_5 \rightarrow O_2$
$O_5 \rightarrow O_2O_3$	$O_5O_1 \rightarrow O_3$.*

The proof that the corners $(A_1)'$, $(B_1)'$, $\cdots$ of $\mathsf{DI}P$ and the corners $(A')_1$, $(B')_1$, $\cdots$ of $\mathsf{IDI}P$ coincide consists in various applications of the two correlations.

It is easily found from (1) that if we successively apply Ω and Ω^{-1} in the following manner: $\Omega^{-1}[\Omega(ACEBD)]$, the resulting projectivity carries the original pentagon P: A, B, $\cdots$ into the diagonal pentagon $\mathsf{D}P$: A', B', $\cdots$; so that using the symbol $\sim$ to denote projective and later also correlative relationship,

$$P \sim \mathsf{D}P.$$

The projectivity just constructed necessarily transforms the inscribed pentagon $\mathsf{I}P$ of P: A_1, B_1, $\cdots$ into the inscribed pentagon $\mathsf{IDI}P$ of $\mathsf{D}P$: $(A')_1$, $(B')_1$, $\cdots$; therefore

$$\mathsf{I}P \sim \mathsf{IDI}P.$$

* The fact that the five points O_1, O_2, $\cdots$ and the five lines O_3O_4, O_4O_5, $\cdots$ are really elements corresponding to each other by correlation is recognised by joining any one of the points, say O_1, to the remaining ones. The figure shows immediately that the anharmonic ratio of the four lines O_1O_2, O_1O_3 O_1O_4, O_1O_5 is equal to the anharmonic ratio of the four points in which the line O_3O_4 is intersected by the lines O_4O_5, O_5O_1, O_1O_2, O_2O_3.

In the same way as above, by constructing $\Omega^{-1}[\Omega(A_1C_1E_1B_1D_1)]$, we find

$$\mathsf{I}P \sim \mathsf{DI}P.$$

So we obtain

$$\mathsf{ID}P \sim \mathsf{DI}P \tag{2}$$

whereby, as we proved above, corresponding corners of $\mathsf{ID}P$ and $\mathsf{DI}P$ lie on the same sides of $\mathsf{D}P$[$(A')_1$ and $(A_1)'$ on $C'D'$ and so on].

It is well known that a conic uniquely determines a polarity in which a point of the conic and the tangent at that point are corresponding elements. If for the determining conic we take the one inscribed to the pentagon $\mathsf{D}P$ it follows that the corners of $\mathsf{ID}P$ and the sides of $\mathsf{D}P$ which they respectively lie on are elements corresponding by correlation. In this correlation we may replace $\mathsf{ID}P$ by $\mathsf{DI}P$ by virtue of (2). Writing the two relations thus obtained out in detail, we finally have

$$\left.\begin{aligned} &(A')_1,(B')_1,(C')_1,(D')_1,(E')_1 \sim \quad (3)\\ &(A_1)',(B_1)',(C_1)',(D_1)',(E_1)' \sim \quad (4)\end{aligned}\right\} C'D',D'E',E'A',A'B',B'C'.$$

The correlation defined by (3) has for its point curve of coincidences the conic tangent to the lines $C'D'$, $D'E'$, $\cdots$ in the points $(A')_1$, $(B')_1$, $\cdots$. The correlation defined by (4) has for its point curve of coincidences the conic tangent to the same lines in the points $(A_1)'$, $(B_1)'$, $\cdots$. Since however, there is only one conic tangent to five lines and since by construction $(A')_1$, $(B')_1$, $\cdots$ are the points in which the conic inscribed to $\mathsf{D}P$ is tangent to the sides $C'D'$, $D'E'$, $\cdots$ of $\mathsf{D}P$, it follows that the points $(A_1)'$, $(B_1)'$, $\cdots$ are identical with the points $(A')_1$, $(B')_1$, $\cdots$ and our proof is complete.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A SUGGESTION FOR SIMPLIFICATION OF ANNUITY FORMULAS

By W. L. HART, University of Minnesota

In any treatment of the theory of annuities certain, a fundamental problem is the preparation of formulas which will apply to the computation of the pres-

ent value and the amount of any annuity, under any arrangement concerning the compounding of interest. In the following note there is presented a single pair of simple formulas well arranged with respect to computation by means of tables, which will apply to all cases which could arise. The essence of the method consists of the recognition of the generality of the familiar formula (restricting the discussion to *amounts* of annuities)

$$S = R\frac{i}{j_p}(s_{\overline{n}|} \text{ at } i), \tag{1}$$

when it is interpreted with respect to the *interest period* rather than the *year* as the fundamental unit of time, and when p is *not* required to be an integer.[1]

Consider an investor whose interest rate per period is i, and whose interest period is specified. Let K be the periodic payment of an annuity whose term is n interest periods. Let p represent *the ratio of the interest period to the payment interval*:

$$p = \text{interest period/payment interval.} \tag{2}$$

Then, the customary methods[2] of the mathematics of investment demonstrate that the amount S of this annuity is given by the expression

$$S = K\frac{(1+i)^n - 1}{(1+i)^{1/p} - 1}. \tag{3}$$

From the generality of the data of the previous paragraph, it follows that formula 3 applies to *any* annuity under *any* arrangement as to the computation of compound interest. The simplicity of the formula results from the use of the *interest period* of the investor as the unit of time, and from the explicit introduction of the *rate per period* instead of the *nominal rate* in the final formula. Every student of the mathematics of investment has used formula 3 in instances where p is an integer. It is emphasized here that the use of this formula is equally justifiable when p is *not* an integer, because by definition p is free to assume any positive value except zero. The following illustrations exhibit the ease with which formula 3 applies to problems with data chosen as inconveniently as possible.

Illustration 1. Interest is at the nominal rate 6%, compounded monthly; the annuity consists of payments of \$100 occurring semi-annually for $5\frac{1}{2}$ years.

[1] For the details of this interpretation, for p an integer, see W. L. Hart, *The Mathematics of Investment*, D. C. Heath and Co., 1924, page 47.

[2] Hart, loc. cit., page 46.

The amount of the annuity is obtained from formula (3) with $p=1/6$, $K=\$100$, $i=.005$ and $n=66$.

Illustration 2. Interest is at the rate 4%, compounded semi-annually; the annuity consists of payments of \$5000 occurring at the end of each 6 years for 72 years. The amount of the annuity is obtained from formula (3) with $p=1/12$, $K=\$5000$, $i=.02$ and $n=144$.

The reader can easily verify that any of the usual problems in the applications of annuities would give rise to one or another of the following values of p, as defined by formula 2: $p=1$, 2, 3, 4, 6, 12, 13, 26, 52, 1/2, 1/3, 1/4, 1/6, or 1/12. When $p=1$, formula 3 reduces to $S=K(s_{\overline{n}|}\ at\ i)$. For the other usual values of p, we adapt formula 3 for computation by means of *one* auxiliary conversion table. First let $R=pK$, and $j_p=p[(1+i)^{1/p}-1]$. Then, from formula 3 we obtain the familiar formula 1. For each desirable value of i, suppose that the values of $i\div j_p$ are tabulated for all of the usual *fractional* as well as integral values of p. By use of formula 1, the table for $i\div j_p$, together with the table for $(s_{\overline{n}|}\ at\ i)$, would serve for obtaining the amount of *any* annuity under *any* arrangement as to compound interest, except when n is not an integer, or when i is unusual. These unusual cases can be treated directly by formula 3.

It is not claimed that the method presented here is new in principle. The main object of this note was to emphasize that, with a properly prepared table for $i\div j_p$, all usual problems involving the computation of the amounts or present values of annuities can be solved by use of formula 1 and its twin (which relates to present values).

II. A Systematic Method for the Solution of Simultaneous Linear Equations

By H. G. Deming, Department of Chemistry, University of Nebraska

The method to be described is an application to linear equations of Chiò's method for the reduction of determinants.[1] The numerical work is essentially that involved in any other method. Its advantage is that it furnishes a systematic scheme of procedure, with a check against errors at every stage of the work.

We begin by replacing all the constant literal coefficients by their corresponding numerical values and transposing all terms to the left-hand side of the equality sign. The successive steps are as follows:

$$2.86x - 4.78y + 3.19\ z - 1.375 = 0$$

$$6.32x + 2.47y - 5.64\ z - 4.12\ = 0$$

$$3.03x - 5.91v + 1.553z - 1.362 = 0$$

[1] See Whittaker and Robinson, *The Calculus of Observations*, London, 1924, p. 71.

(1) Tabulate the coefficients of the unknown quantities, together with the constant terms (c), as given in bold-faced type in the first few lines of the table on p. 362. The column headed s gives the actual or approximate sum of the preceding quantities in each row.

(2) We suppress one row and one column (other than column c or s) by drawing rectangles around them. In practice, *the column that is suppressed is the column* (other than c or s) *that contains the simplest numbers, preferably zeros; and the row that is suppressed is the row* (if any) *that contains a unit coefficient in the suppressed column.* In the present instance there is little room for choice. We suppress column y and row B at random.

(3) We divide each of the other numbers in the suppressed column by the number (2.47) that is found at the intersection of the suppressed column and suppressed row. The quotients, R, thus obtained, we set down in parentheses in the suppressed column.

(4) We multiply each of the quotients just obtained by each of the numbers in the other columns of the suppressed row. We place the products so obtained immediately under the numbers of the original table and pair them with the latter by braces. Thus the value of R in the first row is -1.935; and the suppressed coefficient in column c is -4.12. The product of these is $+7.973$, which is to be paired with the original number in the first row and column c.

(5) We construct a new table, using differences of the paired numbers in the preceding table. Thus the first number in the new table will be $2.86-(-12.23)=15.09$. The new table will contain one less row and one less column than the preceding table.

(6) We repeat the process, suppressing one column and one row at a time, until we finally arrive at a single row, from which the value of one unknown quantity may be found directly.

(7) We substitute this result in the last row to be suppressed, and so obtain the value of a second unknown quantity; then, substituting in preceding suppressed rows, we find the others.

(8) A continuous check is furnished by the fact that each successive value of s, when calculated from preceding values of s, will be approximately the sum of the preceding numerical quantities in the same line, so long as the work is correct. Thus, in line C, the value of s, calculated from preceding values of s, is -5.011. The actual sum of the preceding coefficients in that line is -5.010. As a final check, substitute the values of the unknowns in some line not suppressed (say A), and note that the sum of the numerical quantities so obtained is approximately zero.

	x	y	z	c	s
A	2.86 −12.23	−4.78 ($R=-1.935$)	3.19 10.913	−1.375 7.972	−0.105 1.877
B	6.32	2.47	−5.64	−4.12	−0.97
	3.03 −15.124	−5.91 ($R=-2.393$)	1.553 13.496	−1.362 9.859	−2.689 2.321
	15.09 R = 0.8312		−7.723 −9.927	−9.347 −9.327	−1.982 −4.164
C	18.154		−11.943	−11.221	−5.010
			2.204	−0.020	2.182

Thus $2.204z-0.020=0$; whence $z=0.009$. Substituting in C, $x=0.624$. Substituting in B, $y=0.092$. Check: Substituting in A,

$$(2.86\times0.624) - (4.78\times0.092) + (3.19\times0.009) - 1.375 = -0.001 .$$

Our next example is one in which five unknown quantities are to be found from a set of five simultaneous linear equations. If 0 and 1 occur in a number of places in the tabulation, an opportunity is afforded for the exercise of a little skill in determining the order in which the rows and columns are to be suppressed. For it is evident that whenever a suppressed column contains a zero, all the numbers occurring in the same row with that zero may be transferred without change to the next following tabulation; and whenever a suppressed row and column intersect in a 1, we may avoid the labor of forming the preliminary quotients, R. When an entire row is exactly divisible by some constant factor, one may carry through such a division and drop out that factor; or one may multiply any entire row by any desired factor, whenever this will result in a convenience. The given equations and the tabular arrangement of the solution are as follows:

$$\begin{aligned}
x_1 \quad &+ 5x_3 - 2x_4 + x_5 - 10 = 0,\\
4x_2 - 3x_3 &+ 7x_4 \quad + 14 = 0,\\
-9x_1 + 8x_2 \quad &+ 10x_4 \quad - 31 = 0,\\
2x_1 + 20x_2 - 13x_3 \quad &+ x_5 + 2 = 0,\\
5x_1 + 4x_2 + 3x_3 + 2x_4 &+ 4x_5 - 1 = 0.
\end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	c	s
A	1	0	5	−2	1	−10	−5
	0	4	−3	7	0	14	22
	−9	8	0	10	0	−31	−22
	2	20	−13	0	1	2	12
	1	0	5	−2		−10	−5
B	5	4	3	2	4	−1	17
	4	0	20	−8		−40	−20
	0	4	−3	7		14	22
	−9	8	0	10		−31	−22
		−180	162	−18		−108	−153
C	1	20	−18	2		12	17
	1	4	−17	10		39	37
		20	−18	2		12	17
		4	−3	7		14	22
		48		−24		−81	−60
		188	−162	28		77	131
		2592		−1296		−4374	−3240
D		−16	1	8		27	20
E		−44		31		95	82
		−2404		1324		4451	3371
		$R=54.636$		1693.7		5190.4	4480.1
				−369.7		−739.4	−1109.1

Thus $-369.7x_4-739.4=0$; whence $x_4=-2$. Substituting in E, $x_2=\frac{3}{4}$. Substituting in D, $x_3=1$. Substituting in C, $x_1=-5$. Substituting in A, $x_5=6$. Check: Substituting in B, $0=0$.

III. A Method for Approximating Roots of Algebraic Equations in Pairs

By H. G. Deming, Department of Chemistry, University of Nebraska

The method to be described in this article consists in reducing an equation of higher degree to an equivalent set of quadratic equations, and hence gives the approximate or exact roots of the original equation in pairs. Complex or real roots may be obtained with equal ease. Moreover, it furnishes two roots with about the same labor that Horner's method requires for one.

Let the coefficient of the first term of an algebraic equation of the nth degree be unity. Detached coefficients will then appear as in the first line of the tabular arrangement below. Now let approximate values of a pair of roots be found by any of the usual methods. If the sum of the true values of these roots be represented by s, and the negative product of the true values by p, we may represent successive approximations to this sum by s_1, s_2, s_3, $\cdots$ and successive approximations to the negative product by p_1, p_2, $p_3 \cdots$.

Referring to the tabular arrangement below, the first coefficient of the original equation (unity) is multiplied by s_1, and the product, s_1, is placed in the second column. The sum of a and s_1 is then represented by a_1. Next a_1 is multiplied by s_1, and the product, a_1s_1, is placed in the third column. In this column is also placed the product of the first coefficient (unity) and p_1. Proceeding in this way, we complete the upper portion of the tabulation. Then, repeating the process, starting from the *first reduced coefficients*, 1, a_1, b_1, c_1, $\cdots$, just obtained, we obtain a further set of quantities, which we shall call partials, and designate by A, B, C, $\cdots$. The final partial, N, need not be calculated.

Note that the work is carried out to the column beyond n, in the third row. This affords a check, for the sum of the quantities in the second row is then s_1S_1, and the sum of those in the third row is p_1S_1, where S_1 is the sum of those in the fourth row (excluding n_1). A similar check is applied to the last three lines of the tabulation.

	1	a	b	$c \cdots l$	m	n			*Sums*
s_1		s_1	a_1s_1	$b_1s_1 \cdots k_1s_1$	l_1s_1	m_1s_1			s_1S_1
p_1			p_1	$a_1p_1 \cdots j_1p_1$	k_1p_1	l_1p_1	m_1p_1		p_1S_1
	1	a_1	b_1	$c_1 \cdots l_1$	m_1	(n_1)			S_1
s_1		s_1	As_1	$Bs_1 \cdots Ks_1$	Ls_1				s_1S
p_1			p_1	$Ap_1 \cdots Jp_1$	Kp_1	Lp_1			p_1S
	1	A	B	$C \cdots L$	(M)				S

Having completed and checked the tabulation, the preliminary values, s_1 and p_1, may be improved by adding the corrections,

$$(1) \qquad dp_1 = \frac{Mm_1 - Ln_1}{L^2 - KM}, \quad ds_1 = \frac{m_1 + Kdp_1}{-L}.$$

These corrections are best worked out roughly with a slide rule. Then, with the corrected values, $s_2 = s_1 + ds_1$, and $p_2 = p_1 + dp_1$, we may go back to the detached coefficients of the original equation and operate on these again, to obtain a set of *second reduced coefficients*, $1, a_1, b_2, c_2, \cdots$. New values for the partials need not be calculated. A second correction for the preliminary values of s and p may then be found by replacing m_1 and n_1, in the preceding equations, by m_2 and n_2. Repeating this process indefinitely, say r times, we may obtain the sum and product of the roots with any required degree of precision.

Finally, each of the reduced coefficients must be corrected by adding small quantities, which are obtained as in the following scheme:

Partials:	1	A	B	C	$\cdots$	J	K	L	M	
Reduced coefficients:	1	a_r	b_r	c_r	$\cdots$	j_r	k_r	l_r	m_r	n_r
Corrections:		ds_r	Ads_r	Bds_r	$\cdots$	Ids_r	Jds_r	Kds_r	Lds_r	Mds_r
			dp_r	Adp_r	$\cdots$	Hdp_r	Idp_r	Jdp_r	Kdp_r	Ldp_r

The reduced coefficients, thus corrected, are the coefficients of an equation of lower degree, whose roots are those of the original equation, excluding the two roots whose sum and product have just been found.

PROOF: The coefficients of the original equation are the sums of all the products that may be formed from its roots (with signs reversed); these roots being taken one at a time in forming a, two at a time in forming b, and so forth. The process outlined, leading to the reduced coefficients, consists in subtracting from these sums all the root-products that contain either one or both the two given roots as factors. It therefore leads to the coefficients of an equation having all the roots of the original equation save the two given roots.

Now if corrections, ds and dp, are added to the preliminary values of s and p, the corrections that accumulate from column to column by the process outlined may be shown to follow the law that is indicated in the second preceding paragraph. The partials, $1, A, B, C, \cdots$ are in fact the *partial derivatives* of the coefficients with respect to s and p separately. But the reduced coefficients, m_1 and n_1, in the two final columns represent errors that have accumulated in these columns because of the use of inexact values of s and p. These are to be reduced to zero, as nearly as may be, in the next operation. Accordingly,

$$m_1 + Kdp_1 + Lds_1 = 0, \qquad n_1 + Ldp_1 + Mds_1 = 0.$$

From these we may obtain the equations that have been given for calculating ds_1 and dp_1. The partials are given exactly by the process we have described

only when s and p are known exactly from the beginning. But in practice even a very rough calculation of their value will suffice, since they are employed only in finding small corrections to be applied to the coefficients.

Example: As a numerical example consider the equation

$$x^6+7x^5-53x^4-10x^3-180x^2-470x-150=0\ .$$

A rough preliminary survey, plotting a small portion of the curve in the neighborhood of the origin, will reveal four approximate roots, -11, -1.3, -0.4, $+5$. But the sum of all the real and complex roots of the equation is the negative of the second coefficient, namely, -7. Comparing this with the sum of the four known roots, we see that another pair of roots must exist, whose sum is approximately $+1$. Again, the product of the real and complex roots of the equation, with signs reversed, is the final coefficient, -150. Comparing this with the product of the four known roots, we see that the product of the missing pair of roots must be about $+5$. Accordingly, we detach coefficients and proceed as follows:

	1	7	− 53	− 10	− 180	− 470	− 150
$s_1 = 1$		1	+ 8	− 50	− 100	− 30	0
$p_1 = -5$			− 5	− 40	+ 250	+ 500	+ 150
	1	8	− 50	− 100	− 30	0	0

Since the final coefficients have both become zero, our assumed values of s_1 and p_1 were exact, hence the original equation has a pair of (complex) roots which are the roots of the quadratic, $x^2-x+5=0$. If the original equation is divided through by this quadratic, we obtain a quartic equation, whose coefficients are those appearing in the final line of the outline above and whose roots are the real roots of the original equation. Let us determine these to four decimal places.

We may begin with the pair of roots whose approximate values have been given above as -1.3 and -0.4. Their sum is $s_1=-1.7$, and their negative product, rounded, is $p_1=-0.5$.

	1	8	− 50	− 100	− 30		*Sums*
$s_1 = -1.7$		− 1.7	− 10.71	+ 104.057	− 1.542		90.105
$p_1 = -0.5$			− 0.5	− 3.15	+ 30.605	− 0.453	26.502
	1	6.3	− 61.21	$m_1 = 0.907$	$n_1 = -0.937$		− 53.003
$s_1 = -1.7$		− 1.7	− 7.82	+ 118.20			+ 108.68
$p_1 = -0.5$			− 0.5	− 2.3	+ 34.765		+ 31.965
	1	$K = 4.6$	$L = -69.53$	$(M = 116.8)$			

$Mm_1 = 105.93$ $\quad Ln_1 = 65.15$ $\quad L^2 - KM = 4300$ $\quad$ (*approximately*).

Thus, from equations (1),

$$dp_1 = +0.0095, \quad ds_1 = +0.014.$$

With the corrected values, $s_2 = s_1 + ds_1 = -1.686$ and $p_2 = p_1 + dp_1 = -0.4905$, we repeat the first part of the preceding work. This gives,

$$m_2 = +0.0220. \qquad n_2 = -0.0499$$
$$dp_2 = -0.00021 \qquad ds_2 = +0.0003$$
$$p_3 = p_2 + dp_2 = -0.4907 \qquad s_3 = s_2 + ds_2 = -1.6857.$$

The two roots sought are accordingly those of the quadratic

$$x^2 + 1.6857x + 0.4907 = 0,$$

namely -1.3115 and -0.3742.

The reduced coefficients obtained in the last operation are corrected as follows:

Partials:	1	4.6	− 6 .53	+ 116.8	− 0.04
Uncorrected coefficients:	1	6.314	− 61.1359	+ 0.0220	− 0.0499
Corrections: $ds_2 = 0.0003$		+ 0.0003	+ 0.0014	− 0.0209	+ 0.0350
$dp_2 = -0.00021$			− 0.0002	− 0.0009	+ 0.0146
Corrected coefficients:	1	6.3143	− 61.1347	+ 0.0002	− 0.0003

A check is afforded by the fact that the two final coefficients now practically vanish. The remaining roots are evidently those of the quadratic

$$x^2 + 6.3143x - 61.1347 = 0,$$

namely —11.5894 and +5.2750.

RECENT PUBLICATIONS

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REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Algebraic Arithmetic. By ERIC T. BELL. American Mathematical Society, Colloquium Publications, Volume VII. New York, 1927. iv+180 pages.

Between the classic arithmetic on the one hand, as developed by the school of Gauss, and the modern analytic theory of numbers on the other hand, in

which limiting operations usually appear both in the arguments and in the conclusions, lies an extensive intermediate region of the theory of numbers "where the methods of algebra and analysis are freely used to yield relations between integers expressed wholly in finite terms and without reference, in the final propositions, to the operations or concepts of limiting processes." To this part of the theory of numbers Bell has given the name *algebraic arithmetic*. It is not desirable that its boundaries should be sharply defined. In it may be placed many results in the theory of numbers which do not belong specifically either to the classical theory or to the modern analytic theory. During recent times, and especially in the hands of Bell himself, this part of the theory of numbers has had a rapid development.

It is the purpose of this book "to outline a few promising directions in which progress may be made toward classifying, extending, and generalizing the methods and results of algebraic arithmetic. The insistence will be upon general methods rather than specific applications, as the latter are so numerous, and so readily made from the general formulations, that it will be sufficient merely to indicate occasionally a few of them to lend concreteness to the abstract theories. What is given here is but a narrow cross section of a very extensive field." As is indicated by this quotation from the first page, this book is devoted almost entirely to a general and abstract formulation of the problems and the methods of this relatively new division of the theory of numbers.

The character of the work may be partially indicated very simply by means of a special but a typical case of one of the principal concepts on which the methods are based, namely, the concept of paraphrases. Suppose that we have obtained in some way, say from the theory of elliptic or theta functions, a linear identity among the sines of the angles a_ix+b_iy for the set 1, 2, 3, ... of values of i. Let $f(a, b)$ be an arbitrary odd function of its two arguments. Then Bell shows that we have a like linear identity among the $f(a_i, b_i)$. Since the function f is at our disposal we may obtain as many arithmetical facts as we please by specializing this function in various ways. From similar identities among cosines other arithmetical facts may be similarly obtained by means of even functions. These results are extended to functions of various types of mixed parity. All this is but one particular type of application of a rich and extended theory set forth here for the first time in its general and abstract form.

The book contains such a wealth of material and in such an abstract form that one can come to a full appreciation of its scope only by an extended study, supplemented by the development of many of its numerous applications. It is safe to predict that there will be a large development of algebraic arithmetic taking its rise from this book and from the researches of Bell on which it is based.

R. D. CARMICHAEL

Mathematical Preparation for Physical Chemistry. By FARRINGTON DANIELS. McGraw-Hill Book Co., 1928. x+308 pages.

This surprisingly interesting book was written by a professional chemist and it is extremely gratifying to the mathematician to find an American chemist who knows so much mathematics.

As a source of problems the teacher or the writer of undergraduate mathematics text-books will find the book of value. Problems from chemistry afford an interesting variation from those more usually given. The reviewer was particularly impressed with the common sense approach to many topics and with the discussion of various simple things frequently omitted from ordinary courses. The book is decidedly an addition to text-book literature. The author says that it was based on a one semester course for sophomores, but he is not proposing to substitute it for the usual courses. In fact it is hard to believe that the necessary time should be taken in the sophomore year for such a course. It seems that it could be given later as a review to students of chemistry who had had the usual courses but who wished to freshen their memories on their mathematics and to study its applications within their special field. Such a book should also be of service to the mature chemist who wishes to study mathematics without a teacher in order to read modern literature on physical chemistry.

The contents is partly indicated by the chapter headings: Large and Small Numbers; Logarithms; The Slide Rule; Graphical Representation of Equations; Graphs of Equations of the Second Degree; Graphs of Logarithmic and Trigonometric Functions; Differential Calculus; Differentiation; Graphs and Calculus; The Differential; Integral Calculus; The Significance of "e"; Differentiation and Integration of Trigonometrical Functions; Integration; The Use of Integration Tables; Geometrical Applications of Integral Calculus; Partial Differentiation; Differential Equations; Infinite Series; Probability; Graphical Methods in Physical Chemistry. There is a variety of tables. Such a book is clearly mathematics and not chemistry. It is in the problems and in some discussions that chemistry particularly enters.

And now to the task of calling attention to things about which a mathematician can not be enthusiastic. These consist primarily, although not wholly, in the use of some terms that are in bad repute with mathematicians and in an undue laxness in the treatment of limits. At times the reviewer is in doubt if the author himself really always thinks of his calculus as a study of certain limits or if in his thinking he is retaining remnants of the abandoned "little zero" point of view. As we all know there is a very real difficulty in being rigorous and at the same time understandable to a sophomore. In fact it is likely that no one of us objects to some frank and admitted lapses from rigor in undergraduate teaching. In the opinion of the reviewer, however, the author goes too far in this direction, particularly when discussing the foundations of

calculus. Extensive quotations might be given; but because of limitations of space reference is made to the book itself and there to but four places: viz., p. 86; first three lines of p. 78; first paragraph of p. 114; lines 7 to 9, p. 115. The reviewer feels that various definitions are too broad, not conforming to common usage, as for example that of calculus (p. 70), and that some are inadequate, as that of a determinant (p. 260). Some terms with apparent uses to which the reviewer takes exception are: "finite numbers" to distinguish certain numbers from differentials; "quantities" as meaning *numbers*; "limiting value" as meaning *limit*; "becomes" sometimes as meaning *is* and sometimes as meaning *approaches*; "equation" as meaning *function*; "drops out" as meaning *approaches zero*; "order of an equation" as meaning *degree of an equation*; "series" as meaning *sequence*; "approaches infinity"; and a quite too frequent use of "evident."

Not many typographical errors were noted.

TOMLINSON FORT

Freshman Mathematics. By G. W. MULLINS and D. E. SMITH. Boston, Ginn and Company, 1927. vi+386 pages.

The purpose of this text is to present a fairly comprehensive survey of the mathematics of first year college work through an approach by way of "the common uses of the science." It is in this sense then an introduction to some of the topics of college algebra, trigonometry, analytic geometry, and calculus which are most immediately applicable to practical affairs. It is designed as a one year course for students who have not necessarily had more than an "elementary course in algebra such as is offered in every high school, a familiarity with the important basal propositions of plane geometry, and a knowledge of the simple rules or formulas of mensuration commonly given in arithmetic or in intuitive geometry." Consequently it does not wholly meet the needs of one who has already received considerable training in the formal developments of algebra, trigonometry, and geometry except by a selection of topics and supplementary material.

In place of the "conventional formal review of algebra" the book has an opening chapter on applications of elementary algebra through a study of the formula, the equation, graphs, and exponents. The binomial theorem, progressions, and logarithms follow in two more chapters. The next chapter, which is on trigonometry, is excellent as far as it goes, beginning with a study of general angles and introducing the trigonometric ratios as functions of the general angle. With practical applications in view, everything leads up to the solution of triangles. Short four place tables are given at the end of the text, but it seems advisable to use a more extended separate set of five place tables with proportional parts in order to develop speed of computation. Analytic trigonometry must be slighted next with one page of discussion comprising very

scant mention of trigonometric identities and equations. All the rich field of fundamental identities, addition formulas, multiple and half angle formulas and inverse functions receive no treatment.

A chapter on analytic geometry with eight sub-headings follows. The increment notation of the calculus is early introduced; the conics and a few miscellaneous loci are studied. A great many things must of course be omitted, such as transformation of coördinates, equations of curves from empirical data, and polar coördinates, which surely must have been overlooked. Pages 239–316 take up the calculus applied to algebraic functions only. Application of the calculus is made to rates, and to maximum and minimum problems. Integration as the inverse of differentiation is considered and applied to finding the area under a curve by use of "rate of change of area." A short chapter on numerical equations, giving approximate graphic solution and a good discussion of Newton's method completes the more usual work. The last chapter is made up of a few basic formulas in practical mensuration in the plane and space, followed by lists of exercises involving the use of these formulas.

Emphasis throughout the text is placed upon practical problems, with exercises in which the results are not simple integers, and with equations and formulas with decimal coefficients. Historical notes, which were found to stimulate interest, are given at appropriate places. The explanation and discussion is quite detailed and clear, making it easily understood by a student who has had little rigorous mathematical training.

The reviewer has had experience with several texts in so-called "unified" or "combined" courses in which the texts were variously designed, and he feels that this text has very well met the purposes for which it was intended. It seems an admirable text for use in the agricultural, biological, and liberal courses for students who want a little working knowledge of elementary mathematics including the calculus but who do not intend to go much farther. The authors of *Freshman Mathematics* certainly have accomplished this aim, for the text has proved a success in courses such as those mentioned above.

The book is well printed with clear type and well drawn figures. No answers are given; thus the student is compelled to depend upon his own accuracy. More emphasis might have been placed upon checks and check formulas; but this might well be left for the instructor to develop as he must have some excuse for his presence at the class meeting.

CLYDE M. HUBER

Exercices de Géométrie Moderne. Vol. I, Géométrie dirigée, 132 pp., 13 francs. Vol. II, Transversales, 81 pp., 9 fr. Vol. III, Division et faisceau harmoniques, 86 pp., 9 fr. Vol. IV, Pôles, polaires, plans polaires, dans le cercle et la sphère, 125 pp., 13 fr. Vol. V, Rapport anharmonique, 101 pp., 11 fr. Vol.

VI, Inversion, 134 pp., 14 fr. Vol. VII, Homographie, 128 pp., 14 fr. Vol. VIII, Involution, 93 pp., 11 fr. Vol. IX, Géométrie projective. Application aux coniques, 172 pp., 15 fr. In-8. By G. PAPELIER. Paris, Vuibert, 1925–1927.

Each of these little volumes is written according to the same plan. The author establishes first the fundamental theorems relating to the subject at hand, and then proceeds to apply them to the solution of various problems. As one might expect, the line of demarcation between theory and application is not always very definite. The theory of orthogonal circles, for instance, and the general theory of coaxal circles, for that matter, are incorporated in the applications of Vol. III. Then again the choice of the problems solved as applications is of necessity a matter of personal preference, perhaps to some extent even a matter of mere chance. The author's selections are as good as any can be, and most of his examples have also a good deal of intrinsic interest.

The treatment is almost exclusively synthetic and presupposes, on the part of the reader, nothing beyond the rudiments of analytic geometry. The proofs and solutions are presented fully and explicitly, but without undue length. The book will interest any college student with a taste for geometry and will be stimulating to anyone engaged in teaching secondary mathematics. It is regrettable that the reader is offered no opportunity to exercise his own sagacity, as there are no unsolved problems in any of these volumes. Long lists of such exercises can be collected with little trouble. And while expressing wishes, may we be allowed to say that it would be desirable to have a similar series of volumes dealing with the geometry of the triangle, and, perhaps, the tetrahedron as well.

NATHAN ALTSHILLER-COURT

The Mathematics of Engineering. By RALPH E. ROOT. The Williams and Wilkins Company. xiii+540 pages. $7.50.

On opening this book the reader is struck with the clear but unconventional type used by the printers. As he reads the book he is struck with the clear but unconventional proofs used by the author. It is amazing how much mathematics is dealt with in this book (calculus, analytics, parts of algebra, differential equations, some complex functions, empirical data, etc.). The book is almost a cours d'analyse. After reading in the preface and the table of contents about the amount of mathematics that is to be discussed in the following pages, you find yourself in the midst of a leisurely and surprisingly full treatment of functions, of limits and of continuity, and you wonder how at this rate the author will ever cover the ground. However, you find that the discussions later on in the book are not so complete; also that the author (during the thirteen years in which he has gradually accumulated the material of the text) has re-

vised his proofs until many of them are really marvels of brevity, ingenuity, and originality. The book is very stimulating and refreshing, the various subjects are handled vigorously and fairly rigorously. The whole text uses mechanical, geometrical, and dynamical ideas that ought to appeal to engineers. It is intended as a second course in analytics and the calculus that will lead up to more advanced topics in mathematics. It seems as though engineers would welcome such a book in their libraries for frequent reading and reference. In spite of all the topics that the careful and complete index shows that the book touches on, the text does not seem cluttered up or crowded, mostly because of the completeness of some of the sections.

Before going on with the good points of the book, we notice that it has entirely too many typographical errors, some of them very serious ones. Also in some places where fullness of discussion would seem to be desirable the author is too brief, almost descending to a handbook-like statement of methods. Thus he is unsatisfactory when he treats of areas on page 222; and his interpretation of successive integration on page 244 does not clarify the hazy ideas gained in an elementary course in the calculus. It seems as though here the engineer should be shown why he can take the different kinds of elements of area, volume, etc. Also would it not be better to *define* volume as the limit of a sum of infinitesimal volumes, moment of inertia as the limit of a sum of infinitesimal moments of inertia, etc., instead of saying that the volume of a surface *can be obtained* as the limit of a sum of volumes and the moment of inertia *can be obtained* as the limit of a sum of moments. On page 82 he speaks of the integral as the limit of a sum of *differentials*, whereas using the ordinary definitions of differential and of infinitesimal we should call the integral the limit of a sum of *infinitesimals*. He does not justify the method of taking the derivatives of implicit functions. On page 371 he speaks of the area of a surface and shows how to obtain it by integration, whereas logically we define the area of a surface as the limit of a sum of infinitesimals and show how this leads up to the integral. On page 170 and 185 the author is not clear as to what he means by "being at infinity." When discussing Taylor's expansion, it would have been well to show under what circumstances one can expand a function in such a series. The book might contain a better discussion of change of variable and of limits in integration, giving the logic of the situation, the conditions that must be satisfied, etc. No mention is made of the discriminant of a conic, although otherwise the testing and reducing of nondegenerate conics is well done.

The fact that the book has so very many good points makes one notice with surprise the occasions chronicled above where "Homer nods." It would be difficult even to catalogue the interesting points in the text. When discussing the graphs of curves the author points out carefully that in the applications of such functions as $y = \cos x$ the variable x very seldom represents an angle, but

may be time, temperature, length, etc.; also he notes that only certain parts of curves are useful in applications. How often do professors of engineering complain that their students do not learn the above facts in mathematics! On page 32 he gives such problems as "suggest several functions $f(x)$ for which $\lim_{x\to 0+} f(x) = +\infty$." Similarly in Ex. 3 on page 27 we find "show that $\lim_{x\to 0} \sin x = 0$. (Show the way in which the choice of δ depends on the given ϵ)." Would that more such searching examples were given in text books, where the student has to invent illustrations of theorems or has to make up problems and then solve them. On page 43 we find a definition of the derivative that must delight the heart of an engineer, namely "The derivative of a function of x with respect to x is the instantaneous rate of change of the value of the function measured in terms of the rate of change of x at the same instant."

The figures throughout the book are excellent and are very cleverly used in the proofs of formulas. For example on page 54 $(d/dx) \sin x = \cos x$ is proved very deftly from a figure, also for $(d/dx)e^x = e^x$ a figure is used (on page 59). Moreover $(d/dx)e^x$ is discussed before $(d/dx) \log x$. This is certainly an innovation. The proof depends on considering all the curves $y = \alpha^x$ with α having different values, then choosing the value of α for which the corresponding curve has unit slope for $x = 0$. The only criticism one might make is that it is somewhat of an assumption to say that there exists such a curve $y = \alpha^x$ with unit slope for $x = 0$. On page 164 the formulas for rotation of axes follow beautifully from a figure. Good graphs of curves are drawn for the discussion of derivatives, of maxima and minima, of points of inflection, of integrals (areas, etc.). But the most astonishing of all is the way he treats conics on page 169, defining them from a figure as sections of a cone, using this to get their definition in terms of foci and directrices, and finally obtaining the simpler forms for their equations.

The scope of the book is well brought out by noting that on page 98 we find a clear discussion of determinants (later on comes Sylvester's "dialytic method" for the elimination of a variable between two equations). Families of curves are treated; asymptotes of hyperbolas are neatly discussed on page 176; the discriminant of a cubic equation in one variable is derived; on page 261 is a fine discussion of indeterminant forms; on page 268 Taylor's theorem is derived by integration; on page 282 we find a brief proof of the continuity of $\sum u_i(x)$ by means of a "majoring series;" on page 291 the function $e^{i\theta}$ is introduced; on page 298 we are set to work mapping complex functions; on page 330 there is a short treatment of lines in space; etc. The problems are very well chosen so as to bring out applications of the theory, to drive home the topics treated, and to lead on to more advanced subjects. For example, on page 210 we find the problem of the best angle at which to elevate a gun in order to get the greatest possible range; while under derivatives we find problems that are really introductory to differential equations. Moreover the variables used throughout the book are such

as fit the problems, such as t for time, s for space. In fact there is nothing pedantic anywhere in the text. At the end of many chapters are lists of references for further study.

Most of the definitions are very good, for instance that of the length of arc (on page 217). The author has such a thorough and virile mastery of his subject that he is able to choose (or invent) the briefest possible discussions, and those best adapted to his purposes. Thus on page 211 he turns the usual procedure right around and first derives the equation of the osculating circle of a curve at a point; then he gets its radius and center. Mathematicians very often fall into the rut of sticking to so-called "classical" definitions and methods of discussion, sometimes one writer of a textbook slavishly copying an earlier one. To read such a unique, hearty book as this one is as refreshing as a draught of cold water to a thirsty traveler.

ALAN D. CAMPBELL

Elementary Solid Geometry. By V. B. NAIK AND S. B. BONDALE. Aryabhushan Press, Poona, India, 1926. vi+220 pages.

This is the work by two professors of mathematics in India, one in Fergusson College and the other in Willingdon College. It covers the course prescribed by the Bombay University for intermediate examinations, and adds a number of other topics that are not required for this purpose. These include the mensuration and certain properties of the sphere, rectangular coordinates, perspective, direction cosines, the tetrahedron and its inscribed and circumscribed spheres, Euler's theorem, regular solids, and the frustum of a cone. It contains certain features that will appeal to teachers of the subject in the New World, as in the case of its simple introduction, based upon commonplace experiments which lead to a feature wholly lacking in Euclid and not sufficiently in evidence in our progressive texts of today; namely, a satisfactory working set of postulates. These assumptions are classified as (1) General Postulates, (2) Straight-Line Postulates, (3) Plane-Surface Postulates, (4) Parallel-Lines Postulates, and (5) a Two-Planes-in-Space Postulate. Interspersed among these assumptions are the definitions needed in the course.

In the preliminary work the authors have shown considerable ingenuity in selecting the simple experiments mentioned. This is seen in such features as the tests to be applied to a surface to determine if it is a plane and to a stretched thread to see if it is approximately straight; to trying to fold and crease a piece of paper on uneven ground; and to explaining why the shadow of the top of a vertical wall is parallel to the foot of the wall.

The proofs are less formally arranged on the page than in American texts, following rather the method of most editions of Euclid. The basic propositions are of course much like those in this country, but they are fewer in number,

the pupil's interests being directed to numerous deductions from a few standard theorems rather than to a few deductions from a larger number. As already stated, the authors extend the work to include several limited explorations of territory not investigated in our textbooks. For this reason and because of the originality shown in many of the deductions, the book is well worth reading.

It can not be expected that typographically or graphically the work should measure up to what is done in Europe or America, but it is a praiseworthy attempt to give, under unfavorable circumstances, not only a clear page but a set of figures that elucidate adequately the text.

American teachers of the present time would object to such a massing of definitions and postulates as is here found, although this occurs after the pupil has been prepared to understand them, so that the objection is not serious. Probably the chief objection would, however, be that parts of the work are too difficult for the American pupil, although required of the schoolboy or in the early college years on the Indus and the Ganges. If this be the case, we should not be particularly proud of our criticism. High-school teachers or instructors in freshman classes who are interested in the subject might try their pupils upon a few such theorems and exercises as these:

The normal to a surface of revolution at any point in it intersects or is parallel to the axis.

Two surfaces which touch one another at a point have a common normal at that point.

A sphere rests on a hole of the shape of an equilateral triangle of side 5 in. If the radius of the sphere is 3 in., what is the maximum height of the sphere above the plane of the hole?

In a regular tetrahedron in which each dihedral angle is β,

$$\beta = \sin^{-1}(2\sqrt{2})/3 = \cos^{-1}(1/3) = \tan^{-1} 2\sqrt{2}.$$

If OP makes angles x, y, z with the axes of coordinates, $\cos^2 x+\cos^2 y+\cos^2 z=1$.

If two figures in different planes are in perspective, corresponding straight lines either meet in collinear points or are parallel.

Of course these statements are mathematically of no difficulty, but our high-school pupils would hardly care to encounter them on a college entrance examination for honors in mathematics, and our college students would be surprised if they met with them in their courses. Perhaps we can take comfort in thinking that a Hindu boy would not know what to do with an American multiple-response test; but possibly this thought would have the opposite effect, though we would not confess it.

DAVID EUGENE SMITH

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3332. Proposed by R. E. Gaines, University of Richmond.

In a given ellipse the two conjugate diameters are drawn which are equally inclined to the major axis, and a similar variable ellipse touches these two diameters and has its major axis on the same line as that of the fixed ellipse. Find the distance between their centers when the area common to the two ellipses is a maximum.

3333. Proposed by W. H. Roever, Washington University.

Consider the system of two partial differential equations

$$X_i(f) \equiv a_{1i}\frac{\partial f}{\partial x_1} + a_{2i}\frac{\partial f}{\partial x_2} + a_{3i}\frac{\partial f}{\partial x_3} = 0, \quad i = 1, 2,$$

in which the coefficients a_{ji} are functions of x_1, x_2, x_3. From one point of view, a necessary and sufficient condition that these equations have a common integral surface is

$$\Theta \equiv \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ X_1(a_{12}) - X_2(a_{11}) & X_1(a_{22}) - X_2(a_{21}) & X_1(a_{32}) - X_2(a_{31}) \end{vmatrix} \equiv 0.$$

and from another point of view, such a condition is

$$\Omega \equiv U\left(\frac{\partial V}{\partial x_3} - \frac{\partial W}{\partial x_2}\right) + V\left(\frac{\partial W}{\partial x_1} - \frac{\partial U}{\partial x_3}\right) + W\left(\frac{\partial U}{\partial x_2} - \frac{\partial V}{\partial x_1}\right) \equiv 0,$$

in which

$$U = \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}, \quad V = \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix}, \quad W = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}.$$

Show that $\Theta \equiv \Omega$.

3334. Proposed by James Singer, Graduate College, Princeton, N. J.

Evaluate

$$\lim_{n\to\infty}\left[\frac{1}{2^n}\left(\sum \frac{e_1}{2^{2\cdot 1-e_1}} + \frac{e_2}{2^{2\cdot 2-(e_1+e_2)}} + \cdots + \frac{e_n}{2^{2n-(e_1+e_2+\cdots+e_n)}}\right)\right],$$

where $e_i = 0$ or 1, and the sum is extended over all possible choices of the e's; i.e., the sum of the n terms where all the e's except one are zero, plus the $n(n-1)/2$ terms where all the e's except two are zero, $\cdots$, plus the single term where all the e's are unity.

3335. Proposed by Paul Wernicke, Washington, D. C.

Assume, on the sides of a triangle ABC, the points L on BC, M on CA, N on AB. Find the condition for the existence of a real straight line dividing the three joins AL, BM, CN in the same ratio, and, in particular, bisecting them.

3336. Proposed by Otto J. Ramler, The Catholic University of America.

Construct a triangle having given $a-b$, h_b+h_c, and the angle A, where a and b are sides, and h_b, h_c are the altitudes upon the sides b, c, respectively. (Altshiller-Court's College Geometry, p. 28, no. 7).

3337. Proposed by E. B. Escott, Oak Park, Illinois.

Sum the series,

$$\frac{6^3}{2.3.4.5}x^5 - \frac{8^3}{4.5.6.7}x^7 + \frac{10^3}{6.7.8.9}x^9 - \cdots$$

3338. Proposed by N. A. Court, University of Oklahoma.

The two pairs of extremities of two harmonic segments determine an involution in which the midpoints of the segments are two conjugate points.

3339. Proposed by Paul Capron, U. S. Naval Academy.

Discuss the nodes of the parabolic spiral $(\rho-a)^2=ca\theta$.

3340. Proposed by V. M. Spunar, Chicago, Illinois.

Solve

$$\begin{vmatrix} x^2dx, & y^2dy, & z^2dz \\ dx, & dy, & dz \\ x, & y, & z \end{vmatrix} = 0$$

consistent with the equation $xyz=1$.

SOLUTIONS

238 [1916, 19]. Proposed by C. N. Mills, Brookings, South Dakota.

Determine the rational values of x that will render x^3+px^2+qx+r a perfect cube. Apply the result to $x^3-8x^2+12x-6$.

Partial Solution by R. E. Moritz, University of Washington

Since the given expression may be written $(x+p/3)^3+(q-p^2/3)x+(r-p^3/27)$, a rational value of x which renders it a perfect cube is given by $x=(27r-p^3)/9(p^2-3q)$, where it is assumed that p, q, r are rational numbers. In the given numerical example $x=25/18$ gives for the expression the value $(-23/18)^3$. This is not the only solution, for $x=1$, -1 give the values $(-1)^3$, $(-3)^3$.

470 [1915, 228; 1919, 414]. Proposed by R. E. Moritz, University of Washington.

Prove that

$$\theta = (\lambda + (q/p)\mu)\pi, \quad (\lambda = 1, 2, 3, \cdots, q-1;\ \mu = 0, 1, 2, \cdots, p-1),$$

and

$$\theta = (2\lambda - 1)\pi/2 + (q/p)(4\mu \pm 1)\pi/2, \qquad (\lambda = 1, 2, 3, \cdots, (q-1)/2; \quad \mu = 0, 1, \cdots, p-1),$$

determine the same set of points on the curve $\rho=a\cos(p/q)\theta$, where p and q are two odd integers without a common factor, and a is any constant.

Solution by Otto Dunkel, Washington University

For the given equation two points corresponding to θ_1 and θ_2 will coincide in two cases: if $\theta_2-\theta_1$ is an odd multiple of π and $\rho_2=-\rho_1$; if $\theta_2-\theta_1$ is an even multiple of π and $\rho_2=\rho_1$. Designate by (P) the set of points given by the second formula for the angle. We may write this formula as

$$(P) \qquad \theta' = \left[(2\lambda' - 1) + \frac{q}{p}(2\mu' - 1)\right]\frac{\pi}{2}, \qquad \begin{matrix}\lambda' = 1\ 2, \cdots, (q-1)/2, \\ \mu' = 0, 1, \cdots, (2p-1).\end{matrix}$$

If in this formula we replace μ' by $\mu'-p$, the angle θ' is replaced by $\theta'-\pi q$. Also $\cos(p/q)(\theta'-\pi q) = -\cos(p/q)(\theta')$. Since p and q are odd the two points coincide. Hence the values $\mu'=p$, $(p+1), \cdots, (2p-1)$ give the same points as the remaining values $\mu'=0, 1, \cdots, (p-1)$. We shall consider only the points of this set which we shall denote by $(P)'$.

Now make the transformation $\lambda'=\lambda''-(q-1)/2$, $\mu'=\mu''+(p+1)/2$, and we may then write

$$(P)' \qquad \theta' = \left(\lambda'' + \frac{q}{p}\mu''\right)\pi, \qquad \begin{aligned} \lambda'' &= (q+1)/2,\ (q+3)/2, \cdots, (q-1), \\ \mu'' &= -(p+1)/2, -(p-1)/2, \cdots, -1, 0, 1, \cdots, (p-3)/2. \end{aligned}$$

In this last formula λ'' and $q-\lambda''$ give the two angles θ_1' and θ_2' such that $\theta_2'-\theta_1'=(q-2\lambda'')\pi$ and $(p/q)\theta_2'+(p/q)\theta_1'=(2\mu''+p)\pi$. Hence these values give the same point and we may replace λ'' by $\lambda=q-\lambda''=(q-1)/2$, $(q-3)/2, \cdots, 1$. We shall now make a change in the sequence for μ''. If we compare the angles θ_1', θ_2' given by μ'' and $p+\mu''$, we have $\theta_2'-\theta_1'=q\pi$ and $(p/q)\theta_2'-(p/q)\theta_1'=p\pi$. Hence in the sequence for μ'' we may replace the $(p+1)/2$ negative values by $(p-1)/2$, $(p+1)/2, \cdots$, $(p-1)$, and we may write

$$(P)' \qquad \theta' = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad \begin{aligned} \lambda &= 1\ 2, \cdots, (q-1)/2, \\ \mu &= 0, 1, \cdots, (p-1). \end{aligned}$$

Here again λ and $q-\lambda$ give the same points and we obtain the set $(P)'$ if we set $\lambda=(q-1)$, $(q-2), \cdots$, $(q+1)/2$. Hence we have finally

$$(P) \qquad \theta = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad \begin{aligned} \lambda &= 1, 2, \cdots\ (q-1), \\ \mu &= 0, 1, \cdots, (p-1). \end{aligned}$$

Also solved by A. J. CAVANAGH and the PROPOSER.

472 [1915, 267]. Proposed by Paul Capron, U. S. Naval Academy.

The sides of a spherical triangle are a, b, c; the corresponding opposite angles are A, B, C; p and P are the polar distances of the inscribed and circumscribed circles; $a+b+c=2s$; $A+B+C=2S$. From a geometric figure, by the formula for solving right spherical triangles, show that

$$\tan^2 p = \operatorname{cosec} s \sin(s-a) \sin(s-b) \sin(s-c); \tag{1}$$

$$\cot^2 P = -\sec S \cos(S-A) \cos(S-B) \cos(S-C). \tag{2}$$

Thus establish the usual formulas for the tangent of the half-sides and half-angles.

Also show that

$$\frac{\text{sine of angle}}{\text{sine of the opposite side}} = \frac{\cot P \cos S}{\tan p \sin s}. \tag{3}$$

SOLUTION BY OTTO DUNKEL, Washington University

The formula (1), the law of tangents for angles, and the formula for the tangent of the half-angle are established in the desired manner in Kenyon and Ingold's *Trigonometry*, pp. 125–127.[1] The formula (2) and the corresponding other formulas may be obtained by the use of the polar triangle with the aid of the theorem that the pole I' of the circumscribed circle for the triangle ABC is the pole of the inscribed circle for the polar triangle $A'B'C'$, and the pole I of the inscribed circle for ABC is the pole of the circumscribed circle for $A'B'C'$. This is easily proved geometrically. Formula (2) will be deduced in the desired manner, and from the result and (1) the formula (3) will be obtained. Suppose first that I' lies within the triangle ABC; then from the three isosceles triangles with the common vertex I', $A=\beta+\gamma$, $B=\gamma+\alpha$. $C=\alpha+\beta$, $S=\alpha+\beta+\gamma$. Hence $\alpha=S-A$, $\beta=S-B$, $\gamma=S-C$. Then from two of the isosceles triangles we have

$$\cos(S-A) = \tan(a/2) \cot P, \qquad \cos(S-B) = \tan(b/2) \cot P,$$

$$\frac{\cos(S-A)}{\cos(S-B)} = \frac{\tan(a/2)}{\tan(b/2)}. \tag{1}$$

If I' lies outside and on the side of BC opposite to A we have merely to replace α by $-\alpha$. The formulas (1) are then true in both cases. Now consider the triangle $\bar{A}\bar{B}\bar{C}$ which with ABC makes up the lune of

[1] The same process, essentially, is given by E. von Szücs, *Ebene und sphärische Trigonometrie auf ganz neuer Grundlage*, Zeitschrift für Mathematik und Naturwissenschaftlichen Unterricht, vol. 42 (1911), pp. 529–533.

angle A. For this triangle $\overline{A}=A$, $\overline{B}=180°-B$, $\overline{C}=180°-C$, $\bar{a}=a$, $\bar{b}=180°-b$, $\overline{S}=180°-S+A$, $\overline{S}-\overline{A}=180°-S$, $\overline{S}-\overline{B}=S-C$. Hence the last formula in (1) applied to this triangle gives

$$\frac{\cos S}{\cos (S-C)} = -\frac{\tan (a/2)}{\cot (b/2)}; \tag{2}$$

and from (1) and (2) we have at once

$$\tan^2 (a/2) = -\frac{\cos S \cos (S-A)}{\cos (S-B)\cos (S-C)}, \quad \cot^2 P = -\sec S \cos (S-A)\cos (S-B)\cos (S-C). \tag{3}$$

From the first formula above we obtain by two trigonometric reductions

$$\sec^2 (a/2) = \sin B \sin C/[\cos (S-B)\cos (S-C)].$$

Then, since $\sin^2 a = 4\tan^2(a/2)/\sec^4(a/2)$, we obtain

$$\sin^2 a = 4\cos^2 S \cot^2 P/\sin^2 B \sin^2 C.$$

Since $90°<S<270°$, we have

$$\frac{\sin a}{\sin A} = -\frac{2\cos S \cot P}{\sin A \sin B \sin C}.$$

This proves the law of sines. The corresponding formula is

$$\frac{\sin A}{\sin a} = \frac{2\sin s \tan p}{\sin a \sin b \sin c},$$

and the two combined give the result, after using the law of sines,

$$\frac{\sin A}{\sin a} = -\frac{\cos S \cot P}{\sin s \tan p}.$$

2686 [1918, 118]. Proposed by Edwin R. Smith, Iowa State College.

Given the difference equation,

$$u(x+1) - u(x) = \log\left(1 - \frac{x}{sp}\right) - \log\left(1 + \frac{x+1}{sq}\right),$$

where s is a positive integer, p and q are proper fractions such that $p+q=1$, and x is small compared to sp and sq. Determine $T(x)$, if $u(x)=\log T(x)$.

Solution by B. P. Hoover, Carnegie Institute of Technology

Consider the difference equation obtained by differentiating the given equation with regard to x. We have

$$u'(x+1) - u'(x) = -1/(sp-x) - 1/(sq+x+1),$$

where we notice that if the terms of the second member are integrated between the limits 0 and x and between -1 and x, respectively, we get the terms in the right member of the original equation.

A solution of the equation $F(x+1)-F(x)=1/x$, commonly called $\psi(x)$, is the following infinite series

$$F(x) = \sum_{i=0}^{\infty}\left(\frac{1}{i+1} - \frac{1}{x+i}\right)^{1}$$

Let $f(x)$ and $\phi(x)$ be solutions of $f(x+1)-f(x)=-1/(sp-x)$ and $\phi(x+1)-\phi(x)=1/(sq+x+1)$, respectively. Replacing x by $sp-x$ in $F(x+1)-F(x)=1/x$, we get $F(sp-x+1)-F(sp-x)=1/(sp-x)$, or, $F(sp-x)-F(sp-x+1)=-1/(sp-x)$. Hence we may take $F(sp-x+1)$ as our $f(x)$ and write

$$f(x) = \sum_{i=0}^{\infty}\left\{\frac{1}{i+1} - \frac{1}{sp-x+1+i}\right\}.$$

[1] See, e.g., N. E. Nörlund's *Differenzenrechnung*, p. 102.

Similarly, we may take for our $\phi(x)$,

$$\phi(x) = \sum_{i=0}^{\infty} \left\{ \frac{1}{i+1} - \frac{1}{sq+x+1+i} \right\}.$$

These series converge uniformly with regard to x in the common interval $-sq-1+\epsilon_1 < x < sp+1-\epsilon_2$, where ϵ_1 and ϵ_2 are any positive numbers such that $-sq-1+\epsilon_1 < sp+1-\epsilon_2$. Hence, integrating term by term the series for $f(x)$ between the limits 0 and x, and the series for $\phi(x)$ between the limits -1 and x, and subtracting the latter from the former resulting series, [since we have taken $u'(x) = f(x) - \phi(x)$], we are led to the following series, which we denote by $u_1(x)$, which is valid in the same interval as before:

$$u_1(x) = \sum_{i=0}^{\infty} \left\{ -\frac{1}{i+1} + \log\left(\frac{sp-x+1+i}{sp+1+i} \cdot \frac{sq+x+1+i}{sq+i} \right) \right\}.$$

The difference, $u_1(x+1) - u_1(x)$, should differ from the right member of the first given equation by a constant. This difference is easily calculated by direct substitution and found to be $\log(p/q)$. Hence, we have only to add to $u_1(x)$ the quantity $-x \log(p/q)$ to have a solution of the first given equation. Denoting this solution by $u(x)$, we write

$$u(x) = -\left[x \log \frac{p}{q} + \sum_{i=0}^{\infty} \left\{ \frac{1}{i+1} - \log\left(\frac{sp-x+1+i}{sp+1+i} \cdot \frac{sq+x+1+i}{sq+i} \right) \right\} \right].$$

Finally, if $u(x) = \log T(x)$, we have $T(x) = \exp u(x)$, where the series given above replaces $u(x)$.

2891 [1921, 184]. Proposed by D. F. Barrow, Philomath, Ga.

Let A', A'', A''', and P denote, respectively, the vertices of a triangle and any point in its plane; and let P', P'', P''', denote the feet of the perpendiculars from P upon the sides opposite A', A'', and A'''. Now suppose each of the lines PP', PP'', and PP''' to revolve about P through an angle α; and let P_α', P_α'' and P_α''' denote the intersections of this new triad of lines with the corresponding sides of the triangle. As α varies, find the envelope of the variable circle through P_α', P_α'' and P_α'''.

Solution by Roscoe Woods, University of Iowa

There is no loss of generality if P be taken at the origin. Let $x \cos\theta_i + y \sin\theta_i = p_i$ ($i = 1, 2, 3$) be the equations of the sides of the triangle opposite the vertices A', A'' and A''' respectively. The coordinates of P' and P_α' are

$$(p_1 \cos\theta_1,\ p_1 \sin\theta_1) \quad \text{and} \quad [p_1 \cos(\theta_1 + \alpha) \sec\alpha,\ p_1 \sin(\theta_1 + \alpha) \sec\alpha],$$

respectively. By symmetry, we may write the coordinates of the points P'', P''', P_α'', and P_α'''. The equation of the variable circle through the points P_α', P_α'', and P_α''' is readily found to be of the following form when arranged so as to exhibit the parameter α,

$$\cos 2\alpha[A(x^2+y^2) + Bx - Cy] + \sin 2\alpha[Cx + By] + A(x^2+y^2) + Bx - Cy - D = 0,$$

where

$$A = \sum p_1 p_2 \sin(\theta_2 - \theta_1), \qquad B = \sum p_1 p_2 (p_2 \sin\theta_1 - p_1 \sin\theta_2),$$
$$C = \sum p_1 p_2 (p_2 \cos\theta_1 - p_1 \cos\theta_2), \quad D = 2 p_1 p_2 p_3 \sum p_3 \sin(\theta_2 - \theta_1),$$

and where the summation signs cover the cyclic advance of the subscripts 1, 2, and 3.

Let us note that $D = 0$ is the condition that the variable circle pass through the origin, P. If $A = 0$ the circle degenerates into a straight line for all α's. This line through the points P_α', P_α'', and P_α''' is known as Simson's generalised line, since the lines PP_α', PP_α'', and PP_α''' make equal angles with the sides of the given triangle. It is known that P lies on the circumcircle of the triangle $A'A''A'''$ in this case.

Using the regular method of obtaining the envelope, we find that it consists of the following parts

$$[A(x^2+y^2) + Bx - Cy]^2 + (Cx + By)^2 = 0,$$
$$2D[A(x^2+y^2) + Bx - Cy] + (Cx + By)^2 - D^2 = 0.$$

The only real point of the first is the origin and evidently this is extraneous. The second factor is a

conic which becomes a repeated line when $D=0$ and a parabola when $A=0$, this parabola being the envelope of all the generalised Simson lines associated with a given point P on the circumcircle of the triangle $A'A''A'''$.

2918 [1921, 327]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find planes which cut four given lines in four concyclic points.

Solution by Roscoe Woods, University of Iowa

In a space of three dimensions, a quadric surface may be defined as the aggregate of all points on all the transversals that can be drawn to three skew lines. See H. F. Baker, *Principles of Geometry*, Vol. 3, p. 1 ff. It is clear that each of these lines lies on the quadric. Let $l_i (i=1, 2, 3, 4)$ be the four given lines in space and choose any three of them, say $l_k (k=1, 2, 3)$ to determine a quadric Q as outlined above. The line l_4 cuts this quadric in two points, L and M, which can be real and distinct, coincident, or imaginary. It could happen that l_4 would lie entirely on the quadric.

The sections of a quadric by a system of parallel planes are conics, similar and similarly placed. Hence if one of the conics be a circle, all the sections made by the planes parallel to this one are circles. Now through each real finite point in space there pass six planes which intersect a given non-composite, non-spherical quadric in circles. Therefore, from each of the six systems of parallel planes so found there are two planes that cut the four lines $l_i (i=1, 2, 3, 4)$ in four concyclic points, namely, those determined by the points L and M. For the reality of these planes and the method of determining them, one should consult such a book as Snyder and Sisam, *Analytic Geometry of Space*, p. 96 ff. Because of the complexity of the problem, no attempt has been made to write down the solution for four general lines or to discuss any of the special cases that might arise, except to mention the fact that when l_4 lies entirely on the quadric Q all the planes of the six systems cut the four given lines in concyclic points.

Note by Otto Dunkel, Washington University

Let us assume that three of the straight lines l_1, l_2, l_3 do not lie in the same plane, and also that D_1 and D_2 are two points on l_1 and l_2, respectively, such that D_1D_2 does not cut and is not parallel to l_3. Take D_1D_2 as the x-axis of rectangular space coördinates with the origin at its mid-point, and the plane through D_1D_2 parallel to l_3 as the xz-plane. Let l_3 have the inclination γ to the xy-plane, let its trace in this plane be (a, b), and set $OD_1=d$, $OD_2=-d$. Then the circles through D_1 and D_2 and points on l_3 lie on the surface whose equation is

$$by(x^2 + y^2+z^2 - d^2) + d^2y^2 - (ay + b \cot \gamma z)^2 - b^2(y^2 + z^2) = 0.$$

Hence the line l_4 cuts this surface in one or three real points, and these points determine one or three real circles which satisfy the conditions of the problem. One or more of these circles may, however, degenerate into a straight line. Since D_1 and D_2 have each one degree of freedom it appears that there are an infinite number of constructions.

Particular constructions may be obtained by considering l_1, l_2, l_3 as three skew lines and as generators of the quadric surface containing the straight lines which touch them. If l_4 cuts this quadric in a point, the two real circular sections of the quadric through this point give two of the required circles. With the four lines we have in general four such quadrics. The quadric surface, however, is only a very special case of the surfaces containing l_1, l_2, l_3 such that all of its sections parallel to an arbitrarily given plane are circles. This more general surface is of the fourth order which degenerates into the quadric for special orientations of the given plane.

Also solved by Rufus Crane.

2977 [1922, 270]. Proposed by Florence P. Lewis, Goucher College.

A point moves in such a way that its polars with respect to two given conics intersect at right angles. Prove that the locus of this point of intersection is a rational quartic curve through the circular points and find its double points.

Solution by N. A. Court, University of Oklahoma

The point of intersection P' of the polars p_1, p_2 of the point P with respect to two given conics

(C_1), (C_2) is the conjugate of P with respect to both conics, hence P' corresponds to P in a quadratic transformation the singular points of which are the vertices of the triangle ABC which is self-polar to both (C_1) and (C_2).[1]

If the lines p_1, p_2 are perpendicular to each other, then the locus of P is, in general, a conic (C), as has been shown by the present writer some years ago. (See this MONTHLY, vol. 23, 1916, p. 255). Consequently the locus of P' is a quartic curve (Q) admitting the points A, B, C as double points.

Let i_1, i_2 be the polars of I, one of the circular points at infinity, with respect to (C_1), (C_2). The two polars of the point $H=(i_1 i_2)$ with respect to these two conics will pass through I, i.e., these two lines are parallel isotropic lines; but two such lines are also perpendicular to each other; hence H belongs to the conic (C),and therefore the conjugate I of H with respect to (C_1), (C_2) lies on (Q). Similarly for the other cyclical point I'. Thus (Q) passes through the circular points of the plane.

Also solved by J. H. NEELLEY.

3246 [1927, 156]. Proposed by Daniel Kreth, Wellman, Iowa.

From a point within a square, lines drawn to three of the vertices are 30, 40, and 50. Required the side of the square. A geometrical solution is desired.

SOLUTION BY CHARLES B. BUERGER, Pittsburg, Pennsylvania

The distances to three consecutive vertices are taken in the order $a=40$, $b=30$, $c=50$. Lay off any convenient length OE and the equal length OF at right angles to OE. Let G and G' divide OE internally and externally in the same ratio $OG/GE=b/a$. Then the circle with the diameter $G'G$ is the locus of all points P such that $OP/PE=b/a$. In a similar manner let H and H' divide OF internally and externally in the same ratio $OH/HF=b/c$, and construct the circle with the diameter $H'H$. Let J and M be the points of intersection of the two circles. Then $JE:JO:JF=a:b:c$. Lay off on OJ the length $OK=b$, and draw KL parallel to JE cutting OE or its prolongation in L. Then OL is a side of the required square. A second solution is given by a similar construction applied to M. In the case of the numerical example one square will contain the point and the other will not.

NOTE BY OTTO DUNKEL, Washington University

An analytical solution of this problem has already been printed 1928, 94. A geometric construction is easily read off from that solution. If the order is a, b, c as above (in the previous solution the order of the letters is b, a, c), then a right triangle $Q'BQ$ is constructed with the right angle at B so that $Q'B=QB=b$, and on the hypothenuse $Q'Q=2^{1/2}b$ the triangle $Q'AQ$ is constructed with the side $AQ=a$, $Q'A=c$. Then $BA=s$ is a side of the required square. There are in general two constructions for s. We have merely to rotate about B the triangle $BQ'A$ through 90° so that Q' falls on Q, and then A falls at a vertex C of the square, etc. As pointed out in that solution, if Q is inside the completed square, then s, its side, is the longer of the two lengths determined by the construction. It is easily seen that there are real constructions when the triangle QAQ' exists, that is if $0 \leqq a-c \leqq 2^{1/2}b \leqq a+c$, since we can assume without loss of generality that $a \geqq c$. But it does not follow that the larger square will always contain its point. Moreover there may be more than two solutions if we are allowed to take the lengths a, b, c in any order. In the numerical problem, the order may be taken as 50, 30, 40 or as 50, 40, 30. In each case there is a pair of solutions one of which gives a square with the point inside the square. But no real solution is given by the order 40, 50, 30. The analytical determination of s is simpler without the use of the complicated formula for the area of a quadrilateral which was used in the first solution. For, if we use two sides of the square as coördinate axes, the coördinates of the required point (x, y) are given by $2xs=s^2+b^2-a^2$, $2ys=s^2+b^2-c^2$, $x^2+y^2=b^2$. It then follows that $2s^2=a^2+c^2 \pm 4S$, where S is the area of the triangle with the sides a, c, $2^{1/2}b$. A necessary and sufficient condition that the point lies in the square is that s^2 be greater than or equal to the larger of the absolute values of b^2-c^2 and b^2-a^2, and that a, b, c satisfy the conditions for a real triangle as above. The numerical example $a=37$, $b=16$, $c=16$ gives a case for which neither square contains its point. For $a^2-b^2=1113$; $c^2-b^2=0$; $s=31.85, 24.71$; $s^2=1014.42, 610.58$. In

[1] Ch. Michel, *Compléments de Géométrie Moderne*, Vuibert, Paris, 1926, p. 53; Hilda Hudson, *Cremona Transformations*, Cambridge University Press, 1927, p. 33.

this test the value of s^2 must first be computed. An examination of the figure used in the construction gives another form of test which is rather complicated in its statement. The order is a, b, c and $a \geqq c$ as before. There are two cases according as $b \leqq 2^{1/2}c$ or $b > 2^{1/2}c$. Then necessary and sufficient conditions that the point lies within or on the boundary of the larger square are furnished in the first case by the inequalities

$$(2^{1/2}b - c)^2 \leqq a^2 \leqq b^2 + (b+c)^2, \quad b \leqq 2^{1/2}c.$$

In the second case the following inequalities must be satisfied

$$2b^2 - c^2 - 2c(b^2 - c^2)^{1/2} \leqq a^2 \leqq b^2 + (b+c)^2, \quad b > 2^{1/2}c.$$

3269 [1927, 335]. Proposed by J. Rosenbaum, Milford, Conn.

On the sides A_1B_1, B_1C_1, C_1A_1, of a triangle, $A_1B_1C_1$, the points A_2, B_2, C_2 are taken so that

$$A_1A_2 : A_2B_1 = B_1B_2 : B_2C_1 = C_1C_2 : C_2A_1 = \lambda,$$

thus forming a second triangle $A_2B_2C_2$. In this manner, by repeating the operation on the successive triangles a sequence of triangles $T_1, T_2, \cdots$, is obtained. Find under what conditions will (a) T_n be similar to T_1; (b) the shape of T_n approach a limit as n becomes infinite. (c) Find the point to which the three vertices, A_n, B_n, C_n converge when λ is a positive, proper fraction.

Solution by Paul Wernicke, Washington D. C.

For convenience, I call the given triangle $A_0B_0C_0$ and use barycentric coordinates referred to it or, what is the same, vectors A, B, C from any origin outside its plane to its vertices A_0, B_0, C_0. Then, for the first derived triangle T_1, I put $\lambda = p/q = B_0A_1/A_1C_0 = C_0B_1/B_1A_0 = A_0C_1/C_1B_0$, etc., putting A_{n+1} on B_nC_n, etc. Also, $p+q=1$. The vertices of T_n are found to be

$$A_n = h_nA + k_nB + l_nC, \quad B_n = l_nA + h_nB + k_nC, \quad C_n = k_nA + l_nB + h_nC,$$

where h_n, as well as k_n and l_n, is the result of adding every third term in the expansion $\sum_\nu \binom{n}{\nu} p^{n-\nu}q^\nu$ ($\nu = 0 \cdots n$) of $(p+q)^n$, h_n beginning with the first term when $n \equiv 0$, with the third when $n \equiv 1$, and with the second when $n \equiv 2$ (mod. 3); k_n begins with the second term when $n \equiv 0$, with the first when $n \equiv 1$ and with the third when $n \equiv 2$. l_n begins with the third term when $n \equiv 0$, with the second when $n \equiv 1$ and with the first when $n \equiv 2$.

Every T_n has the same centroid as T_0, viz. $(A+B+C)/3$, as the three coefficients used add to $(p+q)^n = 1$. This is also quite evident mechanically. The ratio of areas, Δ_1/Δ_0, is $p^3+q^3 = p^2 - pq + q^2 = 1 - 3pq = 1 - 3\lambda(\lambda+1)^{-2}$ and the ratio Δ_n/Δ_0 is

$$r_n^2 = (1-3pq)^n = \begin{vmatrix} h_n & k_n & l_n \\ l_n & h_n & k_n \\ k_n & l_n & h_n \end{vmatrix} = \begin{vmatrix} 1 & k_n & l_n \\ 1 & h_n & k_n \\ 1 & l_n & h_n \end{vmatrix} = \left\{ \begin{matrix} h_n^2 + k_n^2 + l_n^2 \\ - k_nl_n - l_nh_n - h_nk_n \end{matrix} \right\}.$$

The vector side B_nC_n of T_n is

$$C_n - B_n = a_n = h_n(C - B) + k_n(A - C) + l_n(B - A) = h_na + k_nb + l_nc, \text{ whence}$$

$$a_n^2 = h^2a^2 + k^2b^2 + l^2c^2 - 2klbc \cdot \cos A - 2lhca \cdot \cos B - 2hkab \cos C$$

$$= (h^2 + kl - lh - hk)a^2 + (k^2 + lh - hk - kl)b^2 + (l^2 + hk - kl - lh)c^2,$$

where h, k, l are supposed to be written with the subscript n. We have similar expressions for b_n^2 and c_n^2.

We find that $a_n^2 + b_n^2 + c_n^2 = r_n^2(a^2+b^2+c^2)$, the meaning of the summation Σ being readily discernible. It is intuitively evident that only if $\sum |a_n| = r_n \sum a$ can there be (straight) similarity (i.e. $A_nB_nC_n \sim ABC$ and not to ACB or another permutation). If $A_nB_nC_n$ were similar to ACB, then $A_{2n}B_{2n}C_{2n} \sim A_nC_nB_n \sim ABC$ and if $A_nB_nC_n \sim BCA$, then $A_{3n}B_{3n}C_{3n} \sim ABC$. Therefore, if any similarity were obtained, then $A_{6n}B_{6n}C_{6n} \sim ABC$.

From the above, the answers to the three questions become:

(a) There can be congruence if $\lambda = 0$ or if $\lambda = \infty$, only, and similarity only in the two trivial cases, (1) $\lambda = 1$ or $p = q = 1/2$ and (2) $a = b = c$. In both cases all triangles T_n are similar (straight).

(b) When $\lambda > 0$, T_n approaches the equilateral shape as $n \to \infty$, while when $\lambda < 0$, T_n approaches a straight line (at infinity).

(c) When $\lambda>0$, since all T_n have the same centroid and the area diminishes without bounds, this centroid is the point towards which the vertices converge. In this connection it is of interest to note that the three triangles cut off by the sides of T_{n+1} and each having one of the vertices A_n, B_n, C_n, have each pq times the area of T_n.

3272 [1927, 380]. Proposed by L. M. Berkeley, New York City.

Solve the differential equation,

$$(\tan x + m \sin y)dy = (\sin y - m \tan x) \cos y dx,$$

where m is a constant.

I. Solution by John P. Dalton, University of the Witwatersrand, Johannesburg, S. A.

Two solutions of this problem are given in the article, *On the integration of a certain type of first order differential equation*, in this Monthly, vol. 35 (1928), pp. 189–191.

II. Solution by Paul Wernicke, Washington, D. C.

Multiply the given equation by $\cos x$. It may then be written:

$$\sin y \cdot \cos y \cdot \cos x \cdot dx - \sin x \cdot dy - m(\cos y \cdot \sin x \cdot dx + \cos x \cdot \sin y \cdot dy) = 0.$$

The parenthesis multiplied by $-m$ is $-d(\cos x \cdot \cos y)$. Put $\cos x \cdot \cos y = u$. The first two terms in (1) do not constitute a complete differential, but give $-d(\tan y/\sin x)$ when multiplied by the integrating factor $(\sin x \cdot \cos y)^{-2}$. Put $\tan y/\sin x = v$. Then $(\sin x \cdot \cos y)^2 = (1-u^2)/(1+v^2)$, so that (1) becomes

$$(1 + v^2)^{-1}dv + m(u^2 - 1)^{-1}du = 0,$$

giving $\tan^{-1}v + (m/2) \log (1-u)(1+u)^{-1} = \text{constant}$, in which the values for u and v are to be substituted.

3273 [1927, 381]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The three points of intersection of the symmedians of a triangle with its circumcircle determine a triangle whose circles of Apollonius are identical with those of the given triangle.

Solution by Paul Wernicke, Washington, D. C.

The Apollonian circle (E) through A for the triangle ABC has its center E on BC, and it cuts the circumcircle (K) of ABC orthogonally at A and at a second point denoted by A'. The polar of E with respect to (K) is therefore AA', and thus the pencil $A\ (BA'\ CE)$ is harmonic. Hence AA' is the symmedian through A as is well known. In a similar manner the corresponding points B' and C' are located. It is also well known that BB' and CC' meet in a point L on AA'. According to the definition of a polar, BC and $B'C'$ meet at a point on the polar of L; this point must be E, the intersection of BC and the polar of L, so that E is on the line $B'C'$. It was shown above that the Apollonian circle (E) of the triangle ABC is characterized by the facts that it intersects the circumcircle orthogonally at A and has center on BC. Since the same circle has the corresponding property with respect to the triangle $A'B'C'$, it follows that these two triangles have the Apollonian circle (E), and similarly the other two Apollonian circles, in common.

Also solved by William Hoover and J. H. Neelley.

3274 [1927, 381]. Proposed by C. N. Mills, Normal, Illinois.

The axes of three mutually perpendicular right circular cylinders intersect in a common point. If the radii are equal, find the common volume.

Solution by Emma M. Gibson, Senior High School, Springfield, Missouri

Let the equations of the cylinders whose axes are the x, y, and z axes of coordinates be, respectively,

$y^2+z^2=R^2$, $x^2+z^2=R^2$, and $x^2+y^2=R^2$. The volume common to two of the cylinders, say, $x^2+z^2=R^2$ and $y^2+z^2=R^2$ is $16R^3/3$. The third cylinder cuts off from this volume in the first octant a volume,

$$v = 2\int_{2^{-1/2}R}^{R} dx \int_{(R^2-x^2)^{1/2}}^{x} dy \int_0^{(R^2-y^2)^{1/2}} dz = \frac{R^3}{3}(3\cdot 2^{1/2}-4).$$

Hence, the volume common to the three cylinders is

$$(16/3)R^3 - (8/3)R^3(3\cdot 2^{1/2} - 4) = 8R^3(2 - 2^{1/2}).$$

Also solved by J. M. EARL, MICHAEL GOLDBERG, ENRIQUE LINARES, Jr., J. B. REYNOLDS, and PAUL WERNICKE.

3275 [1927, 381]. Proposed by J. A. Shohat, University of Michigan.

Solve the differential equation,

$$xydy/dx = y^2 + a^2 - ax \pm [(y^2 + a^2 - ax)^2 - y^2(y^2 + a^2 - 2ax)]^{1/2}.$$

SOLUTION BY FREDERIC H. MILLER, Cooper Union Institute of Technology, New York, N. Y.

The given equation reduces to

$$xydy/dx = y^2 + a^2 - ax \pm a[(x - a)^2 + y^2]^{1/2}.$$

Now introduce a new variable, $z=[(x-a)^2+y^2]^{1/2}$; then $xy\,dy/dx=xz\,dz/dx-x^2+ax$ and $y^2=z^2-(x-a)^2$. Discarding $z=0$, we get $x\,dz/dx=z\pm a$; whence $z\pm a=ex$,

or

$$[(x - a)^2 + y^2]^{1/2} \pm a = ex,$$

e being the constant of integration.

This last equation may be identified with that of a conic with focus at $(a, 0,)$ and with eccentricity e.

Also solved by H. H. DOWNING, J. D. TAMARKIN and PAUL WERNICKE.

3277 [1927, 381]. Proposed by R. H. Sciobereti, Berkeley, California.

Find the motion of a weighing particle of mass 1 moving in a vertical plane and attracted by a point of this plane with a force k/r^2, k being a positive constant. Consider applications of the result to the two special cases defined by the following initial conditions: (*i*) at $t=t_0$, the coordinates of M are (0, b), $b>0$, and velocity, given by $v_0^2=2(gb^2+k)/b$, is parallel to the axis of x; (*ii*) at $t=t_0$, the coordinates of M are $(0, b)$, with $-(k/g)^{1/2}<b<0$, and $v_0^2=2(gb^2-k)/b$, with v_0 parallel to the axis of x. What can be said of the stability of the motion in these two particular cases?

SOLUTION BY THE PROPOSER.

Let us take a system of rectangular axes having its origin at the center of attraction O, the axis of y being the vertical line through O and directed positively upward. The position of the particle M will be determined by its distances r and y from the origin and the axis of x respectively, or by a set of coordinates (u, v) defined by the two relations

$$r + y = 2u, \qquad r - y = 2v,$$

so that the curves $u=$ const. and $v=$const. represent a system of confocal parabolas, with focus at O. Now the equations of motion in finite form will be derived from a complete integral of the Hamilton-Jacobi partial differential equation. In order to form this equation we shall calculate the kinetic energy T, a quadratic function in u' and v', and the force function $U=kr^{-1}-gy$, or in terms of the coordinates (u, v), $U=k(u+v)^{-1}-g(u-v)$. Furthermore $2T=x'^2+y'^2$ which after a few algebraic transformations becomes $2T=(u+v)(u'^2u^{-1}+v'^2v^{-1})$, and since the time does not appear explicitly the Hamilton-Jacobi equation takes the form $H=T-U=h$, where H is a function of the quantities u, v, $p_1=\partial T/\partial u'$ and $p_2=\partial T/\partial v'$; hence replacing the variables p_1 and p_2 by W_u and W_v respectively, we shall find for the kinetic energy

$$2T = (u + v)^{-1}(uW_u^2 + vW_v^2),$$

and for the partial differential equation,

$$(u+v)^{-1}(uW_u^2+vW_v^2)=2k(u+v)^{-1}-2g(u-v)+2h.$$

It can be integrated by a separation of variables; that is, there exists a complete integral of the form $W=W_1(u)+W_2(v)$, since according to Stäckel's theorem, six functions $f_1(u)$, $f_2(u)$, $\phi_1(v)$, $\phi_2(v)$, $\psi_1(u)$ and $\psi_2(v)$ may be found such that

$$u(u+v)^{-1}=\Delta^{-1}(\partial\Delta/\partial f_1), \qquad v(u+v)^{-1}=\Delta^{-1}(\partial\Delta/\partial\phi_1), \tag{1}$$

$$2U=2k(u+v)^{-1}-2g(u-v)=\Delta^{-1}[\psi_1(u)(\partial\Delta/\partial f_1)+\psi_2(v)(\partial\Delta/\partial\phi_1)], \tag{2}$$

where $\Delta=(f_1\phi_2-f_2\phi_1)$. Remembering that $\Delta=f_1(\partial\Delta/\partial f_1)+\phi_1(\partial\Delta/\partial\phi_1)$, we shall find with the help of (1) the relation $u+v=uf_1+v\phi_1$; hence the functions f_1 and ϕ_1 may be taken equal to unity. On the other hand ϕ_2 must be a function of v only, so that selecting for Δ the value $u^{-1}+v^{-1}$, it follows that $\phi_2=v^{-1}$ and $f_2=-u^{-1}$. As to the functions ψ_1 and ψ_2, they are readily determined from (2) and found to be $\psi_1=ku^{-1}-2gu$, $\psi_2=kv^{-1}+2gv$. The partial derivatives $W_u^2=(\partial W_1/\partial u)^2$; $W_v^2=(\partial W_2/\partial v)^2$ are given by the formulas

$$W_u^2=ku^{-1}-2gu+2h-2\alpha u^{-1}, \qquad W_v^2=kv^{-1}+2gv+2h+2\alpha v^{-1},$$

from which the complete integral follows at once:

$$W=\int(ku^{-1}-2gu+2h-2\alpha u^{-1})^{1/2}du+\int(kv^{-1}+2gv+2h+2\alpha v^{-1})^{1/2}dv,$$

where α denotes an arbitrary constant. The equations of motion are therefore

$$\partial W/\partial h=\beta+t=\int F(u)du+\int G(v)dv, \qquad \partial W/\partial\alpha=\gamma=-\int F(u)du/u+\int G(v)dv/v,$$

where

$$F(u)=(ku^{-1}-2gu+2h-2\alpha u^{-1})^{-1/2},\ G(v)=(kv^{-1}+2gv+2h+2\alpha v^{-1})^{-1/2}.$$

From the initial conditions given in (*i*), $u_0=b$, $v_0=0$; hence $h=2gb$; and since $(\partial W/\partial u)_0=0$, it follows that $2\alpha=2gb^2+k$; therefore for this particular case

$$F(u)=u^{1/2}[-2g(u-b)^2]^{-1/2}, \qquad G(v)=v^{1/2}[2g(v+b)^2+2k]^{-1/2}.$$

It is then easy to see that $u=b$ during the whole motion, while v varies in the same direction. The trajectory is the downward parabola $u=b$, the motion being stable, for if the initial conditions are altered by quantities of the first order, the function $u(k-2\alpha-2gu^2+2hu)$ which will vanish for two values of u infinitely near b will be positive for any value of u lying between these roots.

For the initial conditions (*ii*) we shall find in an exactly similar manner $u_0=0$, $v_0=-b$, $U_0=-kb^{-1}-gb^{-1}$; hence $h=2gb$, and since $(\partial W/\partial v)_0=0$, $2\alpha=2gb^2-k$, the functions $F(u)$ and $G(v)$ will reduce to $u^{1/2}[2k-2g(u-b)^2]^{-1/2}$ and $v^{1/2}[2g(v+b)^2]^{-1/2}$, rsepectively. v remains equal to $-b$, and u oscilates between the extreme values 0 and $b+(kg^{-1})^{1/2}$; the path is therefore an arc of the upward parabola $v=-b$, the oscillatory motion being stable for the same reason as before.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor J. W. Lasley, of the University of North Carolina, has been elected chairman of the mathematics section of the North Carolina Academy of Science, and Professor W. W. Elliott, of Duke University, secretary of that section.

Professor HARRY BATEMAN, of the California Institute of Technology, has been elected a fellow of the Royal Society of London.

Williams College has conferred the honorary degree of doctor of science on Professor W. A. BRATTON, of Whitman College.

Columbia University has conferred an honorary doctorate on President W. W. CAMPBELL, of the University of California, director of the Lick Observatory.

The University of Indiana has conferred an honorary degree on Professor J. A. MILLER, of Swarthmore College.

Princeton University has conferred an honorary degree of doctor of science on Professor R. A. MILLIKAN.

Professor D. E. SMITH, retired, of Columbia University, has returned from Europe where he has served as a visiting Carnegie professor of international relations. Representing the Carnegie Endowment for International Peace, he lectured at the University College, London, and at the Universities of Montpellier and Toulouse.

The next annual meeting of the National Council of Teachers of Mathematics will be held at the Hotel Statler, Cleveland, Ohio, on February 22–23, 1929. Fuller details will be announced later in the year.

The final printing of the Rhind Mathematical Papyrus is now in progress, which indicates that the binding may be completed early in the autumn. For a description of this forthcoming publication, see page 445 of the November, 1927 issue of this Monthly.

The Proceedings of the International Mathematical Congress of 1924, which have recently been published in two volumes, may be obtained from the University of Toronto Press. The price is five dollars plus postage to registered members of the Congress and ten dollars plus postage to others. Separate copies of the group photograph of the 230 members of the Congress may be obtained for 50 cents each.

The following graduate courses in mathematics are announced for the academic year 1928–29:

The University of Chicago—By Professer E. H. MOORE: General analysis, III, IV, V; Seminar on the foundations of general analysis, I, II, III. By Professor H. E. SLAUGHT: Differential equations; Definite integrals; Elliptic integrals. By Professor L. E. DICKSON: Advanced topics in algebra and the theory of numbers, I, II; Algebras and their arithmetics. By Professor G. A. BLISS: Calculus of variations, I, II. By Professor A. C. LUNN: Lattices and crystal groups; Theory of sound; Relativity; Mathematical theory of electro-

magnetism. By Professor W. D. MACMILLAN: Analytic mechanics, I, statics, II, dynamics; Celestial mechanics, I; Theory of the potential; Dynamics of rigid bodies. By Professor E. P. LANE: Conjugate sets; Metric differential geometry; Synthetic projective geometry; Projective differential geometry of higher space. By Professor M. I. LOGSDON: Functions of a complex variable; Elliptic functions; Algebraic functions with applications. By Professor L. M. GRAVES: Vectors, matrices, and quaternions; Modern theories of integration; Functions of lines. By Professor R. W. BARNARD: General analysis, I, II; Theory of numbers. By Professor WALTER BARTKY: Modern theories of differential equations, I, II. By Dr. B. W. JONES: Introduction to higher algebra; Theory of numbers. Reading and research courses are offered by Professors Moore and Barnard in the foundations of mathematics and general analysis, and by Professor Lunn in applied mathematics. Thesis work and supervision are offered by Professors Dickson and Barnard in algebra and the theory of numbers; by Professors Bliss and Graves in analysis; by Professor Lane in differential geometry; by Professor Logsdon in algebraic geometry; and by Professors MacMillan and Bartky in applied mathematics.

Assistant Professor C. R. ADAMS, of Brown University, has been promoted to an associate professorship of mathematics.

Dr. H. A. BENDER, formerly instructor in the University of Illinois, has been made assistant professor of mathematics at the University of Akron.

Associate Professor R. W. BRINK, of the College of Science, Literature, and the Arts of the University of Minnesota, has been promoted to a professorship of mathematics and has been appointed chairman of the department of mathematics for the year 1928–29.

Professor J. A. BULLARD has resigned his position at the United States Naval Academy to accept the professorship of mathematics in the college of engineering, University of Vermont, *vice* Professor EVAN THOMAS, recently retired.

Associate Professor ALAN D. CAMPBELL has been promoted to a full professorship of mathematics at Syracuse University.

Miss ELIZABETH CARLSON has been promoted to an assistant professorship of mathematics at the University of Minnesota.

Dr. M. G. CARMAN has been appointed head of the department of mathematics at Murray State Teachers College, Murray, Ky.

Associate Professor A. F. CARPENTER, of the University of Washington, has been promoted to a professorship of mathematics.

Mr. WILLIAM FITCH CHENEY, JR. has been promoted to an assistant professorship of mathematics at Tufts College.

Dr. N. B. CONKWRIGHT has been promoted to the rank of associate at the University of Iowa.

Assistant Professor LENNIE P. COPELAND has been promoted to an associate professorship at Wellesley College.

Assistant Professor C. H. CURRIER has been promoted to an associate professorship of mathematics at Brown University.

Assistant Professor W. E. EDINGTON has been promoted to an associate professorship of mathematics at Purdue University.

Dr. RAYMOND GARVER, of the University of Rochester, has been appointed assistant professor of mathematics at the University of California at Los Angeles.

Assistant Professor G. H. GRAVES has been promoted to an associate professorship of mathematics at Purdue University.

Dr. CHARLES C. GROVE is transferring from the Polytechnic Institute of Brooklyn, where he has been since Jan. 1, 1925, to the Willoughby Branch of the College of the City of New York.

Professor W. L. HART, of the University of Minnesota, has been granted a sabbatical leave of absence for the year 1928–29. He will spend most of the year on the Pacific coast.

Dr. VERN JAMES has been appointed head of the department of mathematics at Menlo Junior College.

Mr. F. C. JOHAH has been appointed assistant professor of mathematics at Adelbert College.

Dr. E. G. KELLER has been appointed assistant professor of mathematics at the University of Texas.

Assistant Professor CLARIBEL KENDALL, of the University of Colorado, has been promoted to an associate professorship of mathematics.

At Columbia University, Dr. B. O. KOOPMAN was recently promoted to an assistant professorship of mathematics, not to a full professorship as was erroneously announced in the June-July issue of this Monthly.

Associate Professor E. P. LANE has been promoted to a professorship of mathematics at the University of Chicago.

Dr. C. G. LATIMER, of the University of Kentucky, has been promoted to a professorship of mathematics.

Dr. R. G. LUBBEN has been appointed adjunct professor of mathematics at the University of Texas.

Associate Professor JOHN H. MCDONALD has been promoted to a professorship of mathematics at the University of California at Berkeley.

Dr. A. J. MARIA has been appointed assistant professor of mathematics at the University of Illinois.

Assistant Professor R. W. MARRIOTT has been promoted to a professorship of mathematics at Swarthmore College.

Dr. G. M. MERRIMAN has been appointed assistant professor of mathematics at Grinnell College.

Professor CLIFFORD N. MILLS, of Illinois State Normal University, has been granted a leave of absence for 1928–29 for graduate study at the University of Michigan.

Assistant Professor D. S. MORSE has been promoted to an associate professorship of mathematics at Union College.

Assistant Professor MARSTON MORSE has been promoted to an associate professorship of mathematics at Harvard University.

Associate Professor F. D. MURNAGHAN, of Johns Hopkins University, has been promoted to a professorship of applied mathematics.

Dr. J. R. MUSSELMAN, of Johns Hopkins University, has been appointed professor of mathematics in the Woman's College of Western Reserve University.

Assistant Professor J. A. NORTHCOTT has been promoted to an associate professorship of mathematics at Columbia University.

Mr. H. S. POLLARD has been appointed to an assistant professorship of mathematics at Miami University.

Professor P. R. RIDER, of Washington University has been appointed Sterling Research Fellow at Yale University for the year 1928–29.

Dr. W. E. ROTH, of the University of Wisconsin, has been appointed professor in the university's extension division at Milwaukee.

Professor JOSEPH EUGENE ROWE has resigned his position as head of the department of mathematics and director of extension at the College of William and Mary to accept the presidency of the Clarkson Memorial Institute of Technology at Potsdam, N. Y.

Assistant Professor H. A. SIMMONS has been promoted to an associate professorship at Northwestern University.

Assistant Professor HERMAN L. SMITH has been promoted to an associate professorship of mathematics at Louisiana State University.

Dr. M. H. STONE has been promoted to an assistant professorship of mathematics at Harvard University.

Professor OSWALD VEBLEN, of Princeton University, has been appointed to a research professorship at that university.

At the University of Iowa, Dr. L. E. WARD has been promoted to an assistant professorship of mathematics.

Assistant Professor WARREN WEAVER, of the University of Wisconsin, has been promoted to a professorship of mathematics.

Dr. G. T. WHYBURN has been promoted to an adjunct professorship of mathematics at the University of Texas.

Professor W. M. WHYBURN, formerly of the Texas Technological College, and recently National Research Fellow, has been appointed assistant professor of mathematics at the University of California at Los Angeles.

The following appointments to instructorships in mathematics are announced:

University of Arkansas, Mr. D. P. RICHARDSON;
Brooklyn Polytechnic Institute, Mr. D. E. WHITFORD;
Bryn Mawr College, Mr. H. H. PIXLEY;
Cornell University, Mr. L. J. PARADISO;
Hunter College, Miss R. L. ANDERSON;
University of Illinois, Dr. BESSIE I. MILLER;
Johns Hopkins University, Dr. L. M. BLUMENTHAL;
Lafayette College, Mr. J. H. FITHIAN;
Lehigh University, Dr. H. B. HAMMATT, Messrs. KEELER and HUGH STANLEY, Dr. W. J. TRJITZINSKY, Mr. A. N. VAN ARNAM;
University of Michigan, Mr. F. S. BEALE;
University of Minnesota, Dr. J. M. EARL;
Oregon State Agricultural College, Miss MELVA K. THOMPSON;
University of Rochester, Miss ROSE A. WHELAN;
Sweet Briar College, Miss JULIA W. BOWER;
Williams College, Mr. H. L. DORWART;
Winthrop College, Miss ALICE A. GRANT;
University of Wisconsin, Mr. J. I. VASS.

Professor R. E. ALLARDICE, of Stanford University, died May 29, 1928, at the age of sixty-six.

Mr. PHILIP FITCH, Assistant principal of North High School, Denver, Colorado, died suddenly on July 23. He was secretary of the Rocky Mountain Section of the Mathematical Association of America.

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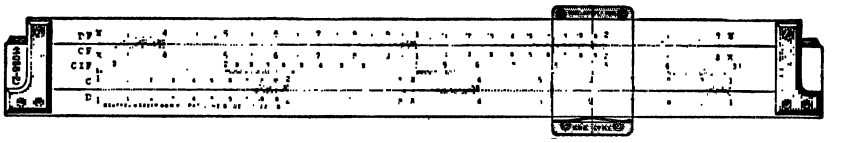
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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December 28-29, 1928

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 31.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, April 28.

OHIO, Columbus, Ohio, April 5.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Dunham, N. C., March.

SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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SUGGESTIONS TO AUTHORS

Much needless expense and many errors can be avoided. The editors of several mathematical journals have agreed upon the following suggestions.

1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: $\not{1}$. Similarly, O means "zero"; to indicate "cap O," backspace and overprint period; thus: $\dot{O}$.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma \cdots$ (all) A B $\Gamma \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma \cdots}$ (all except $^{\iota}$ and ^{o}).

★ Light Greek Inferiors—$_{\Delta\Lambda\Sigma\Omega}$ and $_{\alpha\ \beta\ \gamma \ldots}$ (all except $_{\iota}$ and $_{o}$).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \varrho\ \vartheta\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q}\ \mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b}\ \mathfrak{A\ B\ D}$.

Script (special font $\mathcal{A\ B\ C} \cdots$ (all). No lower case manufactured.

* Hebrew—ℵ א ב troublesome to handle.

★ Dashed Italics—$\overline{A}\ \bar{a}\ \overline{B}\ \bar{b}\ \overline{C}\ \bar{c}\ \overline{E}\ \bar{e}\ \overline{F}\ \bar{f}\ \overline{G}\ \bar{g}\ \overline{H}\ \bar{\imath}\ \bar{\imath}\ \overline{K}\ \bar{k}\ \overline{M}\ \bar{m}\ \overline{N}\ \bar{n}\ \overline{P}\ \bar{p}\ \bar{q}\ \overline{R}\ \bar{r}\ \bar{s}\ \bar{t}\ \bar{u}\ \bar{v}\ \overline{X}\ \bar{x}\ \overline{Y}\ \bar{y}\ \overline{Z}\ \bar{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$.

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \overline{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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THE FIFTH ANNUAL MEETING OF THE TEXAS SECTION

The fifth annual meeting of the Texas Section of the Mathematical Association of America was held at Texas Agricultural and Mechanical College, College Station, Texas on Saturday, January 28, 1928. For the first time since its organization, this section held an independent meeting of its own, all previous meetings having been held in conjunction with the Mathematics Section of the Texas State Teachers Association at the time of its regular annual meeting at Thanksgiving.

Owing to the absence of the chairman of the Section, Professor W. M. Whyburn, who was at Harvard University for the year as National Research Fellow, Professor G. C. Evans presided over the morning session at ten o'clock, and Professor H. J. Ettlinger over the afternoon session at two o'clock. The afternoon session took the form of a round table discussion with the papers of Professor Bray and Dean Puryear forming the basis. Those attending the meetings were the guests of Texas Agricultural and Mechanical College in the private dining room of the Mess Hall, for luncheon. The program committee for the meeting consisted of Professor G. C. Evans, chairman, Professor Hillel Halperin and Professor H. J. Ettlinger. The local committee of arrangements at College Station consisted of Professor D. C. Jones, chairman, Mr. A. A. Blumberg and Mr. J. W. Ross.

The attendance was forty-one including the following twenty-two members of the Association: B. T. Adams, F. Ayres, W. L. Ayres, L. W. Blau, A. A. Blumberg, H. E. Bray, A. C. Dean, J. L. Dorroh, H. J. Ettlinger, G. C. Evans, G. W. Evans, L. R. Ford, H. Halperin, M. Hickey, J. R. McKee, J. N. Michie, L. Oxsheer, W. L. Porter, J. J. Quinn, W. A. Rees, F. W. Sparks, G. T. Whyburn.

The following officers were elected for the coming year: Chairman, G. T. WHYBURN, University of Texas; Vice-chairman, F. W. SPARKS, Texas Technological College; Secretary-Treasurer, H. J. ETTLINGER, University of Texas. The officers, acting as an executive committee, were instructed to select the time and place of the next annual meeting. As an outgrowth of the discussion of the meeting in the afternoon, the executive committee was also instructed to appoint a state-wide committee on "Requirements for high school mathematics teachers." This committee was to make an investigation of minimum requirements for high school mathematics teachers and make recommendations to this Section, to the Mathematics Section of the Texas State Teachers Association and to the Texas State Department of Education. The following were appointed on this committee: F. W. Sparks, Texas Technological College, chairman; Dean C. E. Puryear, Texas A. and M. College; E. R. C. Miles, Rice Institute; Mary Decherd, University of Texas; D. C. Jones, Southern Methodist University; J. W. Harrell, Baylor University; J. J. Quinn, St. Edward's University; A. R. Wapple, Southwestern University; and L. W. Blau, University of Texas.

The following papers were presented:

1. "A line conic camera" by Professor L. R. FORD, Rice Institute.

2. "Accessibility problems" by Professor G. T. WHYBURN, University of Texas.

3. "Forced vibrations with small resistance approximating resonance" by Mr. L. W. BLAU, University of Texas.

4. "Accessible points on the boundary of a three dimensional region" by Mr. J. J. GERGEN, Rice Institute (by invitation).

5. "Some elementary aspects of Heaviside's operational methods" by Professor H. J. ETTLINGER, University of Texas.

6. "N-ic surfaces in S_3 generated by triples of N-ic involutions on a line" by Mr. F. AYRES, Texas Agricultural and Mechanical College.

7. "The golden section" by Dean J. J. QUINN, St. Edward's University.

8. "A prescribed course for freshmen and their mathematical habits," by Professor H. E. BRAY, Rice Institute.

9. "Preparation for College mathematics" by Professor Chas. E. PURYEAR, Dean of the College, Texas Agricultural and Mechanical College (by invitation).

Abstracts of these papers follow:

1. Professor Ford exhibited a line-conic camera, in which the lens is replaced by two slits. The image of a point moving along a line will be a conic section, since the photographic plate intersects the locus of a line which touches three skew lines. Several interesting photographs were exhibited taken by means of this camera.

2. The notions of accessibility and *regular accessibility* of the boundary points of a domain, as well as certain accessibility propositions which hold only in the plane, were discussed by Professor Whyburn with a view of possible extensions to 3-space. A certain "tubular accessibility" in 3-space was offered which seems to carry a number of the well known theorems in the plane about simple continuous arcs and arc-wise accessibility over into higher dimensions. The relation between the regular accessibility of the boundary points of a domain and the connectedness im kleinen of the domain plus certain subsets of its boundary was also discussed. It was shown that although there exists a plane domain having an isolated non-accessible boundary point, each non-*regularly* accessible boundary point of a plane domain must belong to a continuum of such boundary points.

3. The paper by Mr. Blau appeared in full in the June-July 1928 issue of this MONTHLY.

4. Let T be a three dimensional open simply connected region on the bounday of which the Green's function with pole at an interior point O vanishes. Let T be mapped upon the interior T' of the unit sphere, a level surface corresponding to a surface of a sphere, and an orthogonal trajectory to a radius of T'. Let A be an accessible point on the boundary of T, and OA the curve leading to A. Then if OA_t be any trajectory leading to A such that the curve $OA+OA_t$ can be shrunk down to A without leaving T, the image of OA_t in T' is a radius which terminates at some one of the points of a set E of zero measure.

5. In his operational calculus for solving linear differential equations, Heaviside makes use of a simple discontinuous function, called the *unit function* and defined as follows: $f(t)=0$, when $t\leqq 0$, $f(t)=1$, when $t>0$. This is an elementary type of horizontal or step function making use only of the values 0

and 1. Professor Ettlinger discussed some of the properties of the operator p^{-1} operating on the unit function which is equivalent to integration between the limits minus infinity and t. Positive integral powers of this operator generate the integral power function and linear combinations generate polynomials in t. Conditions were derived under which a power series in this operator will generate an analytic function of t over the entire t-axis.

6. In this paper, Mr. Ayres has shown that certain nondegenerate n-ic surfaces can be generated by means of triples of n-ic involutions, or linear series of points; namely the first polar surface of any one with respect to the n-ic degenerates into $n-1$ planes through the line determined by the other two. These points and the lines determined by them lie on the first, $\cdots$, $(n-3)$rd Hessian and corresponding Steinerian surfaces. For the case $n=2$, it was shown that every quadric surface can be generated in the above manner.

7. Dean Quinn presented a new construction for locating the golden section point of a given segment. This paper will appear in full in a fall number of the Texas Mathematics Teachers' Bulletin, published by the University of Texas.

8. Professor Bray outlined a method of handling freshman students in mathematics at Rice Institute which consists of lectures and "supervised" working of problems. Professor Bray gave a summary of the characteristic ailments of freshman mathematical students. This paper will appear in full in a fall number of the Texas Mathematics Teachers' Bulletin.

9. Dean Puryear presents a summary of what the high school graduate should know of mathematics upon entrance to college. He also made a plea for a course in algebra in the last year of the high school course. The paper appears in full in the May 1928 number of the Texas Mathematics Teachers' Bulletin.

H. J. ETTLINGER, *Secretary*

THE TWELFTH ANNUAL MEETING OF THE KENTUCKY SECTION

The twelfth annual meeting of the Kentucky Section of the Mathematical Association of America was held April 21, 1928 at the University of Louisville, Louisville, Ky. Professor Guy Stevenson presided at both sessions. At noon a delightful dinner was served in the Home Economics Building.

At the business meeting the minutes of the previous meeting were read and approved. Professor C. G. LATIMER, University of Kentucky, was elected chairman for the year 1928–1929. Professor A. R. FEHN, Centre College, was elected secretary for the five years, 1928–1933. A vote of thanks and appreciation was given Professor R. D. Carmichael, University of Illinois, for his lecture

and his interest in the Kentucky Section. Resolutions, drawn up by P. P. Boyd, of the University of Kentucky, expressing regret that the Section is to lose Professor C. H. Richardson, Georgetown College, were passed; Professor Richardson will be at Bucknell, Penn., next year. The Section expressed its appreciation to the University of Louisville for the hospitable reception and the delightful dinner given the attending members.

There were twenty eight present including the following members: Dean P. P. Boyd, J. L. Clayton, J. M. Davis, D. G. Dearman, H. H. Downing, A. R. Fehn, R. E. Hill, C. G. Latimer, E. L. Rees, C. H. Richardson, G. E. Stevenson, B. B. Vance.

Papers were presented as follows:

1. "The Steinerian transformation" by Dean P. P. BOYD, University of Kentucky.

2. "Resolutions of singularities" by Professor M. B. TOLAR, Georgetown College, (by invitation).

3. "On the representation of integers by indefinite forms" by C. G. LATIMER, University of Kentucky.

4. "Curves of accumulation" by Professor H. H. DOWNING, University of Kentucky.

5. "Relativity" by Professor R. D. CARMICHAEL, University of Illinois.

6. "Correlation between mental and mathematical tests and first semester grades" by Miss LYDIA FREMD, University of Kentucky (by invitation).

7. "Classification of freshmen" by Professor R. E. HILL, University of Louisville.

8. Business Meeting.

Abstracts of these papers follow:

1. Dean Boyd presented for examination a copy of the Elements of Euclid, done in vulgar tongue under the direction of the most famous Mathematician Federigo Commandinus of Urbino,—carefully revised to date—printed by Flaminio Concordia,—MDCXIX—under the direction of George Antonio Tugegnieri of Fossombrone." The volume was given to the speaker by a friend who found it in Spain a year or two ago. Selections were quoted from the lengthy prologue, setting forth the nature of mathematics, its uses and beauties, and defending the name of Euclid from his detractors. Sample demonstrations were also given.

2. This paper outlined briefly the analytical development of the theory of quadratic inversion and its application to higher singularities. The geometrical aspect was presented in an analysis of the singularities on the curve $x^4y^3=z^7$.

3. In this paper Professor Latimer considered the forms where a is a positive integer less than 100 with no square factor greater than 1 and in the form $8n+5$

or $8n+7$. By a method used in a former paper (Annals of Mathematics, vol. 28) it was shown that each of these forms represents every integer.

4. In this paper attention was directed to the direct method of solving the problem of the calculus of variations. The principal thought in the paper was centered in the theorems of Ascoli, Argela, and Hilbert in which the existence of functions and curves of accumulation are proved. The treatment was that of Tonelli in *Fondimenti di calcolo delle variationi.*

5. Professor Carmichael outlined briefly the historical development of the theory of relativity, sketched its foundations in non-technical language, discussed rapidly its general bearing on the development of science, and analyzed its influence on philosophical, scientific, and ethical thought.

6. The Problem in question form:

(a) What is the relation of the mathematical and mental-test scores made by the University of Kentucky freshmen of 1927 to the first semester grades in College Algebra made by the same students?

(b) What weight, discovered by means of the regression equation, should be attached to each score to obtain the optimum correlation?

(c) What is the test, or the elements of the test, which furnish the best prognosis of the standing of the students in mathematics?

Data: The study includes 140 cases, a wholly unselected group, as it includes every student who had a grade recorded with the registrar for the first semester of college algebra and whose test records were complete. Tests were given under the supervision of the Psychology Department. They were Woody-McCall, Mixed fundamentals, Otis mathematics tests, Reasoning, Iowa training tests, Iowa high school content, College classification and Otis mental tests.

Results: Median quartile scores for all tests with grade groups.

	W-Mc	Otis	Iowa Tr.	Iowa H. S.	Mental	Combined
A	1.32	1.44	2.30	2.15	1.95	1.65
B	1.02	.95	1.48	1.33	1.32	1.15
C	.69	.91	1.20	.65	.93	.80
D	.24	.51	.27	.34	−.17	.15
E	−.15	−.50	.15	−.55	−.85	−.30

The median quartile scores descend by a gradual consistent decline as the grades pass from a higher to a lower grade, which points to an unequivocal relationship between test scores and grades. No median of any test is greater than, or equal to, the median of a higher grade.

A. R. FEHN, *Secretary*

THE FIFTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held in Jackson, Miss., March 30–31, 1928, Professor S. T. Sanders of Louisiana State University presiding.

The following is the list of the delegates at the meeting: H. E. Buchanan, J. B. Cole, Mrs. A. P. Daspit, W. L. Duren, Jr., J. E. Hardin, J. R. Hitt, Mabel Hutchins, George Litchfield, I. Maizlish, B. E. Mitchell, C. N. Moore, I. C. Nichols, R. L. O'Quinn, W. C. Roaten, W. J. Rothfuss, S. T. Sanders, C. Shroeder, C. D. Smith, H. L. Smith, P. K. Smith, N. Smylie, W. B. Stokes, J. B. Temple, J. F. Thompson, N. Touchstone, B. A. Tucker, B. M. Walker, J. F. Welch, W. P. Webber, R. S. Wynn. In addition to the above thirty delegates, there were a few who failed to register.

The meeting of the Section was held in conjunction with the meeting of the Louisiana-Mississippi Section of the National Council of Teachers of Mathematics. On the afternoon of the thirtieth, at a joint meeting of the two sections, several papers were read bearing upon high school mathematics. The college teachers had an opportunity to hear first hand statements on problems in high school mathematics teaching.

At Mississippi College, on the evening of the thirtieth, a dinner was tendered to both sections. Dr. C. N. MOORE, of the University of Cincinnati, the speaker of the evening, spoke upon "Mathematics and Civilization." The speaker cited many interesting cases from the history of science where the scientist has had to turn to the mathematician to give the final word in the solution of his problem. President B. M. Walker, of the A. and M. College of Mississippi, President J. W. Provine, of Mississippi College, and Chairman Sanders also spoke.

At a joint meeting of the sections, on the morning of the thirty-first at Belhaven College, five papers from the college group were read. Following this session a business meeting was held by the section of the Association. Prof. B. E. MITCHELL, of Millsaps College, was elected chairman; Professor S. T. SANDERS, of Louisiana State University, Editor of the "Mathematics News Letter"; Professor J. R. HITT, of Mississippi College, and Professor W. B. STOKES, Vice-chairmen; and Professor J. P. COLE, of Louisiana Polytechnic Institute, Secretary-Treasurer.

Upon adjournment of the business meeting Millsaps College tendered both sections a barbecue.

The titles of the papers read are as follows:

1. "A quadratic one-to-one transformation between two planes," by President B. M. WALKER, A. and M. College of Mississippi.

2. "The construction of magic squares," by Professor J. R. HITT, Mississippi College.

3. "A problem in geometry," by Professor J. P. COLE, Louisiana Polytechnic Institute.

4. "Some remarks on periodic functions," by Professor PAUL WEBBER, Louisiana State University.

5. "A table of chords by Ptolemy," by Professor H. E. BUCHANAN, Tulane University.

Abstracts of these papers follow:

1. The transformation system of equations was first deduced to be $x'\colon y'\colon z'=x^2+yz\colon xz\colon xy$, and following immediately the system of inverse equations was deduced. The treatment of the problem included ordinary points, related points, fundamental points, transition curve or "Uebergangs Curve," and the double curve or "Doppia Curva." Related points were shown to be connected by a quadratic Cremona one-to-one correspondence. It was further shown that to a fundamental point together with the totality of the paths of approach to it in one plane corresponds a fundamental line in the other plane. The one-to-two correspondence was then established by means of projection using a quadric surface and two centers of projection arbitrarily chosen so that one should lie on the surface and the other be external to the surface. The one-to-two correspondence was further established by means of poles and polars. The problem of a one-to-two correspondence was reduced to the problem: given a pole of a cubic to find the polar line; and the line coordinates defined in terms of the coordinates of the given pole are the equations of transformation. The inverse problem being: given a polar line to find its two poles.

2. The author showed (*a*) a general method of constructing magic squares whose square roots are odd; (*b*) a general method of constructing squares whose square roots are multiples of 4; (*c*) special methods of constructing squares whose square roots are multiples of 3 or multiples of 4.

3. The author divided his discussion into three parts. The problem proper was to construct a square so that each side would contain a given point. In the first part four points were taken and connected by straight lines. These lines were then bisected and these points were then connected, forming a parallelogram. On the longer diagonal (assuming one longer than the other) a circle was constructed with it as a diameter. The arc on one side of the diameter was then bisected. This point and one of the other points of the parallelogram were connected and the line extended to the other arc of the circle. This other intersection of the line with the circle was then joined to the end of the diameter of the circle. Through two of the original points, lines were drawn parallel to this last line, and through the other two points lines were drawn perpendicular to the line. The figure so constructed was the required square. In the second part it was shown that if the parallelogram was a rectangle it would have to be a square in order that a square might be drawn through the original

points. In the third part it was shown that if the rectangle was a square an infinite number of squares could be drawn through the four original points.

4. Mr. Webber in his remarks on periodic functions presented his generalized formulae defining functions having n periods and referring to space of n dimensions and called attention to the analogy of form with the fundamental formulae of the Weierstrassian theory and further showed that in the case of $n=2$ his formula reduces identically to Weierstrass's; and when $n=1$, to an analogous form for the cotangent.

5. Professor Buchanan's paper was an attempt to explain the work of Ptolemy on chords. A statement of his results was given in modern notation which showed that he was in possession of practically all of computational trigonometry.

P. K. SMITH, *Secretary*

THE FOURTH ANNUAL MEETING OF THE MICHIGAN SECTION

The fourth annual meeting of the Michigan Section of the Mathematical Association of America was held at Ann Arbor, April 29, 1927.

The following officers were elected for the year: A. L. NELSON, College of the City of Detroit, Chairman; W. W. DENTON, University of Michigan, Secretary-Treasurer; T. O. WALTON, Kalamazoo College, Member of the Executive Committee.

The following program was presented:

1. "Areas on the sphere" by Professor G. Y. RAINICH.
2. "Marginal notes" by Professor T. H. HILDEBRANDT.
3. "An application of Thiele's semi-invariants to the sampling problem" by Professor C. C. CRAIG.
4. "Focal point construction for a certain type of problem in the calculus of variations" by Professor V. G. GROVE.
5. "A convergence proof in the theory of linear differential equations" by Professor J. A. NYSWANDER.

W. W. DENTON, *Secretary*

THE NINTH ANNUAL MEETING OF THE ILLINOIS SECTION

The ninth annual meeting of the Illinois Section of the Mathematical Association of America was held at Eastern Illinois State Teachers College, Charleston, on May 4–5, 1928. The chairman, Professor C. N. Mills, presided. There was an attendance of 51 at the meetings, including the following 26 members of the Association: Beulah Armstrong, Edith I. Atkin, H. W. Bailey,

O. E. Brown, R. D. Carmichael, C. E. Comstock, D. R. Curtiss, Elinor B. Flagg, A. E. Gault, Frances Harshbarger, Martha Hildebrandt, E. C. Kiefer, J. M. Kinney, Harry Levy, E. B. Miller, C. N. Mills, Mary W. Newson, J. A. Nyberg, H. P. Rogers, H. A. Simmons, H. E. Slaught, C. H. Smiley, E. H. Taylor, Mildred E. Taylor, C. A. Van Velzer, and Alice Winbigler.

In the absence of the Secretary, Professor Newson was appointed secretary *pro tem.* The following officers were elected for the ensuing year: A. E. GAULT, Bradley Polytechnic Institute, Chairman; C. I. PALMER, Armour Institute of Technology, Vice-Chairman; C. N. MILLS, Illinois State Normal University, Secretary-Treasurer.

The next meeting will be held on May 3–4, 1929, at Carthage College, Carthage, Illinois. A motion was passed that the incoming secretary-treasurer pay Professor Rietz, from the section treasury, for his travelling expenses incurred for the meeting in May, 1927.

Professor Mills proposed a plan for making it easier for members to pay their sectional dues. A vote of thanks to the Eastern State Teachers College was passed.

The following papers were read:

1. "Derivatives whose orders are not positive integers," by Professor C. E. VAN VELZER, Carthage College.

2. "Some mathematical aspects of public utilities management," by Dr. A. R. SCHWEITZER, Chicago.

3. "The ideals of the teacher of mathematics," by Professor MARY NEWSON, Eureka College.

4. "The typewriting of mathematical lectures," by Mr. ORLEY E. BROWN, Northwestern University.

5. "Recent developments in differential geometry," by Professor HARRY LEVY, University of Illinois.

6. "Stepping stones," an address open to the public, by Professor R. D. CARMICHAEL, University of Illinois.

7. "Expository paper on certain boundary value problems," by Dr. F. R. BAMFORTH, University of Chicago (by invitation).

8. "Plane curves defined by certain projectivities," an expository presentation, by Miss MARTHA HILDEBRANDT, Proviso High School, Maywood, Illinois.

9. "An expository paper on analysis situs," by Dr. H. R. BRAHANA, University of Illinois (by invitation).

10. "Symposium on the teaching of algebra": (*a*) "Continuation of previous report," by Professor C. E. COMSTOCK, Bradley Polytechnic Institute; (*b*) "Some special phases of the subject," by Mr. J. A. NYBERG, Hyde Park High School, Chicago, Illinois.

Abstracts of these papers follow:

1. In this paper an effort was made to find definitions which would give meanings to derivatives whose orders were negative integers or positive or negative fractions, these definitions to conform to the index and distributive laws and at the same time to be consistent throughout the whole range of functions. Such definitions were found which, in the case of transcendental functions were usually expressed in a series whose terms contained gamma functions. In some cases, however, the fractional derivatives of transcendental functions were expressible in a finite form. Charts were shown of a number of fractional derivatives and some of the properties of these curves were pointed out.

2. Dr. Schweitzer's paper had to do with the isolation of the guiding principles of the organization and operation of the light and power corporations, and with the investigation of their mathematicological significance from a theoretical standpoint. He shows that these guiding principles are the same as those of mathematical sciences, as established in his dissertation. This identity leads naturally to the concept of continuities underlying the discrete data presented by the statistical reports of the light and power corporations. The study of such continuities at the present time centers about the modern mathematical investigations into biological mass formation and growth, with application to monopoly and holding companies.

3. A preliminary study seeking what objectives are common to all stages of the study of mathematics. Certain illustrations were given tending to show that the great features of such a system of objectives are the same, whether the child leaves the school early or continues the study of these branches through college or perhaps beyond. Mrs. Newson also gave other illustrations from her experience in teaching, showing the need of more care in grafting the new subjects on the old, as of algebra on arithmetic, and of continual vigilance to avoid the danger of a superficial study of processes which leaves the student unable to apply what he has learned.

4. Mr. Brown has developed two appliances for the Hammond typewriter which facilitate its use in typewriting mathematics. One of these is a new type set or shuttle designed to be supplementary to 135 C, which is the principal mathematical shuttle now available. Shuttles now carried for sale were designed as complete units, but either overlap almost completely or else have different styles of type. Any one shuttle is limited to 120 characters, so that one generally adequate shuttle is impossible. The new shuttle typewriter together with the old one carries 240 characters, with no duplication excepting punctuation. The other appliance is a shift key placed in the keyboard which interchanges the two sets of type in the machine.

5. Levi-Civita, when he introduced the concept of parallelism in a Riemann space, made a contribution of fundamental importance to geometry, and the

development in the succeeding ten years of Non-Riemannian geometries has been a direct consequence of his investigations.

In an n-dimensional space of coordinates $x^1, x^2, \cdots x^n$, an arbitrary analytic transformation of those coordinates

$$x^i = f^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n) \qquad i = 1,2, \cdots, n,$$

induces a linear transformation on the differentials,

$$dx^i = \sum_{j=1}^{n} \frac{\partial f^i}{\partial \bar{x}^j} d\bar{x}^j,$$

Since a set of differentials $(dx^1, dx^2, \cdots, dx^n)$ defines a direction, the relations among directions at a point are those of projective geometry.

But there is *a priori* no association of directions at different points. One can by postulating a Riemann metric,

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j,$$

define, as did Levi-Civita, parallelism of vectors. There is however no logical justification for subordinating the concept of parallelism to that of distance, and the work of Weyl, Eisenhart and Veblen, Schouten, Cartan, and others has been to take as the starting point a definition parallelism of vectors by a means of linear connection

$$d\xi^i = \sum_{j,k=1}^{n} \Gamma^i_{jk} \xi^i dx^k \qquad i = 1,2, \cdots, n,$$

where Γ^i_{jk} are not the Christoffel symbols of the Riemannian metric but arbitrary functions of the coordinates.

A rather extensive bibliography on the subject may be found in the two recent works of L. P. Eisenhart, *Riemannian Geometry*, (Princeton University Press), and *Non-Riemannian Geometry*, (American Mathematical Society).

6. Professor Carmichael discussed various mathematical activities, such as clubs, national and sectional meetings of the Mathematical Association of America, meetings of the American Mathematical Society, and the publication and the purchase of books, as successive stepping stones to the higher mathematical achievements and to some of the more profound values of human life.

7. This paper is an expository account of some of the important topics in the theory of boundary value problems. The discussion is confined to problems in which there is a system of two ordinary linear homogeneous differential equations of the second order, and a set of two algebraic equations which are linear and homogeneous in the values of the two dependent variables at the end points of the interval $a \leqq x \leqq b$. The coefficients of the differential equations

have continuous first derivatives with respect to x on the interval (ab), and contain a parameter λ linearly; the coefficients of the boundary equations are constants. In the study of a boundary value problem it is necessary to find out whether there exist values of λ for which there are non-trivial solutions of the differential equations satisfying the boundary conditions. Such parameter values, if they exist, are called the characteristic numbers of the problem, and the non-trivial solutions, which correspond to them, the characteristic solutions. Other subjects of interest in the study of boundary value problems of the type mentioned above are the so-called Green's matrix, self-adjoint boundary value problems, the reduction of problems to normal forms, the oscillations of the characteristic solutions, and the expansion of a set of two arbitrary functions simultaneously in terms of these solutions.

8. This paper discusses the locus of intersection of the corresponding lines or the envelope of the lines joining the corresponding points when a projectivity has been established between the elements of two one-dimensional forms of the first, second, or higher order in a plane.

9. The paper by H. R. Brahana explained the position of analysis situs in relation to Euclidean and projective geometries, contained a brief mention of some of the famous problems, and pointed out some of the difficulties encountered in passing from two dimensional analysis situs to that of a higher number of dimensions.

10 (*a*). The history of the entrance requirements in mathematics in American colleges was sketched. Attention was called to the change in entrance requirements as pending before the colleges of the North Central Association, reducing the total number of units from 15 to 12 which comprise the work of the senior high schools. The question at issue was the effect of this new legislation upon the status of algebra as a college entrance requirement.

10 (*b*). This paper was published in the *Mathematics Teacher*, Dec., 1927, and is a discussion of a paper with the same title by Professor W. D. Reeve of Columbia University, published in the *Mathematics Teacher*, March, 1927. While the latter paper calls attention to all the desirable objectives in a third semester's work in algebra, Mr. Nyberg discusses how many of these objectives can be attained in a semester's work of 89 days. Attention was also called to the fact that colleges allow their students to register for courses in trigonometry and in college algebra even when the student has had only one year of algebra in high school. On the other hand no high school allows a pupil to take these courses unless the pupil has completed one and one half years of algebra.

MARY W. NEWSON, *Secretary pro tem.*

THE HUMAN SIGNIFICANCE OF MATHEMATICS[1]

By DUNHAM JACKSON, University of Minnesota

As students and teachers of mathematics, we are confronted all the time by the question: What is it all about? What are we doing it for? Are we accomplishing something of which we can give an intelligible account? Or are we merely doing what our predecessors did because somebody else had done it before them? Are we, like Christopher Robin,[2]

". . . . following an elephant
That's following an elephant
That's following an elephant
That isn't really there"?

I do not believe we are in this unprofitable state. If I did, I should not have taken this occasion to say so. But it is no confession of weakness to recognize that the problem is always with us. In its nature it is one for each generation that comes along. We are not asking whether it is important that mathematical knowledge should exist and be on record. That is admitted without question. There are tons of mathematical books in our libraries, and we accept the fact of their existence without uneasiness. The question is why we and our pupils should make it a chief concern of our lives to study those books, and look beyond the printed pages for the meaning that is behind them, and try to do the things that the authors have left undone.

I do not pretend to offer a final and compelling answer, resolving all doubts and removing all difficulties. If I had such an answer, I should not have been so unmindful of my obligation to society as to keep it a secret all this time. But as years pass I do find certain convictions increasingly confirmed in my own mind.

The question must be discussed with reference to the circumstances of the people on whose behalf it is asked. A reason that was good under the conditions of twenty-five years ago may not be a good one now, and if it is, that is a new observation, and not a mere reiteration of an old one.

On the other hand, regarding the problem as personal rather than universal, we can limit its scope, and so escape being carried too far afield. I have no intention of arguing the value of mathematical instruction in the public schools, or the importance of the Pythagorean theorem to the man in the street. I am thinking specifically of the members of the Mathematical Association of America, and of the organizations affiliated with it, a few thousand individuals

[1] Address of the retiring president of the Mathematical Association of America, at Nashville, Tenn., December 29, 1927.

[2] The writer offers his apologies to Christopher Robin for a slight adaptation of the latter's phraseology.

at most in a population of a hundred millions. Why should we, few as we are, spend our time doing mathematics, when there is so much else to be done? What shall we tell our students, when they ask us what we are driving at?

With our own peace of mind is involved the effectiveness of our instruction. We are told constantly that motivation is a prime requisite in teaching. Whatever that may imply for the technique of elementary instruction, the college student is a person of some maturity, and does not require motivation hour by hour and page by page. If it is available, so much the better. A young instructor in perplexity came to my colleague Professor Brooke, and inquired: "What do you do when students ask you what this or that particular topic in the calculus is good for in engineering?" To which Professor Brooke replied: "I tell them." But generally speaking, our students have some continuity of purpose, to the extent that they have purpose at all, and the most effective motivation is confidence that the instructor knows what he is about.

There was a time when you could make a good beginning by a reference to a liberal education. We have to recognize now that the thing formerly called by that name no longer exists. There is no body of knowledge or type of intellectual experience which is the common possession of persons of academic training, a bond of fraternity and medium of understanding between one educated man and another. It is possible to be influential in the intellectual life of the community without ever having opened a Latin grammar or a trigonometry. It is possible, as I happen to know from a specific instance, to graduate from an American university and think that Columbus discovered America in 1642. The traditional education survives as an experience of individuals, but not as a badge of a coherent and dominant group in society.

I do not admit for a moment that this is a sign of retrogression or decay. It is an incident of progress into the unknown. There is so much more to be learned that there has to be a division of labor in the learning of it; and there are thousands of people who have never been educated in the old-fashioned way, but whose intellectual attainments the alumnus of the old-fashioned education is compelled to respect. We have achieved democracy, or it has been achieved for us, if not in perfection, at any rate to a degree never contemplated by those who invoked it. And now that we have it, we have the problem of discovering what to do with it. Incidentally, we can not pretend that its destinies are committed to the hands of a class characterized by an ability to solve quadratic equations.

Another quaint phrase which we have heard from the lips of old people is "academic leisure." According to the theory of academic leisure, a professor taught his classes for an hour or two, and then had nothing more to do for the rest of the day. He retired to his desk to study undisturbed, or relaxed in intellectual companionship with students who chose to profit by his counsel,

Nowadays, if a professor has leisure, it is either because his colleagues have discovered by sad experience that he is incompetent outside his own limited sphere, or because he is protectively endowed by nature like the animal of which Hilaire Belloc writes:

"I shoot the hippopotamus with bullets made of platinum,
Because if I use leaden ones his hide is sure to flatten 'em."

There are so many things to be done, things that really matter to somebody, things that are demanded every day and every hour with an urgency well-nigh irresistible! While as for any particular piece of research, one realizes that no hungry world is clamoring for it, and if by an infrequent chance it really is relatively important, it will be done independently by two or three other people in the course of the next few years. The fact that there is a personal satisfaction in doing it, does not meet the issue. The investigator who used to boast that his researches amused only himself, and were of no possible interest to anybody else, may still investigate, but he does not boast. When the gentleman of leisure was the ideal of society, a gentleman whose activities were harmless was a good citizen. But this democracy which has happened so unexpectedly demands that you do something useful, or at least make out a case for the usefulness of what you do. More than that, if you are not satisfied with evasion, it demands a degree of usefulness at least comparable with the alternatives that press for attention. To those of us who have to earn a living, and through force of circumstances would not be gentlemen of leisure even if that profession still enjoyed its old prestige, there is significance in Gottlieb's words in "Arrowsmith": "Why should the world pay me for doing what I want, and what they do not want?" The relentless question for the individual is not why he should do mathematics, but how he can persistently refuse to do a hundred other things that leave no time or energy for mathematics.

Well, in the first place, I am not abashed to say that I believe that mathematical training has a great deal to do with thinking straight. The force of this contention has been so much exaggerated on the one hand, and so much depreciated on the other, that stress on its residual merit is not out of place. Let it be granted that prejudice and error have been known to survive exposure to courses in geometry. It is no less true that malignant organisms in the body frequently survive exposure to the most approved medical treatment. Let it be granted that the decisions of every-day life are seldom reached by the formal application of logic, and that those who know how to think logically need constantly to be on their guard not to let their reasoning faculties corrupt their better judgment. Still it is of inestimable benefit to any nation to have a few men within its borders who know the difference between a sound argument and a specious one, and to have abroad in its population a wide-spread consciousness that there are norms of thought transcending factional expediency. It

is my firm conviction that our average in this respect is higher for the presence among us of those of our colleagues and pupils, however few they may be, who consciously look for the wider significance of the standards with which their mathematical experience has familiarized them.

Mathematics not only demands straight thinking; it grants the student the satisfaction of knowing when he is thinking straight. There is no other field in which he can so soon stand on his own feet and face the world with confidence, assured that on his own ground his authority is final. Not many of your students, to be sure, ever reach this stage. Most of them keep on running to you with each successive bit of work, to ask you if it is right. But now and then, with your encouragement and guidance, one of them comes of age, having learned the meaning of self-criticism without self-distrust, and takes his place among those who know whereof they speak. He is potentially a man fit to exercise authority, with a full realization of the responsibility that goes with it.

A distinguishing characteristic of mathematics is its universality, its independence of time, place, and circumstance. The philosophers have made a great deal of this. They have ascribed to mathematical truth an absolute quality, transcending human experience and human existence. Perhaps they are right; nobody can prove the contrary. The alleged absoluteness of mathematics has always made it a refuge for souls who wanted to shut themselves away from the world. Probably the science always will owe much to those who see in it an escape from human limitations. But the universal persistence of mathematics has an appeal also for those of us who can not grasp so profound a metaphysical formulation. Even if we cannot attain anything beyond human experience, there is a compelling power in processes of thought which we know we have in common with intelligent men of all races and all times. Throughout the bewildering diversity of human nature, we have in the recognition of the facts of mathematics something as dependable and predictable as any phenomenon in the physical universe. By the exercise of our mathematical faculties we are brought into community of experience with Thales and Newton and their every-day associates, and their successors whose great-grandfathers have not yet been born. To this extent, the universal brotherhood of man is not a hypothesis or an aspiration, but an intimately accessible fact.

Still another phrase that needs to be rescued from abuse and restored to its proper place of dignity is "the enlargement of human knowledge." The words cover, if not a multitude of sins, at any rate a multitude of futilities. I may take this occasion to report, as a result of my own research (executed by a few seconds' manipulation of a calculating machine), that the product of the numbers 63471928 and 84023963 is 5333162929810664. I am confident that this fact is not to be found in the literature, and that it was never known until I

discovered it. If there is exalted merit in such an addition to knowledge, I am ready to propose an unlimited number of topics for investigation.

But the point that I really want to make is that this *reductio ad absurdum* is as far from being final as the shibboleth against which it is directed. Collectively and in the long run we are finding out things that the world wants to know, however ephemeral most of our individual contributions may be. To mention specific instances, the methods in exterior ballistics developed by Moulton and Bliss during the war depend on the Picard process of successive approximations and the introduction of an adjoint system of differential equations, familiar products of modern mathematical research. The preliminary work on which the theory of relativity is based was as much mathematical as physical. The quantitative relations which make the theory of statistics forbiddingly abstruse to the non-mathematician are commonplaces in the study of orthogonal functions and approximation by means of them. While the inventors of the statistical measures in question do not appear to have been extensively familiar with this particular part of the mathematical background, and while their achievement is all the more remarkable on that account, the general theory is available for the enlightenment of the student.

More fundamentally, the calculus is an essential part (one hesitates to say an integral part) of our civilization, and it is the research of the last hundred years that has given us a real understanding of the meaning of limits, and assured us that the processes of the calculus will bear critical examination. I believe that in fifty years more we shall have achieved a similar clarification of the concepts of probability. As in the case of the notion of limit, which can be defined in thirty seconds, but can be really appreciated, even by gifted graduate students, only after months of study, and took decades in reaching general acceptance, it will not be a question merely of a few formulations and theorems, but of years of accumulated experience, the gradual suppression of what is irrelevant and the emergence of what is essential. Incidentally, in going about our business toward the attainment of this end, we shall not only be offering the products of our own study, but shall be bringing together the biologist, the psychologist, the economist, and the physician, and making available to each the common elements in the experience of all.

No less important perhaps than what we contribute to the technique of the natural sciences is what we learn directly about the workings of our own minds. When the attempt has been made to analyze the processes of rational thought, the case of mathematical reasoning has always been cited as typical. Our activity is a continuing laboratory experiment, which is no nearer finality than the rest of the science of psychology, and I believe there is much still to be learned from it. We gradually recognize, for example, that the achievement of a mathematical proof in our consciousness is an experience rather than an

act, and that clarification is as likely to accompany relaxation following effort as to result immediately from the effort itself; but we can only guess how far these observations may ultimately be of theoretical and practical importance. Again, in trying to trace the inheritance of mental traits, we are likely to come down to specific cases by looking for something that can be identified as mathematical ability. One does not expect that the relations will be direct and simple. On finding that the highest score in an examination in my calculus class was made by a sister of a girl who had been similarly distinguished in the corresponding class a year or two earlier, and being informed that a third sister was taking freshman mathematics, I inquired as to her progress there, and was told that she wasn't doing so well in trigonometry, but got a B+ in English for a theme about trigonometry as the course that she disliked most. But however individual manifestations vary, it may turn out that there are fundamental differences in the chemical constitution of the nervous system which make sustained intellectual concentration practicable for one person, and insuperably distasteful to another. And whatever the answer may be, we are as likely as anybody to have something to offer toward the solution.

On the plane of social rather than individual psychology, there is a fascinating subject of inquiry in the relation between mathematical advances and the general consciousness of the age that produces them. Is there any underlying reason why Newton should have been a product of the age following Shakespeare, and Gauss a contemporary of Beethoven? Conversely, the main currents of mathematical progress must continue to have a profound influence on the habits of thought of the race. Who can estimate how much the emancipation of the intellect owes to the putting of the circle-squarer in his proper place, and the evidence of Gauss, Bolyai, Lobachewsky, and Riemann that the authority of Euclid did not finally seal the book of knowledge? The modern mathematician, like his predecessors, has done his part for the liberation of the human spirit.

In conclusion, let me emphasize that the advancement of mathematics is a great human enterprise in which we are all participants, and not mere spectators. You are making a contribution if you carry through a piece of research, or if you accept a committee appointment which leaves a colleague free to carry on his research, or if you study what he has done and give him the benefit of your appreciation and criticism, or if you merely work over the existing material and add to the fund of experience from which comes the collective assurance that we are on the right track. The science shares the vitality of all men in whose consciousness it exists.

A GEOMETRICAL PROBLEM IN WHICH ORDER OF CHOICE IS IMPORTANT

By W. C. GRAUSTEIN, Harvard University

Order of choice is frequently an essential element in analysis. It is only rarely, however, that it occurs in geometry. Recently the author encountered an instance of it which is of unusual interest because of the importance of the geometrical question at issue.

The question is simple to formulate. Can a collineation between two distinct planes in space always be established by a single projection? In other words: if a projective correspondence between the two planes is given, is it always possible to move the planes rigidly into new positions in which the lines joining corresponding points go through a point or are parallel?

The question, thus formulated—in the usual way, is enigmatic. It cannot be answered categorically with either "yes" or "no." It pertains to projective geometry and yet it is not purely projective in that a metric geometry must either be at hand or be introduced in the space in order to make rigid motions possible. From a historical point of view, it belongs to the hybrid class of questions originating during the period in which projective geometry was not as yet divorced from metric elements. The answer to it depends on whether it is considered from this essentially metric point of view or from the point of view of pure projective geometry.

From the metric point of view we start with an extended Euclidean space, i.e., with a projective space in which a definite Euclidean geometry has been introduced, and consider in this space an arbitrary projective correspondence between two arbitrarily chosen distinct finite planes. Our question must then be answered in the negative.

From a purely projective point of view, the point of departure is a projective space in which is given the projective correspondence between the two planes. Here the metric is introduced into the space *subsequent* to the projective correspondence, and it is always possible to choose it so that our question can be answered affirmatively.

In establishing these statements it will be necessary to recall briefly the salient metric properties of collineations.[1]

Metric point of view. In this case our space is metric from the start, and we choose in it arbitrarily a collineation between two finite planes, M and M'. It is convenient to distinguish three cases, according as the collineation is non-affine, affine but not conformal, or a transformation of similarity.

The only projections in metric space which establish non-affine collineations

[1] For an exhaustive treatment of these properties, the reader is referred to Sturm, *Die Lehre von den geometrischen Verwandtschaften*, II, §§40, 42.

are the central projections of one of two intersecting planes on the other. Hence, a necessary condition that it be possible to establish a given non-affine collineation between the planes M and M' by a single projection is that there exist at least one finite line in M which is transformed as by a rigid motion. This condition is also sufficient. If L and L' are lines in M and M' which correspond so that distance is preserved, the plane M' can be moved rigidly so that L' coincides with L and every point of L' coincides with the corresponding point of L. If, then, A, A' and B, B' are two distinct pairs of corresponding points, not lying on L, the corresponding lines AB and $A'B'$ intersect on L and hence the lines AA' and BB' meet in a point Q. The projection from Q of M on M' establishes between M and M' the given collineation.

As a matter of fact, there are two distinct finite lines in M which are transformed as by a rigid motion by a non-affine collineation. If the non-affine collineation, expressed in terms of homogeneous metric coordinates in the two planes, has the equations

$$x_i' = \sum_{j=1}^{3} a_{ij}x_j, \qquad (i = 1,2,3) \quad |a_{ij}| \neq 0,$$

they are the lines

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = kx_3,$$

parallel to the vanishing line, for which[2]

$$(a_{31}^2 + a_{32}^2)k^2 = (A_{13}^2 + A_{23}^2),$$

where A_{ij} is the cofactor of a_{ij} in $|a_{ij}|$.

THEOREM 1. *A non-affine collineation between the points of two distinct finite planes in metric space can be established by a central projection of one of two intersecting planes on the other. Conversely, every projection of this type establishes a non-affine collineation.*

If the given collineation is a transformation of similarity, the facts are obvious.

THEOREM 2. *A transformation of similarity of one of two distinct finite planes in metric space into the other can be established by a projection of one of two parallel planes on the other, and conversely.*

There remain, on the one hand, the collineations which are affine but not conformal, and on the other hand, the parallel projections of one of two intersecting planes on the other. It is true that a parallel projection which relates to one another two intersecting planes establishes a non-conformal affine trans-

[2] Since we are restricting ourselves to the geometry of reals, neither $a_{31}^2+a_{32}^2$ nor $A_{13}^2+A_{23}^2$ can vanish.

formation. But it is *not* true that every non-conformal affine transformation can be established by a single projection. Herein lies the reason for the fact that when our question is considered from the metric point of view, it must be answered in the negative.

It is evident that a necessary, as well as sufficient, condition that it be possible to establish a given non-conformal affine transformation between M and M' by a single projection is that there exist at least one line in M which is transformed so that distance is preserved. It is, moreover, well known that an affine transformation multiplies by the same constant factor, k, the lengths of all line-segments in M with the same direction. Hence, if C is a *unit* circle in M and E is the ellipse in M' into which C is carried, the factor k by which all distances in the direction of a radius of C are multiplied is precisely the length of the semi-diameter of E which corresponds to the radius of C. Thus, k varies from the semi-minor axis b to the semi-major axis a of the ellipse E:

$$b \leqq k \leqq a,$$

and therefore takes on the value 1 if and only if 1 lies in the closed interval from b to a.

THEOREM 3. *There are two, one, or no pencils of real parallel lines which are transformed by a non-conformal affine transformation so that distance is preserved, according as unity lies within, is an end point of, or lies without the interval from b to a, where b and a are the semi-axes of an ellipse which corresponds to a unit circle by the transformation.*

For the transformation

$$x' = a_1x + b_1y + c_1, \qquad y' = a_2x + b_2y + c_2, \qquad a_1b_2 - a_2b_1 \neq 0,$$

the value of k for the direction of slope-angle θ is found to be

$$k^2 = (a_1^2 + a_2^2)\cos^2\theta + 2(a_1b_1 + a_2b_2)\cos\theta\sin\theta + (b_1^2 + b_2^2)\sin^2\theta.$$

It is evident from this expression, and also geometrically, that non-conformal affine transformations actually exist for which k never takes on the value unity.

Projective point of view. Since, here, our space is in the beginning projective, and metric geometry is not introduced until after the projective transformation of M into M' is chosen, we are free to adapt the metric geometry to the given transformation. In particular, we take as the fixed plane—the plane at infinity—of the metric geometry a plane which meets M and M' in lines which do *not* correspond by the given transformation. These lines become, then, the lines at infinity in M and M', and the lines to which they correspond become vanishing lines. Consequently, the given projective transformation of M into M', considered with reference to the resulting metric space, is a non-

affine collineation. But, by Theorem 1, a non-affine collineation between M and M' can be established by a single projection.

THEOREM 4. *If a projective correspondence between the points of two distinct planes M and M' in projective space is given, and there is introduced in the space a metric geometry referred to a properly chosen fixed plane, then the planes M and M' can be moved rigidly into new positions in which the lines joining corresponding points go through a point.*

STEREOGRAPHIC PROJECTION OF A QUADRIC

By CHARLES A. RUPP, Pennsylvania State College

1. Introduction. By projecting plane configurations stereographically onto a quadric, we obtain a number of interesting configurations which deserve to be better known. Thus we rediscover the theorem of Hesse[1] that a Pascal hexagon upon a quadric is likewise a Brianchon hexagon, and readily find several other interesting theorems. This paper aims to show the ease with which theorems are transferred from the plane to a quadric, and to give independent proofs of some of the outstanding theorems thus attained.

Klein[2] was one of the first to use stereographic projection as a means of finding new theorems, and Segre,[3] in his laureate thesis, suggests that the method is apt to be fruitful in a general study of configurations upon hyperquadrics. The present paper is a part of such a general study, but is restricted to non-degenerate quadrics in S_3.

2. Coordinate Machinery. We shall adopt the notation of Bertini[4] for the important spaces in the projection. For the center of the stereographic projection take a point P on the quadric ϕ, and project the points of the quadric upon a plane π which does not pass through the point P. We denote by τ the tangent plane to the quadric ϕ at the point P, and by σ the line of intersection of the planes π and τ. The intersection of ϕ and its tangent plane τ is the pair of rulings through the point P, and is a degenerate conic called ϕ'. The line σ also meets ϕ in a degenerate conic, called ϕ'', which is the pair of points A and B. We have given names to three quadratic spaces, ϕ, ϕ', and ϕ'', and to three linear spaces, π, τ, and σ. We shall see their significance as soon as we have set up the equations of the projection.

Let the coordinates of the points P, A, and B be respectively (0, 0, 0, 1),

[1] Hesse, *Werke*, p. 58; p. 651; p. 676.

[2] Klein, *Ueber einen liniengeometrischen Satz*, Mathematische Annalen, vol. 22 (1883), p. 325.

[3] Segre, *Studio sulle quadriche*, Mem. dell'Acad. di Torino, vol. 36 (1884).

[4] Bertini, *Proj. Geom. Mehrdimensionaler Räume*, Chapter VI, Section 16.

(1, 0, 0, 0), (0, 0, 1, 0), and for the fourth vertex of the coordinate tetrahedron choose the point C common to the second rulings of ϕ which go through A and B. This tetrahedron is both inscribed in and circumscribed to the quadric ϕ. By a proper choice of the unit point, we can reduce the equation of ϕ to the form $x_0x_2 - x_1x_3 = 0$. The equations of the linear spaces π, τ, and σ are respectively

$$x_3 = 0, \quad x_1 = 0, \qquad x_3 = x_1 = 0.$$

Recalling that ϕ' is the intersection of ϕ and τ, we have

$$\left.\begin{aligned} x_0x_2 - x_1x_3 &= 0 \\ x_1 &= 0 \end{aligned}\right\} \text{ or } \begin{cases} x_1 = x_2 = 0 & (PA) \\ x_1 = x_0 = 0 & (PB). \end{cases}$$

ϕ'' was defined as the intersection of ϕ and σ; hence

$$\left.\begin{aligned} x_0x_2 - x_1x_3 &= 0 \\ x_1 = x_3 &= 0 \end{aligned}\right\} \text{ or } \begin{cases} x_1 = x_3 = x_2 = 0 & (A) \\ x_1 = x_3 = x_0 = 0 & (B). \end{cases}$$

3. Equations of Projection. The projection going from ϕ to π is straightforward. A point M of ϕ of coordinates (y_0, y_1, y_2, y_3) has for its image a point M' of π of coordinates $(y_0, y_1, y_2, 0)$. It is on the return trip from π to ϕ that we first encounter irregularities. The transform of a point M' of π of coordinates $(x, y, z, 0)$ is found by getting the piercing points on ϕ of the line PM', one of which is P and the other is the true transform, the point M of ϕ. We find

$$x_0 : x_1 : x_2 : x_3 = yx : y^2 : yz \cdot xz$$

which are the equations of the stereographic projection.

If $y=0$, we get only the point P, no matter what point on the line we take. This line is the line σ, and is a fundamental line for the projection, its transform being the fundamental point P. If, besides $y=0$, we also have x or $z=0$, the transformation becomes indeterminate. This means that the points A and B are fundamental points in the plane π. Let us examine more closely what happens to A.

Consider a line S_1' of π through A, of equation $y=\lambda z$. The transform of a point on this line has coordinates

$$\lambda zx : \lambda^2 z^2 : \lambda z^2 : xz = \lambda x : \lambda^2 z : \lambda z : x.$$

From this we see that as the point M' moves up the line S_1' towards A, the transform on ϕ approaches the point of coordinates $(\lambda, 0, 0, 1)$, which is a definite point on the line $x_1=x_2=0$, or PA. Thus the fundamental point A is transformed into the fundamental line PA, and in such a way that we attain a definite point of PA as we approach A in a definite direction. In like manner the transform of the fundamental point B is the fundamental line PB. Thus

we have seen that the transform of the quadratic space ϕ'' is the quadratic space ϕ'.

It is readily shown that a point on the fundamental line σ comes from a tangent to the quadric ϕ at P, which the Italian geometers would express by saying that the transform of a point on σ is a point on ϕ "infinitesimally close" to P.

What is the transform of the line S_1' considered above? The equation of its projecting plane is $x_1-\lambda x_2=0$, which meets ϕ in a conic which breaks up into the line $x_1=x_2=0$, or PA, and the line S_1, which we may regard as the true transform of S_1', whose equations are $x_1-\lambda x_2=0$ and $x_0-\lambda x_3=0$. The transform of any line in π through one of the points of ϕ'' is a generator of the quadric ϕ. It readily follows, as Klein and Clebsch have pointed out, that the net of rulings on ϕ is projected into the net of the two pencils of lines of π whose vertices are at the fundamental points of the plane. Since there are but two of these fundamental points, there are but two families of lines upon the quadric. The well known properties of these families, that two rulings of the same family never meet, and that two rulings of different families always meet, follow at once from the properties of the two pencils in π.

It may be interesting to note here that in the more general theory of hyperspace, any S_m' of π which contains an S_{m-1} of ϕ' is the image of a linear space S_m of ϕ.

4. Theorems. The transform of the line joining a pair of points of π is a conic through P and the transforms of the points. A pair of lines of π which meet in a point M' are the images of a pair of conics on ϕ through P, which also meet at the point M. The transform of a triangle of π is a configuration upon ϕ which we may call a *triconic*. A triconic is a set of three conics on ϕ passing through the point P. The three conics are termed the sides of the triconic, and each pair has a point in common (besides P), which defines the three vertices of the triconic. In like manner, the transform of a hexagon of π is a set of six conics and six points of ϕ which we call a *hexaconic*. A conic of π will correspond to a quartic on ϕ which has a double point at P, whose nodal tangents will be fixed by the points where the fundamental line σ intersects the original conic of π. The image of a general conic on ϕ is a conic of π through the fundamental points A and B.

With these preliminaries behind, let us consider some of the classic configurations of the plane. Desargues's theorem on triangles says that two triangles which are perspective from a point are also perspective from a line, and conversely. When we project these two triangles up onto the quadric, we obtain

THEOREM 1. *If two triconics on ϕ are such that the intersections of corresponding conics lie on a conic through the point P, then the conics joining pairs of op-*

posite vertices meet in P and another point. There are a number of special cases of this theorem, arising from the various relations which the Desargues line may have to the fundamental points and line of π. We shall state one corollary that occurs when the Desargues line and σ coincide.

COROLLARY 1. *If two triconics on ϕ are such that corresponding sides share the same tangent at P, then the conics joining pairs of corresponding vertices meet in P and still another point.*

Pascal's theorem states that the necessary and sufficient condition that the vertices of a hexagon lie upon a conic is that the points of intersection of opposite sides be collinear. This leads to

THEOREM 2. *The necessary and sufficient condition that the vertices of a hexaconic lie upon a quartic on ϕ through P is that the points of intersection of opposite conics lie on a conic of ϕ through P.*

A special case of this theorem is when the hexaconic degenerates to a hexagon. Consider a [skew] hexagon upon ϕ, whose sides we call 1, 2, 3, 4, 5, 6. It will be convenient to number the vertices also, the scheme being that point 1 is the intersection of lines 1 and 2, $\cdots$, point 6 is the intersection of sides 6 and 1. Clearly lines 1, 3, and 5 belong to one family of rulings of ϕ, and the other sides to the other family. Since we plan to investigate this configuration in some detail, we take the rulings 1, 2, 4, and 5 to be the edges of the coordinate tetrahedron CA, AP, CB, and BP, and let the rulings 3 and 6 pass through the unit point, which we know to be on ϕ. We shall denote by primes the images of these points and lines, and we shall study the configuration we get by projection upon π to see if it gives us any information about the corresponding configuration on ϕ. It is this systematic use of stereographic projection, both to arrive at the knowledge of the existence of certain configurations of linear spaces upon a quadric, and to ascertain the properties of given configurations upon the quadric, which is the *raison d'être* of this paper.

When we project upon π the skew hexagon formed of six rulings of ϕ, we see that the points $1'$ and $2'$, the images of two vertices of the skew hexagon, coincide at the point A, and in like manner, $4'$ and $5'$ coincide at B. The image hexagon is accordingly degenerate. It is a Brianchon hexagon, for its sides are tangent to the conic ϕ'', and it is also a Pascal hexagon, for its vertices lie on the conic formed by the pair of lines CA and CB. Thus it has a Pascal line and a Brianchon point, and since this projected hexagon has shrunk to a complete quadrangle, it turns out that the Brianchon point is a vertex of the diagonal triangle, and the Pascal line is the opposite side. In the plane π, then, the Brianchon point and Pascal line are pole and polar with respect to both the Brianchon conic ϕ'' and the Pascal conic.

Let us now give a couple of definitions. A Pascal skew hexagon is one whose pairs of opposite planes meet in three coplanar lines, this plane being the Pascal plane of the hexagon. A Brianchon skew hexagon is one whose diagonals have a point in common. We may now return to the quadric and utilize the information we gleaned from affairs in π.

THEOREM 3. *A skew hexagon upon a quadric is both a Pascal and a Brianchon hexagon. The Brianchon point and the Pascal plane are polar with respect to the quadric.*

This theorem was discovered some eighty-five years ago by Hesse[1], whose attack, if we may judge by his proof, was purely synthetic. Since we have nearly all the material needed for a direct analytic proof, we go on to make one.

Upon the quadric ϕ of equation $x_0x_2 - x_1x_3 = 0$ we have taken six generators, of equations

$$\begin{array}{llll} 1 & x_2 = x_3 = 0 & 2 & x_1 = x_2 = 0 \\ 3 & x_1 = x_2,\ x_0 = x_3 & 6 & x_0 = x_1,\ x_2 = x_3 \\ 5 & x_0 = x_1 = 0 & 4 & x_0 = x_3 = 0. \end{array}$$

These six rulings meet in nine points, whose coordinates are

$$\begin{array}{lllllll} 1 & (1,0,0,0) & 2 & (1,0,0,1) & 3 & (0,1,1,0) \\ 4 & (0,0,1,0) & 5 & (0,0,1,1) & 6 & (1,1,0,0) \\ 7 & (0,0,0,1) & 8 & (0,1,0,0) & 9 & (1,1,1,1) \end{array}$$

where the first six points have already been defined as vertices of the hexagon, and the last three points are P, C, and U.

We proceed to verify that the hexagon has a Brianchon point. The pairs of opposite vertices are the points 1, 4; 2, 5; and 3, 6. The point whose coordinates are (1, 0,–1, 0) is clearly a linear combination of the coordinates of the points taken in pairs; hence this point is on the three diagonals, and is the Brianchon point of the hexagon. The pairs of opposite planes are the planes fixed by the pairs of lines 12 and 45. We shall designate these six planes of the hexagon by the lower number in the pair of sides defining them. For the lines of intersection,

$$\left\{\begin{array}{l} \text{Plane } 1 \quad x_2 = 0 \\ \text{“ } \quad 4 \quad x_0 = 0 \end{array}\right. \quad \left\{\begin{array}{l} \text{Plane } 2 \quad x_1 - x_2 = 0 \\ \text{“ } \quad 5 \quad x_1 - x_0 = 0 \end{array}\right. \quad \left\{\begin{array}{l} \text{Plane } 3 \quad x_0 - x_3 = 0 \\ \text{“ } \quad 6 \quad x_2 - x_3 = 0, \end{array}\right.$$

which lines are precisely those fixed by the points 78, 97, and 89. It is obvious from the displayed equations that the plane $x_0 = x_2$ contains all three lines, and hence is the Pascal plane.

[1] Hesse, loc. cit.

The polar plane of y with respect to ϕ has the equation

$$y_2x_0 - y_3x_1 + y_0x_2 - y_1x_3 = 0.$$

The polar plane of the point (1, 0, 1, 0) is $x_0-x_2=0$. This completes the analytic proof of the theorem.

We note in passing that the line joining opposite vertices of the skew hexagon is the conjugate of the line of intersection of opposite planes, which follows from the self-duality of the hexagon.

Since generators of the same family do not meet, we can arrange the six rulings into only six hexagons, not sixty. These six are obtained by holding three alternate sides fixed, and permuting the others. In the plane π we have, then, two conjugate Steiner points, which happen, as a glance at the figure of the complete quadrangle in π shows us, to be points already obtained, namely, the points of coordinates $(1, 0, -1)$ and (1, 1, 1). The six Brianchon points are likewise arranged by threes, upon two lines whose equations are seen to be $x-y+z=0$ and $x-z=0$. We expect that there will be some relations between the Pascal planes and the Brianchon points of the skew hexagons, but we cannot go back directly from π because the points and planes in the space figure are not on ϕ. We shall have to find the coordinates of the Brianchon points and the Pascal planes directly from the knowledge of the sides of the hexagons.

The six possible hexagons have their sides arranged in the orders

1 2 3 4 5 6,	1 4 3 6 5 2,	1 6 3 2 5 4,
1 6 3 4 5 2,	1 2 3 6 5 4,	1 4 3 2 5 6,

and their vertices are the points

1 2 3 4 5 6,	8 3 9 5 7 1,	6 9 2 7 4 8
6 9 3 4 7 1,	1 2 9 5 4 8,	8 3 2 7 5 6.

The six Brianchon points have the coordinates

(1,0,−1,0),	(0,1,1,1),	(1,1,0,1),
(1,1,1,0),	(1,0,1,1),	(0,1,0,−1),

and the six Pascal planes have the equations

$$x_0 - x_2 = 0, \quad x_0 - x_1 - x_3 = 0, \quad x_1 - x_2 + x_3 = 0,$$
$$x_0 + x_2 - x_3 = 0, \quad x_0 - x_1 + x_2 = 0, \quad x_1 - x_3 = 0.$$

The first three Brianchon points are seen to lie on a line whose equations are $x_0-x_1+x_2=x_1-x_3=0$, and this same line lies on each of the second trio of Pascal planes. The second triple of Brianchon points lies on the line whose equations are $x_0-x_2=x_1-x_2+x_3=0$, and through this line pass the first three Pascal planes. These lines are obviously conjugate with respect to ϕ. They may be

called Steiner lines, for they are common to the triples of Pascal planes. We may now collect our results in the form of

THEOREM 4. *If six rulings of a quadric form a skew hexagon, they form six hexagons, each of which has a Pascal plane and a Brianchon point which are pole and polar with respect to the quadric. The six Pascal planes pass by threes through two polar lines, and the six Brianchon points lie by threes on the same two lines.*

FAMILIES OF CURVES OF PURSUIT, AND THEIR ISOCHRONES

By ALFRED J. LOTKA, Statistical Bureau, Metropolitan Life Insurance Company

A family of curves of pursuit may be defined by the condition that all curves of the family shall meet the path of the pursued point in the same fixed point.

Let P_1 be the point pursued, which moves with a constant velocity v_1 and in a straight line P_1O. Let P_2 be the pursuing point, which moves always in the direction P_2P_1, with a constant velocity v_2. If $v_2 > v_1$, P_2 will ultimately catch up with P_1 at some fixed point O. We may speak of O as the point of capture.

Now we may ask, for a given ratio $v_1/v_2 = \epsilon$, and a given direction of v_1, what is the locus of all those *positions at time t* of the pursuing point, which will result in capture at the given point O?

The problem can be solved by applying the equation of curves of pursuit and expressing the constants of integration in terms of the initial configuration and velocity components. This was the method originally followed by the writer. A briefer and more direct method communicated to him by Professor F. R. Sharpe is the following:

Denoting by x, y, the rectangular coördinates of P_2, with the origin at O and the x axis along OP_1, and denoting by s the length of the curve of pursuit from O to P_2, we have, evidently,

$$x = \epsilon s - \rho \cos \phi \qquad y = \rho \sin \phi$$

where ϕ is the angle P_2P_1O, and ρ is the radius vector P_1P_2. Differentiating we have:

$$(dx/ds) = \epsilon - (d\rho/ds) \cos \phi + \rho \sin \phi (d\phi/ds) = -\cos \phi,$$

$$(dy/ds) = \qquad (d\rho/ds) \sin \phi + \rho \cos \phi (d\phi/ds) = \sin \phi.$$

Solving we have

$$(d\rho/ds) = 1 + \epsilon \cos \phi, \qquad \rho(d\phi/ds) = -\epsilon \sin \phi.$$

Hence

$$(1 - \epsilon \cos \phi)(d\rho/ds) + \epsilon\rho \sin \phi (d\phi/ds) = 1 - \epsilon^2.$$

Integrating,

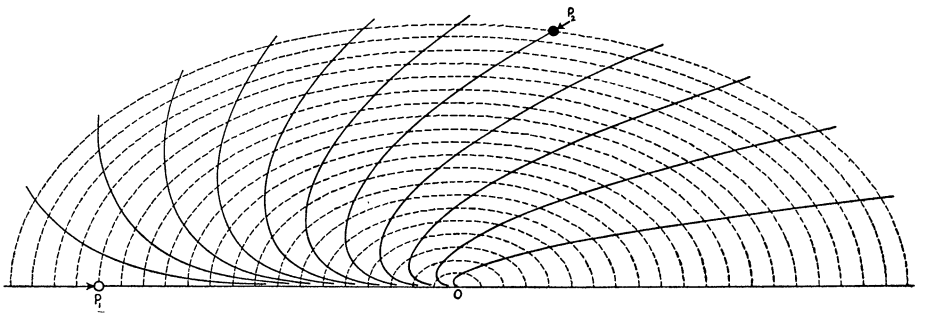
$$(1 - \epsilon \cos \phi)\rho = (1 - \epsilon^2)s \quad \text{and} \quad \epsilon s = \frac{\rho\epsilon}{1 - \epsilon^2}(1 - \epsilon \cos \phi) = D.$$

For a given value of D, and regarding ρ, ϕ, as polar coördinates of the point P_2, this represents an ellipse of eccentricity ϵ or v_1/v_2, and with its center at O and focus at P_1, a distance D from O. But the pursuing point, which at time t is at P_2, will ultimately meet the pursued point at O. Evidently this will take place at time $t+(D/v_1)$. In other words, points which at time t lie on the ellipse defined as above, will, after a lapse of time D/v_1, all simultaneously converge in the fixed point O. Or, conversely, all points on curves of pursuit of the character defined, which terminate at the fixed point O, lie, *at time t*, on an ellipse of eccentricity ϵ, with center at O, and one focus at the pursued point P_1. Thus the ellipses so defined are the loci of positions of equal time on the family of curves of pursuit. We may term these loci *isochrones*.

If we denote by the term *centrocline* the locus of all points in a family of curves, at which the slopes of the curves pass through a single point or *center*, it is clear that the *isochrone* ellipses are also the *centroclines* of the family of curves described, the corresponding *center* being that focus of the ellipse, at which the moving point is located.

This suggests a graphic method of mapping the family of curves of pursuit, analogous to the mapping of integral curves by means of isoclines.

A map of the family of curves of pursuit so plotted is shown on the accompanying diagram. The process hardly needs description. It may be remarked, however, that with a little resourcefulness in manipulation (using a loop of thread—fishing line is best because "inextensible"—and four pins, two at the focus, and two to establish "setting" points) a complete map of many curves can be drawn in less time that would be required to plot a single curve of pursuit by a point-by-point method.



Point P_2 moving with constant velocity v_2 and starting anywhere on outer ellipse, "pursues" point P_1 which starts at one of the foci, with constant velocity v_1. Capture takes place always at the same point O, the center of the ellipse. The concentric ellipses are *isochrones*, the curves of pursuit their *isoclinal trajectories*.

The following theorems are obviously implied in the developments set forth above, or follow by simple deduction:

THEOREM: *The isochrones of a family of curves of pursuit defined by a fixed point of capture, a pursued point moving with constant rectilinear velocity* v_1 *and a pursuing point moving with constant velocity* v_2 *directed always toward the pursued point,* are ellipses of eccentricity v_1/v_2, *with center at the fixed point of capture, and with one focus at the pursued point.*

THEOREM: *The isochrones of the family of curves specified are also its centro-clines.*

THEOREM: *If the center of an ellipse of eccentricity* $v_1/v_2 = \epsilon$ *remains fixed while one focus approaches this center at a constant rectilinear velocity* v_1, *and if at the same time each point of the ellipse "pursues" the moving focus at a constant velocity* v_2, *then the ellipse, while shrinking continually, retains its eccentricity* v_1/v_2 *until it has completely shrunk into a point at the fixed center.*

THEOREM: *The foco-clinal trajectories*[1] *of a family of concentric ellipses of equal eccentricity make everywhere equal intercepts between two such ellipses.* (This theorem should be visualised by inspection of the figure, preferably with a pair of compasses handy. Where the curves of pursuit are relatively flat, the compasses give immediate evidence of the truth of the theorem.)

The case here discussed has a certain concrete background. We may view the matter in the following light: An organism P_1, of a species which serves as food to an organism P_2, becomes aware of the presence of the latter when at a distance ρ from the same, and immediately flees in a straight line to the nearest cover or refuge, distance D away. If at the moment of sighting, the polar co-ordinates ρ, ϕ of P_2 are such that

$$\rho > D(1 - \epsilon^2)/\epsilon(1 - \epsilon \cos \phi),$$

then the fleeing organism will escape, reaching cover before its pursuer. If on the other hand

$$\rho < D(1 - \epsilon^2)/\epsilon(1 - \epsilon \cos \phi),$$

then capture will occur. In other words, capture will or will not occur, according as the organism P_2 is inside or outside the ellipse

[1] This term seems self-explanatory. The curves of pursuit form a family which cross the ellipses everywhere at such an angle that the tangent at any point to a curve of pursuit passes through the focus of the ellipse crossed at that point. Thus the curves of pursuit are trajectories of the ellipses, inclined, at each crossing point, towards the focus.

$$\rho = D(1 - \epsilon^2)/\epsilon(1 - \epsilon \cos \phi)$$

at the time it is sighted by the pursued organism.

The present communication is of the nature of a preliminary step toward a more complete analysis of certain biological problems on which work is in progress.

AN EXPRESSION FOR THE SUMMATION $\sum_{m=1}^{n} m^p$

By R. S. UNDERWOOD, Texas Technological College

1. In this paper I shall derive by induction an expression for the summation $\sum_{m=1}^{n} m^p$ as a polynomial in n, whose coefficients incidentally define the Bernoulli numbers in the beautifully symmetric quadratic recursion relation:

$$(2r + 3)\frac{B_{2r+2}}{(2r + 2)!} = \frac{B_2 B_{2r}}{2!(2r)!} + \frac{B_4 B_{2r-2}}{4!(2r - 2)!} \cdots + \frac{B_{2r} B_2}{(2r)!2!}.$$

This equation was thought to be possibly new by others who looked over my work as well as by myself until I located it in slightly different form in N. Nielsen's *Nombres de Bernoulli*. However, the method by which I arrived at this result is totally unlike all other treatments of the Bernoulli numbers I have found, and the constants k_r which appear naturally by this method are of basic interest in themselves, since the equation connecting them [see (2) below] is stripped even of the factorials which appear apparently in all Bernoulli number recursion relations.

THEOREM; *Let* $S^{(p)} = \sum_{m=1}^{n} m^p = 1^p + 2^p + \cdots + n^p$ (p *an integer*), *and let*

$$(1) \quad \begin{bmatrix} 1 \\ p \end{bmatrix} = p(p - 1) = \frac{p!}{(p - 2)!}; \quad \begin{bmatrix} r \\ p \end{bmatrix} = p(p - 1) \cdots (p - r) = \frac{p!}{(p - r - 1)!}.$$

Then

$$(A) \quad S^{(p)} = \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + k_1 p n^{p-1} - k_3 \begin{bmatrix} 2 \\ p \end{bmatrix} n^{p-3} + k_5 \begin{bmatrix} 4 \\ p \end{bmatrix} n^{p-5} - \cdots,$$

(*ending with the* n^2 *or* n *term*), *where the constants* k_r (r *odd*) *can be evaluated successively as follows*:

$$k_1 = \frac{1}{12}\,;\quad k_3 = \frac{1}{3+2}(k_1k_1) = \frac{1}{6!}\,;\quad k_5 = \frac{1}{5+2}(k_1k_3 + k_3k_1) = \frac{4}{3\cdot 8!}\,;\cdots$$

$$k_r = \frac{1}{r+2}(k_1\,k_{r-2} + k_3k_{r-4} + \ldots + k_{r-2}k_1)\,; \tag{2}$$

or, for quicker computation where r is large,

$$k_r = \frac{2}{r+2}\left(k_1k_{r-2} + k_3k_{r-4} + \cdots + k_{(r-5)/2}k_{(r+3)/2} + \frac{1}{2}k^2{}_{(r-1)/2}\right), \quad \text{when } r = 4n-1\,;$$

$$k_r = \frac{2}{r+2}\left(k_1k_{r-2} + k_3k_{r-4} + \cdots + k_{(r-3)/2}k_{(r+1)/2}\right), \quad \text{when } r = 4n+1. \tag{3}$$

The first nine values of k_r are

$$\frac{1}{12}, \frac{1}{6!}, \frac{4}{3\cdot 8!}, \frac{3}{10!}, \frac{10}{12!}, \frac{691}{15\cdot 14!}, \frac{280}{16!}, \frac{10{,}851}{5\cdot 18!}, \text{ and } \frac{438{,}670}{21\cdot 20!}.$$

These first nine constants enable one to write immediately the sums $S^{(p)}$ for every value of p up to and including 18. For instance,

$$\begin{aligned} S^{(18)} &= \frac{1}{19}n^{19} + \frac{1}{2}n^{18} + \frac{19}{12}n^{17} - \frac{2\cdot 17}{5}n^{15} + 34n^{13} - \frac{3\cdot 13\cdot 17}{5}n^{11} + \frac{5\cdot 13\cdot 17}{3}n^9 \\ &\quad - \frac{2\cdot 17\cdot 691}{5\cdot 7}n^7 + 2\cdot 3\cdot 7\cdot 17n^5 - \frac{3617}{2\cdot 5}n^3 + \frac{43{,}867}{2\cdot 3\cdot 7\cdot 19}n \\ &= \frac{1}{3990}[210n^{19} + 1{,}995n^{18} + 5{,}985n^{17} - 27{,}132n^{15} + 135{,}660n^{13} \\ &\quad - 529{,}074n^{11} + 1{,}469{,}650n^9 - 2{,}678{,}316n^7 + 2{,}848{,}860n^5 \\ &\quad - 1{,}443{,}183n^3 + 219{,}335n]. \end{aligned}$$

PROOF: Formula (A) yields a known result for $p=1$. Assuming it true up to and including $p=p$, with $\left[{r \atop p}\right]$ and k_r defined as in (1) and (2), we have

$$S^{(p)} = \frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + k_1pn^{p-1} - k_3\left[{2 \atop p}\right]n^{p-3} + \cdots \pm k_r\left[{r-1 \atop p}\right]n^{p-r} \mp \cdots. \tag{A}$$

Then

$$
\begin{aligned}
S^{(p+1)} &= 1\cdot 1^p + 2\cdot 2^p + 3\cdot 3^p + \cdots + n\cdot n^p \\
&= nS^{(p)} - \left[1^p + (1^p + 2^p) + (1^p + 2^p + 3^p) + \cdots \right. \\
&\quad \left. + \{1^p + 2^p + \cdots + (n-1)^p\}\right] \\
&= nS^{(p)} - \left(\frac{1}{p+1}1^{p+1} + \frac{1}{2}1^p + k_1 p 1^{p-1} - \cdots\right) \\
&\quad + \left(\frac{1}{p+1}2^{p+1} + \frac{1}{2}2^p + k_1 p 2^{p-1} - \cdots\right) \\
&\quad + \cdots + \left\{\frac{1}{p+1}(n-1)^{p+1} + \frac{1}{2}(n-1)^p + k_1 p (n-1)^{p-1} - \cdots\right\}\Bigg] \\
&= nS^{(p)} - \left[\frac{1}{p+1}(S^{(p+1)} - n^{p+1}) + \frac{1}{2}(S^{(p)} - n^p)\right. \\
&\quad \left. + k_1 p (S^{(p-1)} - n^{p-1}) - \cdots\right] \\
&= nS^{(p)} - \left[\left(\frac{1}{p+1}S^{(p+1)} + \frac{1}{2}S^{(p)} + k_1 p S^{(p-1)} - \cdots\right) - S^{(p)}\right].
\end{aligned}
$$

$$
\frac{p+2}{p+1}S^{(p+1)} = \frac{2n+1}{2}S^{(p)} - k_1 p S^{(p-1)} + k_3\begin{bmatrix} 2 \\ p \end{bmatrix}S^{(p-3)} - \cdots
$$
$$
\mp k_r\begin{bmatrix} r-1 \\ p \end{bmatrix}S^{(p-r)} \pm \cdots
$$

$$
S^{(p+1)} = \frac{p+1}{p+2}P(n),
$$

where $P(n)$ is a polynomial in n.

Making the further definition that $\begin{bmatrix} j \\ p_i \end{bmatrix} = (p-i)(p-i-1)\cdots(p-j)$ (i, j integers and $i \leqq j$), we may arrange $P(n)$ in columns of like powers of n as follows:

n^{p+2}	n^{p+1}	n^p	n^{p-1}	n^{p-2}	n^{p-3}	n^{p-r+3}	n^{p-r+2}	n^{p-r+1}	n^{p-r}	
$\frac{1}{p+1}$	$+\frac{1}{2}$	$+k_1 p$		$-k_2\begin{bmatrix} 2 \\ p \end{bmatrix}$		$+\cdots\cdots$	$\cdot$	$\pm k_r\begin{bmatrix} r-1 \\ p \end{bmatrix}$		$\mp\cdots$
	$+\frac{1}{2(p+1)}$	$+\frac{1}{4}$	$+\frac{k_1 p}{2}$		$-\frac{k_2}{2}\begin{bmatrix} 2 \\ p \end{bmatrix}$	$+\cdots\mp\frac{k_{r-2}}{2}\begin{bmatrix} r-3 \\ p \end{bmatrix}$			$\pm\frac{k_r}{2}\begin{bmatrix} r-1 \\ p \end{bmatrix}$	$\cdots$
$+k_1 p$	$\}$	$-\frac{1}{p}$	$-\frac{1}{2}$	$-k_1\begin{bmatrix} 1 \\ p_1 \end{bmatrix}$		$+\cdots\cdots$	$\cdot$	$\pm k_{r-2}\begin{bmatrix} r-2 \\ p_1 \end{bmatrix}$		$\mp\cdots$
$+k_3\begin{bmatrix} 2 \\ p \end{bmatrix}$	$\}$			$+\frac{1}{p-2}$	$\frac{1}{2}$	$+\cdots\cdots$	$\cdot$	$\pm k_{r-4}\begin{bmatrix} r-2 \\ p_3 \end{bmatrix}$		$\mp\cdots$
$+k_5\begin{bmatrix} 4 \\ p \end{bmatrix}$	$\}$					$-\frac{1}{p-4}-\cdots$	$\cdot$	$\pm k_{r-6}\begin{bmatrix} r-2 \\ p_5 \end{bmatrix}$		$\mp\cdots$
$\cdots\cdots$						$\cdots\cdots$	$\cdots$			
$\cdots\cdots$						$\cdots\cdots$	$\cdots$			
$+k_{r-2}\begin{bmatrix} r-3 \\ p \end{bmatrix}$	$\}$					$\pm\frac{1}{p-r+3}\pm\frac{1}{2}$		$\pm k_1\begin{bmatrix} r-2 \\ p_{r-3} \end{bmatrix}$		$\mp\cdots$
$+k_r\begin{bmatrix} r-1 \\ p \end{bmatrix}$	$\}$							$\mp\frac{1}{p-r+1}$	$\mp\frac{1}{2}$	$\mp\cdots$
$\vdots$										$\mp\cdots$

It will be noted that there is one positive and one equal negative term in n^{p-m} for every odd m; hence these terms disappear, and the exponents of all terms of $S^{(p+1)}$ except the second are even or odd (the last terms ending in n^2 or n) according as p is even or odd.

The coefficient of $n^{(p-r+1)}$ is

$$\pm \frac{p+1}{p+2}\left\{ k_r \begin{bmatrix} r-1 \\ p \end{bmatrix} + k_1 k_{r-2} \begin{bmatrix} r-2 \\ p \end{bmatrix} + k_3 k_{r-4} \begin{bmatrix} r-2 \\ p \end{bmatrix} + \cdots \right.$$

$$\left. + k_{r-2} k_1 \begin{bmatrix} r-2 \\ p \end{bmatrix} - \begin{bmatrix} r-2 \\ p \end{bmatrix} k_r \right\}$$

$$= \pm \frac{p+1}{p+2} \begin{bmatrix} r-2 \\ p \end{bmatrix} [k_r(p-r+1) + (k_1 k_{r-2} + k_3 k_{r-4} + \cdots + k_{r-2} k_1 - k_r]$$

$$= \pm \frac{p+1}{p+2} \begin{bmatrix} r-2 \\ p \end{bmatrix} [k_r(p-r) + (r+2) k_r], \text{ by } (2)$$

$$= \pm (p+1) \begin{bmatrix} r-2 \\ p \end{bmatrix} k_r.$$

the form required if (A) is to hold when p is replaced by $p+1$. Q.E.D.

REMARK: It may be worth noting that formula (A) readily gives the sum of any series of which the *nth* term is a polynomial in n. For instance, suppose $S(n) = 1 \cdot 3 \cdot 4 + 2 \cdot 4 \cdot 5 + \cdots + n(n+2)(n+3)$. Then $P(n) = n(n+2)(n+3) = n^3 + 5n^2 + 6n$ and

$$S(n) = \sum_1^n P(n) = S^{(3)} + 5S^{(2)} + 6S^{(1)} = \frac{1}{12}(3n^4 + 26n^3 + 69n^2 + 46n)$$

2. Relation of the k's to the Bernoulli numbers. Writing the result in the form,

$$\sum_{m=1}^{n} m^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + p! \sum_{r=1}^{[p/2]} \frac{(-1)^{r-1} k_{2r-1}}{(p+1-2r)!} n^{p-2r+1},$$

and comparing it with the known result,

$$\sum_{m=1}^{n} m^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{1}{p+1} \sum_{r=1}^{[p/2]} (-1)^{r-1} \binom{p+1}{2r} B_{2r} n^{p-2r+1},$$

where $\binom{p+1}{2r}$ is the usual symbol for a binomial coefficient, it may be seen that

$$(2r)! k_{2r-1} = B_{2r}, \text{ or } k_s = \frac{B_{s+1}}{(s+1)!} \ (s \text{ odd}).$$

This, together with the relation

$$(2r+3) k_{2r+1} = (k_1 k_{2r-1} + k_3 k_{2r-3} + \cdots + k_{2r-1} k_1)$$

which is what relation (2) becomes when r is replaced by $2r+1$, gives the Bernoulli number relation

$$(2r+3)\frac{B_{2r+2}}{(2r+2)!} = \frac{B_2B_{2r}}{2!(2r)!} + \frac{B_4B_{2r-2}}{4!(2r-2)!} + \cdots + \frac{B_{2r}B_2}{(2r)!2!},$$

from which the B's can be readily calculated, starting with $B_2 = 2k_1 = 1/6$.

SOME LINES AND CURVES ASSOCIATED WITH THE PLANE TRIANGLE

By ROSCOE WOODS, University of Iowa

Consider the triangle ABC. Let a, b, c, be the sides opposite the vertices A, B, C, respectively. Take a point L in the plane of the triangle and draw through L lines parallel to the sides a, b, c. These lines determine two points on each side of the triangle. Let P, P', Q, Q', R, R' be the points determined on the sides a, b, c, respectively, so that the parallel to c cuts a in P, the parallel to a cuts b in Q, etc.; and similarly the parallel to b cuts a in P', etc. It is known that when P, Q, R are collinear[1] the point L lies on an ellipse circumscribing the triangle ABC whose centroid is the center of the ellipse. The tangent to the ellipse at each vertex of the triangle is parallel to the opposite side. This ellipse is known as the ellipse of Steiner. A great many properties of this ellipse are known and have been listed.[2] It is easy to show that when P, Q, R are collinear so are P', Q', R' and conversely.

A similarity to the theorem concerning the pedal triangle of the point L is noticed here.[3] That is, L lies on a circle when its pedal triangle has a constant area but when this area is zero this circle becomes the circumcircle. In the present case if the triangle PQR has a constant area, L lies on an ellipse whose center is the centroid of the triangle ABC, but becomes Steiner's ellipse when the area is zero. If L be placed at the centroid of the triangle ABC, it is the centroid of the triangles PQR and $P'Q'R'$. If the triangle ABC is equilateral, the ellipse degenerates into the circumcircle of the triangle. On the other hand if the ellipse of Steiner is projected into a circle the triangle ABC projects into an equilateral triangle. This observation shows that the triangle ABC is a triangle of maximum area inscribed in the ellipse. Hence if we start with the ellipse whose equation is $a^{-2}x^2+b^{-2}y^2=1$, the coordinates of the vertices

[1] Rouché et de Comberousse, *Traité de Géométrie*, (1900 ed.) vol. 2, p. 662. Also Casey, *Analytic Geometry of the Point, Line and Circle* (1893), p. 451.

[2] Mathesis, (2), vol. 5, p. 42, p. 81; (2), vol. 7, p. 88; (4) vol. 1, p. 243.

[3] Coolidge, *A Treatise on the Circle and Sphere*, p. 49; Mathesis, (3), vol. 7, p. 99.

of the single infinity of triangles of maximum area inscribed in the ellipse are given by $a\cos(\theta+2n\pi/3)$, $b\sin(\theta+2n\pi/3)$ for $n=0, 1, 2$. The coordinates of the point L when on the ellipse are $(a\cos\phi, b\sin\phi)$. Many properties of this configuration are easily investigated since both parameters θ and ϕ occur in virtually all the equations with which we deal. If either parameter is fixed and the other varied many loci and envelopes are obtained. It is known that the lines PQR and $P'Q'R'$ envelop hypocycloids of three cusps when L generates the ellipse.[4] Their common point of intersection likewise generates a similar curve. The remainder of this paper gives the parametric equations of these curves and some of their properties and exhibits some other loci believed to be new in this connection.

The equations of the lines PQR and $P'Q'R'$ are, respectively,

$$(1)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta-\phi+(2\pi/3)}{2}\right]+(\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta-\phi+(2\pi/3)}{2}\right]+\sin\left[\frac{3\theta-3\phi-(2\pi/3)}{2}\right]=0,$$

$$(2)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta-\phi+(4\pi/3)}{2}\right]+(\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta-\phi-(4\pi/3)}{2}\right]+\sin\left[\frac{3\theta-3\phi+(2\pi/3)}{2}\right]=0.$$

Setting $\phi=\theta+(2n\pi/3)$ $(n=0, 1, 2)$ in (1) and (2), they furnish the sides of the triangle ABC.

These lines contain both the parameters θ and ϕ. Let θ be held constant and let ϕ vary; then the lines (1) and (2) envelop hypocycloids of 3 cusps. The parametric equations of these curves are found to be

$$(3)\quad \begin{aligned} x\sqrt{3}/a &= +\sin(3\theta-2\phi)+2\sin[\phi+(2\pi/3)],\\ y\sqrt{3}/b &= -\cos(3\theta-2\phi)-2\cos[\phi+(2\pi/3)]; \end{aligned}$$

$$(4)\quad \begin{aligned} x\sqrt{3}/a &= +\sin(3\theta-2\phi)+2\sin[\phi+(\pi/3)],\\ y\sqrt{3}/b &= +\cos(3\theta-2\phi)-2\sin[\phi+(\pi/3)]. \end{aligned}$$

That these are the equations of three-cusped hypocycloids may be shown by rotating the axes through the proper angle, for this will give rise to the standard form of the parametric equations of these curves.

[4] Mathesis, vol. 2, p. 42. M. Barbarin shows that, if the triangle ABC be projected orthogonally on a plane so chosen that it projects into an equilateral triangle, the lines PQR and $P'Q'R'$ project into generalized Simson lines. He treats this topic on p. 106 and p. 122 of this volume.

The coordinates of the cusps of these curves are

$$(5)\quad x/a = \sqrt{3}\sin[\theta + 2(3n+2)\pi/9],\quad y/b = \sqrt{3}\cos[\theta + 2(3n+2)\pi/9];$$

$$(6)\quad x/a = -\sqrt{3}\sin[\theta + 2(3n+1)\pi/9],\ y/b = +\sqrt{3}\cos[\theta + 2(3n+1)\pi/9];$$

where $n=0, 1, 2$. It is now evident that these points all lie on the ellipse whose equation is $a^{-2}x^2+b^{-2}y^2=3$ for all θ.

These hypocycloids pass through the vertices of the triangle ABC. One of them is tangent to the sides of the triangle at the vertices, that is, tangent to c, a, b at A, B, C, respectively; while the other is tangent to b, c, a at A, B, C, respectively. Easy analysis shows that the lines (1) and (2) are never parallel but are perpendicular when $-\cos(3\theta-\phi)=(a^2+b^2)/2(a^2-b^2)$. The angle $3\theta-\phi$ will be real if $a\geqq b\sqrt{3}$, which is also the condition that at least one of the triangles ABC may be a right triangle. It is clear that the lines (1) and (2) meet in a point whose coordinates are given by

$$(7)\quad x\sqrt{3}/a = 2\cos\phi + \cos(3\theta - 2\phi),\quad y\sqrt{3}/b = 2\sin\phi + \sin(3\theta - 2\phi).$$

These are the parametric equations of a three-cusped hypocycloid whose cusps are situated at the vertices of the triangle ABC. The cusp-tangents pass through the centroid of the triangle.

Consider ϕ constant and seek the envelopes of the lines (1) and (2). It is found that these lines revolve around the points whose equations are

$$(8)\qquad \sqrt{3}x/a = \cos(\phi + \pi/6),\quad \sqrt{3}y/b = \sin(\phi + \pi/6);$$

$$(9)\qquad \sqrt{3}x/a = \cos(\phi - \pi/6),\quad \sqrt{3}y/b = \sin(\phi - \pi/6).$$

From these we see that these points always lie on the ellipse whose equation is $a^{-2}x^2+b^{-2}y^2=1/3$.

The lines associated with the extremities of a diameter of the ellipse are considered next. They are found from (1) and (2) by replacing ϕ by $\phi+\pi$ They are

$$(10)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta - \phi - (\pi/3)}{2}\right] + (\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta - \phi - (\pi/3)}{2}\right] + \sin\frac{3}{2}[\theta - \phi + (\pi/9)] = 0,$$

$$(11)\quad (\sqrt{3})\frac{x}{a}\cos\left[\frac{3\theta - \phi + (\pi/3)}{2}\right] + (\sqrt{3})\frac{y}{b}\sin\left[\frac{3\theta - \phi + (\pi/3)}{2}\right] + \sin\frac{3}{2}[\theta - \phi + (5\pi/9] = 0.$$

The lines (1) and (10) meet in a point on the ellipse $a^{-2}x^2+b^{-2}y^2=1/3$. The lines (1) and (11) meet in the point given by

$$(12)\quad x\sqrt{3}/a=-(\sqrt{3})\cos(3\theta-2\phi)-2\sin\phi,\quad y\sqrt{3}/b=\sqrt{3}\sin(3\theta-2\phi)+2\cos\phi.$$

These are the parametric equations of a hypotrochoid. By rotation of axes this curve can be brought into a standard form and its double points easily found. It is evident that (2) and (11) meet in a point on the same ellipse as do lines (1) and (10); their point of intersection being diametrically opposite. Likewise it is clear that the lines (2) and (10) generate a second hypotrochoid.

If ϕ be replaced by $\phi+(\pi/2)$ in (1) and (2) we have lines, which we shall refer to as (13) and (14), associated with the adjacent extremities of a pair of conjugate diameters. The intersection of (1) and (13) give rise to the hypotrochoid whose equations are

$$(15)\qquad \begin{aligned} x\sqrt{3}/a &= -\cos(3\theta-2\phi)+(\sqrt{2})\cos[\phi+(5\pi/12)]\\ y\sqrt{3}/b &= -\sin(3\theta-2\phi)+(\sqrt{2})\sin[\phi+(5\pi/12)]. \end{aligned}$$

Again lines (1) and (14) meet on the same ellipse as do (1) and (10). Similar results are obtained when (2) and (13) and (14) are paired.

The sum of the eccentric angles of four concyclic points on an ellipse is a a multiple[5] of 2π. The fourth point D on an ellipse determined by the circumcircle of an inscribed triangle ABC of maximum area is known as Steiner's point.[6] The eccentric angle D is therefore $-3\theta+2n\pi$. A close resemblance between the equation of line (1) and the equation of the chord LD is readily seen. Hence considering these two lines together we find their common intersection to lie on a conic when θ varies and on a quartic curve when ϕ varies. This quartic has 3 or 1 real branches through D according as D lies within or without the hypocycloid (3). This quartic has branches that extend to the infinite region of the plane. Similar remarks apply to LD and (2). Consider the line $L'D'$ symmetrical to LD with respect to the x-axis. For variable ϕ the common intersection of (1) and $L'D'$ is a quartic curve lying in the finite part of the plane with 3 or 1 real branches through D' according as D' lies within or without the hypocycloid (3). For variable θ this intersection is a conic. For the line passing through the origin always parallel to $L'D'$, the quartic generated is similar to the three-leaved rose and becomes a three-leaved rose when the triangle is made equilateral. The conics in this case are ellipses with their centers on the ellipse $a^{-2}x^2+b^{-2}y^2=1/9$. There are evidently other interesting loci when other lines in the configuration are taken.

[5] C. Smith, *Conic Sections by the Methods of Coordinate Geometry* (1914), p. 169.

[6] Steiner, Gesammelte Werke, vol. 2, p. 691.

where $\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \beta_3$ are the coordinates of the points a_1, a_2, a_3. The equations of B_1a_1, B_2a_2, B_3a_3 are

$$x(x_3 - x_2) + y(y_3 - y_2) = x_1'(x_3 - x_2) + y_1'(y_3 - y_2),$$

.

.

whence

$$\alpha_1(x_3 - x_2) + \beta_1(y_3 - y_2) = x_1'(x_3 - y_2) + y_1'(y_3 - y_2)$$

.

.

The above set of equations then becomes

$$\begin{aligned} S_{ab} &= 2[\textstyle\sum x_1'(x_3 - x_2) + \sum y_1'(y_3 - y_2)] \\ &= 2[\textstyle\sum x_1(x_2' - x_3') + \sum y_1(y_2' - y_3')] = -S_{ba}. \end{aligned}$$

This proves the theorem.

3. Putting $S_{ab}=0$, we obtain the theorem (2) of §1.

It is seen from the preceding that the theorem offered by me belongs to the same category as the theorems which Mr. Paul Wernicke has given in his article on *The theorems of Ceva and Menelaus and their extension.*[1]

II. The Summation of a Class of Series

By Joseph D. Grant, University of Illinois

Many, doubtless, have noticed that the series for $\log_e 2$ is made up of alternate terms from the series:

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots.$$

Here it is proposed to show a simple means for summing certain series made up of terms from the series:[2]

$$(1) \qquad \sum_{x=0}^{\infty} \frac{1}{(ax+d)^{\lfloor n \rfloor a}} = \frac{1}{a(n-1)(ax+d)^{\lfloor n-1 \rfloor a}}\Bigg]_{x=0}, \quad a, \quad d \geqq 1, \quad n \geqq 2,$$

wherein $(ax+d)^{\lfloor n \rfloor a} = (ax+d)(ax+a+d)(ax+2a+d)\cdots(ax+na-a+d)$. Grouping the terms of (1) in sets of c terms each, one has

$$(2) \qquad \begin{aligned} \sum_{x=0}^{\infty} \frac{1}{(ax+d)^{\lfloor n \rfloor a}} &= \sum_{x=0}^{\infty} \sum_{k=0}^{c-1} \frac{1}{(acx+ak+d)^{\lfloor n \rfloor a}} \\ &= \sum_{k=0}^{c-1} \sum_{x=0}^{\infty} \frac{1}{(acx+ak+d)^{\lfloor n \rfloor a}}. \end{aligned}$$

[1] This Monthly, vol. 34 (1927), pp. 468–472.

[2] Chrystal's Algebra, (1920 edition), vol. 2, p. 403.

The series (1) is thus separated into c series which may be distinguished by four variable parameters, a, $ak+d$, c, n. Hence it is convenient to define

$$f(a,b,c,n) = \sum_{x=0}^{\infty} \frac{1}{(acx+b)^{|n|a}}, \text{ where } b = ak + d. \tag{3}$$

The purpose of this paper is to prove that

$$a^{n-1}(n-1)!\,f(a,b,c,n) = \int_0^1 v^{b-1} \frac{(1-v^a)^{n-1}}{(1-v^{ac})}\,dv, \quad n \geqq 2. \tag{4}$$

Restating (2) in terms of $f(a, b, c, n)$, we find

$$f(a,d,1,n) = \sum_{k=0}^{c-1} f(a,ak+d,c,n). \tag{5}$$

If one writes out and examines the general terms of the series it may be seen that

$$f(a,b,c,n) - f(a,a+b,c,n) = a \cdot n \cdot f(a,b,c,n+1). \tag{6}$$

Before proceeding to prove (4), it should be noted that the value there given for $f(a, b, c, n)$ is consistent with (5) and (6).

Consider the identity:

$$\begin{aligned} v^{b-1}(1-v^a)/(1-v^{ac}) \equiv{}& v^{b-1} - v^{a+b-1} + v^{ac+b-1} - v^{ac+a+b-1} + \cdots \\ &+ v^{acw+b-1} - v^{acw+a+b-1} + \frac{v^{acw+b-1}\cdot(1-v^a)}{(1-v^{ac})}. \end{aligned} \tag{7}$$

Multiplying (7) through by dv and integrating throughout between the limits 0 and 1, we find

$$\begin{aligned} \int_0^1 \frac{v^{b-1}(1-v^a)}{(1-v^{ac})}\,dv = {}& \frac{1}{b} - \frac{1}{a+b} + \frac{1}{ac+b} - \frac{1}{ac+a+b} + \cdots \\ &+ \frac{1}{acw+b} - \frac{1}{acw+a+b} + \int_0^1 \frac{v^{acw+b-1}(1-v^a)}{(1-v^{ac})}\,dv. \end{aligned} \tag{8}$$

Combining the terms of (8) in pairs and referring to the second integral as R, we obtain

$$\int_0^1 \frac{v^{b-1}(1-v^a)}{(1-v^{ac})}\,dv - R = a \cdot \sum_{x=0}^{w} \frac{1}{(acx-b)^{|2|a}} \tag{9}$$

From (3) it is seen that

$$\lim_{w\to\infty} \sum_{x=0}^{w} \frac{1}{(acx+b)^{|2|a}} = f(a,b,c,2).$$

R is a function of w and may be written:

$$R = \int_0^1 \frac{v^{\alpha w+\beta}(1 - v^a)}{(1 - v^{ac})}\, dv.$$

For v in the interval (0, 1), $(1-v^a)/(1-v^{ac}) \leqq 1$, since $a \geqq 1$, $c \geqq 1$. Hence $R \leqq \int_0^1 v^{\alpha w+\beta} dv = (\alpha w+\beta+1)^{-1}$.

Since R is always less than, or at the most equal to, a quantity which approaches zero as w approaches infinity, it follows that $\text{limit}_{w\to\infty} R = 0$, and (4) is established for $n = 2$.

Equation (4) may now be established, in general, by mathematical induction, using (6) to pass from n to $n+1$, and the value established for $n = 2$ as the special case.

The formulas (3), (4), (6) above may be generalized to the following, the proof being the same as before.

$$(3') \qquad f(a,b,c,n,\rho) = \sum_{x=0}^{\infty} \frac{1}{(acx + b)^{\lfloor n \rfloor a}} \left(\frac{1}{\rho}\right)^{acx+b}, \qquad \rho \geqq 1,$$

$$(4') \qquad a^{n-1}(n - 1)!\, f(a,b,c,n,\rho) = \int_0^{1/\rho} \frac{x^{b-1}(1 - \rho^a x^a)^{n-1} dx}{(1 - x)^{ac}},$$

$$(6') \qquad f(a,b,c,n,\rho) - \rho^a f(a,a + b,c,n,\rho) = anf(a,b,c,n + 1,\rho).$$

Many interesting results may be derived from (4). One of the simplest follows from the value it gives for the sum in (1):

$$\int_0^1 v^{b-1}(1 - v^a)^{n-2} dv = \sum_{R=0}^{n-2} \frac{{}_{n-2}C_R}{aR + b}(-1)^R = \frac{a^{n-2}(n - 2)!}{(ax + b)^{\lfloor n-1 \rfloor a}}\Bigg]_{x=0}.$$

In conclusion: It has been shown that an interesting class of series may be summed by integration such as is treated in every elementary text in the calculus.

III. Two Notes on Cyclic Cubics

By Raymond Garver, University of California at Los Angeles

1. A General Expression for Cyclic Cubics. Seidelmann[1] has given general expressions which represent all cubic and quartic equations with rational coefficients which have certain Galois groups for the field of rational numbers. His article is simply a summary of results given in his Dissertation, and does not give detailed methods; his general method, however, as outlined near the beginning of his paper, involves the use and manipulation of functions "belonging" to the required group.

[1] Mathematische Annalen, vol. 78 (1918), pp. 230–233.

I wish to show that his expression for the general cyclic cubic with second term removed,

$$x^3 - 3(p^2 + 3q^2)x + 2p(p^2 + 3q^2) = 0, \tag{1}$$

can be obtained in a very elementary manner. In fact if we define a cyclic cubic as an irreducible cubic whose discriminant is a perfect square, we may avoid all reference to groups.

First, it is clear that any cubic with rational coefficients whose second term has been removed by a linear transformation is of the form

$$x^3 + Am^2x + Am^3 = 0,$$

where A and m are rational. If this equation is cyclic, its discriminant is a perfect square, which is equivalent to saying that we must have

$$A = -(c^2 + 27)/4, \text{ where } c \text{ is rational.}$$

If we now put $p = -3m/2$, $q = cm/6$,[1] the equation takes Seidelmann's form. Equation (1) hence represents all cyclic cubics, first, since every cyclic cubic has been shown to be of that form and, secondly, since every cubic of form (1) is cyclic (if it is irreducible), having its discriminant a perfect square.

2. A Normal Form for Cyclic Cubics. There are a number of simple one-parameter normal forms to which a cyclic cubic can be reduced. Cajori[2] gives the form

$$z^3 - 3(m^2 + m + 1)z + (m^2 + m + 1)(2m + 1) = 0,$$

which becomes, on putting $2m+1 = P$, $y = 2z$,

$$y^3 - 3(P^2 + 3)y + 2P(P^2 + 3) = 0. \tag{2}$$

This form resembles (1) closely, but is a normal form and not a general expression.

Cajori's derivation of his normal form is long, and gives no method by which the reduction can be carried out for a *particular* cubic. These two objections seem to be met by the following theorem:

Any cyclic cubic can be transformed into form (2) by a quadratic Tschirnhaus transformation with rational coefficients.

To show this, it is necessary merely to consider the general cyclic cubic in the form (1), and to apply the transformation

$$y = -p^{-1}q^{-1}[x^2 + px - 2(p^2 + 3q^2)].$$

The coefficients of the transformed equation are obtained with the help of formulas which I gave recently in this Monthly:[3] I shall not reproduce the calculations here. Equation (2) is easily obtained upon putting $P = -3q/p$.

[1] i.e., $m = -2p/3$, $c = -9q/p$.

[2] *Theory of Equations*, 1921, pp. 196–197.

[3] Vol. 34 (1927), pp. 521–522.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB TOPICS

1928 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Stanford University, California.

In continuation of previously published lists of centennial dates[1] in the history of mathematics, the following list of important 1928 centennial dates is presented. They may be of interest and value to mathematics teachers as suggestions for program topics or special investigations and reports. Further details may be found in the standard histories of mathematics.

628. Publication of the *Brahma-sphuta-siddhanta* of Brahma-gupta, most prominent Hindu mathematician of the seventh century, with whom mathematics in India reached its high water mark.

1228. Publication of the revised edition of Leonardo of Pisa's *Liber Abaci* which marked the first renaissance of mathematics on Christian soil, introduced Arabian algebra, and brought into general use in Europe the labor-saving Hindu-Arabic numerals, and for centuries was a storehouse of material for later European writers on arithmetic and algebra.

1528. Death of Albrecht Dürer, celebrated artist, author of first printed work on higher plane curves, and earliest occidental writer to call attention to magic squares. See the magic square in his well known painting "Melancholia."

1528. Death of Johannes Werner, author of the first book on conic sections in Christian Europe.

1628. Publication of Adrian Vlacq's fourteen place logarithm tables of numbers from 1 to 100,000, of which 70,000 were computed by him.

1728. Birth of J. H. Lambert, versatile mathematician who first proved the irrationality of pi.

1828. The symbol "e" for the natural base of logarithms introduced by Euler.

[1] This Monthly, vol. 33 (1927), p. 141 for a list of 1927 events and for references to previous issues for corresponding lists for 1926 and 1925.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

Plane Trigonometry, by ARTHUR M. HARDING and GEORGE W. MULLINS. New York, the Macmillan Company. 118 pages.

A concise and conservative text.

College Algebra, by CLAUDE IRWIN PALMER and WILSON LEE MISER. New York, McGraw-Hill Book Company. xiv+377 pages. $2.50.

Another conservative book, unusually full in treatment and sufficiently elastic for a brief or an extended course.

Analytic Geometry, by R. L. BORGER. New York, McGraw-Gill Book Company. xii+334 pages.

Interesting features of this text are the emphasis on ruler and compass construction, the simultaneous study of cartesian, polar and parametric equations, and the introduction of the derivative.

Short Course in Spherical Trigonometry, by PAULINE SPERRY. Richmond, Johnson Publishing Company. xii+60 pages.

"An effort to meet the need for a short yet rigorous text."

Mathematics in Liberal Education, by FLORIAN CAJORI. Boston, Christopher Publishing House. 169 pages. $1.50.

"A critical examination of the judgments of prominent men of the ages" as to the values of mathematics.

The Symmetrical Optical System, by G. C. STEWARD. Cambridge Tracts in Mathematics and Mathematical Physics, No. 25. Cambridge University Press. 102 pages.

Beyond the Electron, by SIR J. J. THOMSON. Cambridge University Press. 43 pages.

A lecture on the new physics, delivered at Girton College.

Theory of Probability, by WILLIAM BURNSIDE. Cambridge University Press. xxx+106 pages.

"A manuscript which Professor Burnside had practically completed some time before his death. It has been edited for the press, with a memoir, by Dr. A. R. Forsyth."

The New Quantum Mechanics, by G. BIRTWISTLE. Cambridge University Press. xiv+290 pages.

"This book is concerned with the development of quantum mechanics during the past two years."

* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

Neuere Methoden und Ergebnisse in der Hydrodynamik, VON C. W. OSEEN (Upsala). Leipzig, Akademische Verlagsgesellschaft M.B.H. xxiv+337 pages.

This is Band I of a series designated as "Mathematik und Ihre Anwendungen in Monographen und Lehrbüchern," edited by E. Hilb of the University of Würzburg. xxiv+337 pages.

Plane and Spherical Trigonometry, by MURRAY J. LEVENTHAL. New York, Globe Book Company. $0.53.

The purpose of this book is to present the essentials of trigonometry in as brief a form as possible.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Researches into the Mathematical Principles of the Theory of Wealth. By AUGUSTIN COURNOT. Translated by Nathaniel T. Bacon, with an essay on "Cournot and Mathematical Economics" and a bibliography of mathematical economics by Irving Fisher. The Macmillan Co., 1927. xxiv+213 pages.

This translation of Cournot's classic of 1838 was first published in 1897. The present volume is an exact reprint, except for the addition by Professor Fisher of a foreword and twelve pages of notes on Cournot's mathematics. With his usual clearness Professor Fisher helps the reader of moderate mathematical attainments over the more difficult steps. In the foreword he says:

> In the twenty years since the book first appeared in English, the mathematical method has become so general in economic and statistical studies that no attempt has been made to bring the bibliography down to date by adding the many items which would be necessary; and there is today little need, as there then was, to emphasize the value of that method, as it is now seldom, if ever, challenged.

This is not the place to go into the criticisms and prolonged discussion of Cournot's theories of duopoly and of foreign exchange. If students continue to believe in the literal inspiration of textbooks, the book should not be used as a text. But the work has a value not merely historical, for it sets an example of precise reasoning in economics which is worthy of emulation.

The publication of this edition is related to a widespread increase in interest in mathematical economics; and by mathematical economics is meant not only statistics, but the development with free use of mathematical tools of the consequences of economic assumptions. The work of Cournot, Jevons, Walras, Pareto, and Marshall has stimulated two main lines of development. The more widely known is the study of economic statistics and the investigation of particular industries, with the object more or less plainly in view of obtaining definite expressions for supply and demand functions. This work has become allied with the financially attractive enterprise of business forecasting, but has encountered practical difficulties; only a few of the functions in question have thus far been evaluated, and these with dubious reliability. On the other hand is the theoretical work represented by Irving Fisher's doctoral thesis, recently

republished, and by his studies of interest rates, currency and the price level; by the works of Edgeworth, Pantaleoni, and Amoroso; and now by C. F. Roos, who has actually uncovered some *mathematically* new problems. Contributions of the deductive type have been made by G. C. Evans and others. Like mathematical physics and laboratory experiment, the two pursuits must henceforth go on side by side, each deriving meaning and guidance from the other.

Incidentally this trend in economics is demanding more and earlier mathematics. The prospective economist who is not familiar with calculus before his sophomore year is now at a disadvantage. Both in economic theory and in statistics the rate of learning could be quadrupled if all the students in the class could handle calculus, if texts were available assuming this condition and as well written as the average book on college algebra, and if instructors were themselves able to take advantage of the situation. The economist has small use for conic sections or for drill in solving triangles, but he does need calculus. The same is true of the sociologist, the political scientist, the psychologist, and the biologist. If alongside the familiar curriculum we could have one in which trigonometry should be regarded primarily as preparation for calculus rather than for surveying, and in which integral calculus were completed in the first college year or earlier, a serious need of these important groups could more easily be met. The graduate student who now so commonly turns aside from the subject of his choice to struggle with elementary mathematics is probably acting wisely, but the diversion of attention from his main interest must make the results less valuable than if his mathematics were already well enough consolidated to allow him to go straight ahead.

HAROLD HOTELLING

Die trigonometrischen Lehren des persischen Astronomen Abu'l-Raihân Muh. ibn Ahmad Al-Bîrûnî, dargestellt nach Al-Qânûn Al-Mas'ûdî von CARL SCHOY. Hannover, Orient-Buchhandlung Heinz Lafaire, 1927. xii+108 pages.

With the death of Carl Schoy on December 6, 1925, scholarship in the field of history of exact science suffered a severe and irreparable loss. In his short life time, Schoy had done much very valuable work in uncovering the buried treasures of Arabian mathematics, including in particular many on astronomical and geographical subjects. In his earlier years handicapped by poverty, and later by an insidious and incurable disease, Schoy's career compels our admiration for his never-failing courage and indomitable will, no less than for his scientific achievements.[1]

The book under review was finished and prepared for the press by the author in 1924, but was not published until after his death, when it was seen through

[1] Biographical sketch, by J. Ruska, *Isis*, vol. 9 (1927), pp. 83–89 with portrait.

the press by J. Ruska and H. Wieleitner. Although it is the most extensive published work of the author, he had designed it to be only a small section in a history of Arabian mathematics which he was planning to write.

Al-Biruni, whose trigonometric work is here published for the first time in any detail, was a Persian astronomer, mathematician, and geographer of the 11th century. He resided at Gazna at the time when it was the brilliant capital of Mahmud the conqueror of India. Al-Biruni traveled widely in India, and his description of this country is one of the greatest works in all Arabian literature. As a scientist he was both original and versatile, and marks, with Ibn Sina (Avicenna) the high-water mark of Moslem science. Thus the trigonometric work of Al-Biruni is of great interest and importance in the history of mathematics.

Schoy has here given us 10 chapters, selected from Al-Biruni's *Al-Qânûn al-Mas'ûdî*. Chapter 1 gives the lengths of the sides of the regular polygons of 3, 4, 5, 6, 8, and 10 sides inscribed in a circle of radius unity. Chapter 2 shows how to find the chord of double an arc from the chord of the simple arc; similarly, the chord of half or a fourth of a given arc, and in general $(1/2)^n$ of a given arc; and the chord of the sum or difference of two arcs whose chords are given. Chapter 3 gives a construction of the regular nonagon, with a computation which gives the numerical value of the length of a side (in sexagesimals) correct to 7 decimal places. At the end of the chapter, as a verification of his geometrical reasoning, Al-Biruni shows how by trigonometric combinations starting with 45° and 30° we can reach the chord of $40°0'0''0'''24^{IV}+$, which gives a value agreeing with that previously obtained, within less than one two-millionth. Chapter 4 gives two methods of finding the chord of 1° when the radius is unity: (1) by starting with s_9 and s_{10}, using $40°-36°=4°$ and halving twice; (2) by using s_{30} and s_{36}, $12°-10°=2°$, and halving. Next, the trisection equation for 3° is obtained in geometric form, and thus the chord of 1° is computed, agreeing to $(1/60)^4$ with the other results.

Chapter 5 gives a computation of π, resulting in the value 3.14174. Schoy remarks that he himself obtained the considerably more accurate value 3.14167 by using Al-Biruni's methods with but a very slight modification. Chapter 6 explains the construction of Al-Biruni's trigonometric table of sines, which is also printed therewith (pp. 37–39.) The angles progress by quarter-degrees; the sines are given to fourths (in sexagesimal notation) thus: $\sin 2°30' = 2'37''1'''47^{IV} = .043618+$. Chapter 7 explains the use of the table in finding the sine or versed sine of a given angle, or the angle corresponding to a given sine or versed sine. Second differences are taken into account in the process. Chapter 8 deals with the "shadows of illuminated objects" and explains the solution of problems involving tangents, after the manner of Abu'l Wafa. A table of tangents at intervals of 1° with first and second differences follows. At

the end of the chapter the law of sines is proved. Chapters 9 and 10 contain the proof of the sine law for a spherical triangle, and a few other theorems derived from it.

The remainder of the book (pp. 64–108) consists of supplementary material, included by the author because of its close relation to Al-Biruni's work. First, Al-Biruni's own application of his theorems on spherical trigonometry to problems of practical astronomy; secondly, Tabit ibn Qurra's and Ibn Al-Haitam's monographs on the regular heptagon; and thirdly, the sine and tangent tables of Ulug Beg.

This book is a very valuable addition to the primary sources of information about Arabian mathematics, and emphasizes the greatness of the loss which its author's premature death has caused.

R. B. McCLENON

Calculus. By H. B. FINE. The Macmillan Co., 1927. 418 pages.

The Calculus. By R. D. CARMICHAEL and J. H. WEAVER. Ginn and Co. 342 pages.

Differential and Integral Calculus. BY A. COHEN. D. C. Heath Co. 540 pages.

These three books cover the usual topics in calculus in about the customary manner.

Fine's book has in addition some material usually reserved for texts in advanced calculus, including chapters on Fourier series and functions of a complex variable. The distinctive feature is the care with which questions of rigor are treated. For example, the author says that the slope of the tangent to a curve at a point is by *definition* the limit, if it exists, of the slope of the secant, and that the area under a curve is by *definition* the limit of a certain sum. It might seem superfluous to mention such things in 1928, but for the fact that the other two books are written as if slopes of curves and areas already existed, previous to definition, and as if mathematicians had been able to find their values by ingenuity and insight. Of course everyone wants books for beginners to appeal a great deal to the intuition and to avoid proofs and distinctions which the students cannot understand. The point is that Fine has avoided saying things which will have to be unlearned and has at the same time kept the appeal to the intuition. He does this by the turn of a phrase, without formality. The formal proofs occupy surprisingly few pages in all, and yet everything is there. The book is condensed, partly by leaving to the teacher some of the pleasure of making illuminating remarks and telling what it is all about, partly by a skillful arrangement of material, and partly by including fewer worked out examples. The exposition does not seem harder to read than other much longer expositions which skip half the reasoning and get the rest mixed up.

The text of Carmichael and Weaver is written in an enthusiastic and vivid style. There is a conscious, and in the reviewer's opinion, successful effort to keep the student's work from degenerating into formal manipulation. For example, more space than usual is given to making the reader associate some geometrical or kinetic picture with the word derivative and the symbol dy/dx, before formal differentiation is taken up. Perhaps as a result of this there are a good many phrases, especially in the opening chapter which will offend the purists. The following is typical, (p. 12)—"By the slope of a curve at a given point we mean the slope of the tangent line to the curve at that point." Although this comes in a section which is frankly untechnical, there are two objections to it. First, the slope of a curve is as vivid intuitionally as the existence of a tangent; and secondly, it is apparently and not actually a definition. It might just as well read—"By the tangent line to a curve at a point we mean a straight line through the point having the same slope as the slope of the curve at that point." Neither sentence adds to the clearness or to the reasoning. On the other hand, the author's habit of stopping occasionally to tell where he is going makes the book easier reading and more interesting.

While Fine confines himself to saying things completely and just once, Cohen, like Carmichael and Weaver, writes as a teacher talks in the classroom, giving motives and purposes, sometimes mentioning mistakes to be avoided, repeating and paraphrasing. "Much emphasis is laid upon applications to geometry, mechanics, and the physical sciences." The exposition is full and both the verbal and algebraic parts are given in enough detail for moderately good students. This text also may be criticized on logical grounds. For one of many possible examples, on page 229, there is a page of algebra ending with the formula for the differential of arc length. Anyone mature enough to read this would also be mature enough to realize that the length of arc requires definition, and to ask if the length of an arc "always concave to its chord" is necessarily less than the length of a broken line on the convex side with the same end points. It would be pedagogically better and logically the same to write at once, $ds^2 = dx^2 + dy^2$. There are numerous misprints. There are plenty of worked out examples, and those left for the student include many not in other texts.

K. W. LAMSON

Analytic Geometry. By E. S. CRAWLEY and H. B. EVANS. F. S. Crofts & Co., 1928. xiii+244 pages. Price $2.00.

The present edition is a revision of the original text written in 1917.

The first seven pages are taken up with a series of formulae and tables.

The first section contains a discussion of oblique coordinates, which is perfectly logical, but is not the usual method of starting the beginner in analy-

tic geometry. No exercises as vitalizing material are given under this section. This plan of giving articles of a theoretical nature without concrete exercises immediately following each article is quite common in the text. The plan of deriving the general formula and then having the student solve exercises by applying the formula is an important way of teaching the subject; but the scheme of having a student go through, in solving a problem, the same process as the author carries out in deriving the formula is also an excellent plan from a pedagogical point of view.

The way of dividing exercises at the end of each chapter into two classes is highly commendable. The first class contains simple and direct applications of the theory given in preceding articles. The second class includes general exercises requiring more maturity on the part of the student. This plan will aid the teacher in the assignment of lessons.

In the chapter on graphs of equations the idea of showing one graph with isolated points corresponding to the pairs of values of x and y, and then another graph with a curve drawn through the points, is highly commendable as a pedagogical scheme. The plan of showing points equally spaced to represent measurements parallel to the x and y axes will be found of aid to the student working at a black board which is not cross-sectioned.

In the chapter on the straight line, the writer of this review feels that there are too many sections on the general derivation of formulae before any exercises are given. The equations of the following forms are given before any exercises are set for the student: two-point, point-slope, slope-intercept, intercept, and normal. Instead of using the normal form of the equation of a straight line to determine the perpendicular distance of a given point from a given line, the authors have a rather unique way of carrying out the derivation of the well known formula

$$d = (Ax_1 + By_1 + C)/\pm (A^2 + B^2)^{1/2}.$$

The scheme consists in writing the equation of the perpendicular from the point (x_1, y_1) to the line

$$Ax + By + C = 0,$$

and then finding the coordinates of the point of intersection of the two lines. Then the distance formula

$$d = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$$

is used. In theory this plan is much simpler than the one usually followed, but the algebraic work is more difficult for the student to follow.[1]

[1] EDITOR'S NOTE: If in the class room we ask our students to suggest methods of attacking this problem, the most frequent suggestion is that of the method discussed here. Perhaps the fact that the process is one that appeals to the student mind justifies its use in spite of the rather formidable algebraic reductions.

Much more extensive treatment of imaginaries is carried out in this text than is usual. However, this should find favor with teachers, since it is highly probable that too little attention to these numbers has been given in current texts.

The introduction to the chapter on conic sections by way of geometric constructions based upon the arc tan e is very interesting. The authors have here a simple operation which holds alike for all the conics.

The chapter on tangents, normals, diameters, poles, and polars contains 20 pages, which is ample space for covering these topics in a thorough manner. Another chapter of about the same length covers the general equation of the second degree.

The discussion of polar coordinates is relegated to a later chapter in the text. The reviewer feels that it is better to introduce this subject early in the course, so that constant application may be made during the term.

The last two chapters of the book cover empirical equations and space geometry.

MYRON O. TRIPP

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3341. Proposed by Paul Wernicke, Washington, D. C.

Given a triangle ABC and a point P, find the conditions for the construction of a triangle having its sides parallel and proportional to the joins PA, PB and PC.

3342. Proposed by R. Goormaghtigh, La Louvière, Belgium.

Let α, β, γ, δ be the points where the straight lines AP, BP, CP, DP, meet the faces of the tetrahedron $ABCD$; the perpendiculars dropped from the vertices A, B, C, D on the lines joining respectively α, β, γ, δ to the orthocentre H meet the corresponding faces of the tetrahedron $ABCD$ in four coplanar points. The plane passing through these four points is perpendicular to PH.

NOTE BY THE EDITORS: This problem is a generalization of Problem **3228** [**1926, 525** and **1928, 42**]. See also the solution of **3258** [**1928, 210**].

3343. Proposed by J. V. Uspensky.

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{2n\pi}^{\infty} \frac{\sin z}{z} dz = \pi - \frac{\pi}{2} \log 2\pi$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{n\pi}^{\infty} \frac{\sin z}{z} dz = \frac{\pi}{2} - \frac{\pi}{2} \log \pi.$$

3344. Proposed by B. F. Yanney, Wooster, Ohio.

Two altitudes of a given triangle and the side from whose extremities these altitudes are drawn meet in collinear points the corresponding sides of the orthic triangle of the given triangle.

3345. Proposed by B. F. Finkel, Drury College.

A mill wheel of radius a revolves so that its rim has a velocity v, and drops of water are thrown off from the rim. Find the envelope of the paths of the drops.

3346. Proposed by Frank Irwin, University of California.

Show that, if $P_1P_2 \cdots P_n P_1$ be any polygon of n sides, the broken line whose segments are parallel and equal to $P_1 P_k$, $P_2 P_{k+1}$, $\cdots$, P_nP_{k-1} $(P_{n+i}=P_i)$ in order will close. Generalize.

Again let S_i be the middle point of P_iP_{i+1}, $i=1, 2, \cdots n$, $(P_{n+1}=P_1)$; show that the broken line whose sides are parallel and equal to P_1S_k, P_2S_{k+1}, $\cdots$, $P_n S_{k-1}$ in order will close. Generalize.

3347. Proposed by R. S. Underwood, Texas Technological College.

If $x^n+y^n=z^n$, where n is an odd positive integer greater than or equal to 3, and x, y, z are positive integers, prove that $x>6K$ and $y>6K$, where K is the product of all distinct factors in n other than 2 and 3. As a corollary, it follows immediately that, if

$$x^p + y^p = z^p \quad (p \text{ a prime} \geqq 3) \quad x > 6p \text{ and } y > 6p.$$

3348. Proposed by A. C. Aitken, University of Edinburgh.

Show that

$$\frac{\pi}{6} = \sum_{1}^{\infty} \operatorname{arccot} 2u_r^2,$$

where $u_r=4u_{r-1}-u_{r-2}$ with $u_1=1$, $u_2=3$.

SOLUTIONS

3280. [1927, 438]. Proposed by Philip Fitch, Denver Public Schools.

If a flexible chain, suspended at the ends from points in a horizontal line, is loaded so that the load varies directly as the square of the distance along the horizontal from the mid-point of the chain, find the equation of the curve made by the chain.

Solution by J. B. Reynolds, Lehigh University.

Assuming a y-axis vertical through the mid-point of the chain and an x-axis horizontal, let T be the tension in the chain at a point P, (x, y) where the tangent to the curve makes an angle ϕ with the x-axis. Then, if the load at the point P is kx^2 we have,

$$(1) \qquad T d\phi/ds = kx^2 \cos\phi, \qquad (2) \quad dT/ds = kx^2 \sin\phi.$$

Dividing these equations and integrating the result we find, $T=T_0 \sec\phi$, in which T_0 is the tension at the mid-point where $\phi=0$.

Eliminating T from (1) we get

$$T_0 \sec^2\phi \, d\phi = kx^2 ds,$$

or since $\tan\phi = dy/dx = p$,

$$T_0 dp/(1+p^2)^{1/2} = kx^2 dx.$$

Upon integrating this equation we find,

$$p = \tfrac{1}{2}[\exp kx^3/3T_0 - \exp(-kx^3/3T_0)]$$

and, therefore, for the required curve,

$$y = \int \sinh (kx^3/3T_0)dx$$

Also solved by PAUL WERNICKE.

3282 [1927, 438]. Proposed by H. Halperin, A. & M. College of Texas, College Station, Texas.

Justify the following method of solving an exact differential equation of the first order, $Mdx+Ndy=0$: Replace y in M and N by ux and dy by udx; integrate the resulting expression with regard to x, considering u as constant; replace u by y/x, equate to a constant, C, and obtain the solution.

SOLUTION BY OTTO DUNKEL, Washington University.

In the theorem of the problem the statement should be made more precise in the form "integrate the resulting expression with regard to x between the limits x and 0, considering u as a constant." In a convex region containing the origin in which M and N are continuous functions of x and y, the integral of $Mdx+Ndy=df(x, y)$ along a chosen path of a suitable form within the region from (0, 0) to the point (x, y) which is fixed for the moment is $f(x, y)-f(0, 0)$. The value of the integral does not depend upon the path, and we may use as a path the straight line $y=ux$, where u is a constant so chosen that the line goes through the other end point. The integral becomes

$$\int_0^x [M(x,ux) + uN(x,ux)]dx = \phi(x,u) - \phi(0,u).$$

Hence, replacing in the above u by y/x, we have

$$f(x,y) = \phi(x,y/x) - \phi(0,y/x) + f(0,0),$$

which is obtained by equating the results of the two integrations. This is an identity in x and y, and it proves the theorem as amended.

If the origin is a singular point as in $-(y+3)dx/x^2+dy/x=0$, the method gives an incorrect result, But we may first make the transformation $x=x'+1$, $y=y'$, and then apply the theorem to the resulting equation.

Also solved by HARRY LANGMAN, E. C. KIEFER, A. PELLETIER, PAUL WERNICKE, AND THE PROPOSER.

3284 [1927, 438]. Proposed by Burrell Morgan, Krellitz, W. Va., and Norman Anning, University of Michigan.

In a given sphere is inscribed the maximum right prism whose bases are regular polygons of a given number of sides. Show that its altitude is independent of the number of its lateral faces.

SOLUTION BY A. PELLETIER, Montreal, Canada.

Let r be the radius of the sphere, n the number of sides in the polygon, and x the radius of the circle circumscribed about the polygon. Then the volume of the prism is expressed by

$$nx^2 (r^2 - x^2)^{1/2} \sin \left(\frac{2\pi}{n}\right).$$

The maximum of this expression takes place when $x^2=2r^2/3$.

Hence the altitude $a=2(r^2-x^2)^{1/2}=2\cdot 3^{-1/2}r$, a value independent of n, the number of lateral faces of the prism.

Also solved by H. C. BRADLEY, HARRY LANGMAN, E. C. KIEFER, W. T. REID, AND J. B. REYNOLDS.

NOTES AND NEWS

LINNAEUS WAYLAND DOWLING, Professor of Mathematics at the University of Wisconsin, died at his home in Madison on September 16. He had been failing in health for some months, and was obliged to relinquish his work early in the second semester of the past year.

Professor Dowling was born at Medina, Michigan, on December 8, 1867. He was a graduate of Adrian College, and in 1895 received the degree of Doctor of Philosophy from Clark University, his doctor's dissertation being "On the forms of plane quintic curves." In the same year he came to the University of Wisconsin as instructor in mathematics, remaining a member of the department until his last illness. In 1905–6 he spent a year at Turin studying under Segré. He was a member of the American Mathematical Society, the Mathematical Association of America, and the Circolo Matematico di Palermo. He was the author of textbooks on analytic geometry, projective geometry, and mathematics of insurance, and edited the mathematical series for the Johnson Publishing Company.

This long service as teacher and friend has been rich in its fruitful associations. Magnetic in personality, genial of manner, peculiarly endowed with the teaching gift, he quickly won and maintained the admiration and devotion of his students. He brought to them a rare combination of discipline and vision, a mingling of carefully developed fundamentals with glimpses of the heights beyond. Many of his students who later became teachers testify to the special inspiration and help in their work received in his classroom.

The days of growing weakness were brightened by daily visits of friends and students, who found him, as always, full of sympathy for their interests. His home was a shrine, to which flocked the many who loved him. These found him with peace and serenity of spirit, surrounded by the things he put first, his family, his friends, and his books. Active in community life, lover of literature, music, and nature, his passing leaves a wide sense of personal loss.

The annual meeting of the National Council of Teachers of Mathematics will be held at the Statler Hotel in Cleveland, Ohio, Feb. 22 and 23, 1929. Mr. Harry C. Barber of The Phillips Exeter Academy is the president of the organization and he will be pleased to answer any inquiries concerning the annual meeting or any other phase of the Council's activity. It may be of interest to note that the National Council has just become an incorporate body and that there is great activity on foot to increase the membership from the present five thousand to at least ten thousand in the next two years. Its official journal is *The Mathematical Teacher*, which is the only journal in this country devoted exclusively to mathematics in the elementary and secondary fields; and it is

highly worth while for every teacher of secondary mathematics to be a member of the Council and a reader of this journal.

At its commencement exercises in June, Dartmouth College conferred the honorary degree of Doctor of Science on Professor CHARLES N. HASKINS of the department of mathematics. This was an expression of appreciation for his work in connection with the building of the new Baker Memorial Library dedicated at that time.

Assistant Professor FRED W. BEAL, of the University of Pennsylvania, has been promoted to a professorship of mathematics.

Assistant Professor C. C. BRAMBLE has been promoted to a professorship of mathematics in the post graduate school of the United States Naval Academy.

Associate Professor J. A. BULLARD, of the United States Naval Academy, has been appointed to a professorship of mathematics at the University of Vermont.

Dr. A. H. COPELAND has been appointed assistant professor of mathematics at the University of Buffalo.

Professor J. C. FITTERER has been appointed associate professor of mathematics at the Colorado School of Mines.

Dr. H. C. HICKS has been appointed assistant professor of mathematics at the University of Oregon.

Assistant Professor DANIEL HULL, of the University of Notre Dame, has been promoted to the rank of professor and head of the department of physics.

Assistant Professor ROGER A. JOHNSON, of Hunter College of the City of New York, has been promoted to an associate professorship of mathematics.

Assistant Professor JOHN R. KLINE, of the University of Pennsylvania, has been promoted to a professorship of mathematics.

Dr. E. E. LIBMAN, of the University of Illinois, has been promoted to as assistant professorship of physics.

Associate Professor J. J. NASSAU has resumed his work at the Case School of Applied Science after a year's leave of absence spent at Cambridge University working principally under Professor Eddington.

The Reverend E. C. PHILLIPS, director of the astronomical observatory of Georgetown University, has been appointed to the office of provincial of the Jesuit Province of Maryland-New York.

Dr. WILLIAM C. RISSELMAN, of the University of Minnesota, has been appointed assistant professor of mathematics in the University of Pittsburg. His work will be at the Junior College branch of the University at Uniontown, Pa.

Associate Professor H. M. ROBERTS, of the United States Naval Academy, has been promoted to a professorship of mathematics.

Assistant Professor J. B. ROSENBACH, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

EDWIN W. SCHREIBER, formerly head of the mathematics department, Proviso Township High School, Maywood, Illinois, is continuing his graduate studies under Professor Karpinski at the University of Michigan. Mr. Schreiber has been honored in being appointed to a university fellowship for the year 1928–1929.

At the University of California at Los Angeles, Assistant Professor H. M. SHOWMAN has been made recorder of the university but will remain a member of the department of mathematics.

Assistant Professor STANLEY P. SHUGERT, of the University of Pennsylvania, has been promoted to a professorship of mathematics.

Professor EVAN THOMAS, of the University of Vermont, has retired.

The following appointments to instructorships in mathematics are announced:

Albion College, Mr. WATSON M. DAVIS;
University of Arkansas, Mr. HARVEY A. WRIGHT;
Brown University, Mr. H. S. THURSTON;
Milton College, Miss FANNIE HOPKINS;
University of Pennsylvania, V. W. ADKISSON, P. A. KNEDLER;
St. Olaf's College, Mr. CLARENCE CARLSON.

CORRIGENDA

The following corrections should be made in the August-September issue of the Monthly:

On page 335, in line 8, the statement, "The words were written by Miss Lutz" should be replaced by this sentence: "The pageant was written by Miss Banes, assisted by five students of her class in the history of mathematics."

On page 361, in line 19, change 7.973 to 7.972; and in the fourth line from the bottom, change -5.011 to -5.010.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

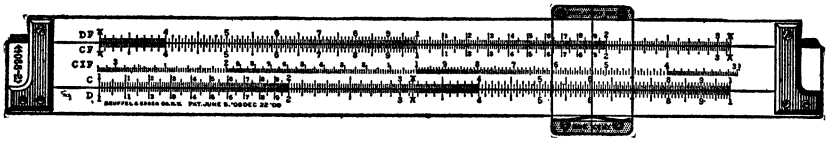
Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December 28-29, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.
INDIANA, Butler University, May 11-12.
IOWA, Grinnell College, May 4-5.
KANSAS, Topeka, Kan., February 4.
KENTUCKY, Louisville, Ky., April 21.
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.
MICHIGAN, Ann Arbor, Mich., March 31.
MINNESOTA, St. Joseph, Minn., May 19.
MISSOURI, Kansas City, Mo., November.
NEBRASKA, Midland College, April 28.
OHIO, Columbus, Ohio, April 5.
PHILADELPHIA, Philadelphia, Pa., December 1.
ROCKY MOUNTAIN, Golden, Colo., April 20-21.
SOUTHEASTERN, Dunham, N. C., March.
SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.
TEXAS, Texas A. and M. College, Jan. 28.

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The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. There will be an important announcement concerning this prize in the November issue of this MONTHLY.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

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The first award, covering the five years preceding 1925, was made to GILBERT AMES BLISS for his paper on "Algebraic Functions and their Divisors" published in the *Annals of Mathematics*. The committee of award consisted of Professor W. C. Graustein, Harvard University, Anna J. Pell-Wheeler, Bryn Mawr College, and E. B. Van Vleck, Chairman, University of Wisconsin.

The next award will be made in 1930 covering the five years preceding.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE THIRTEENTH SUMMER MEETING OF THE ASSOCIATION.

The thirteenth summer meeting of the Mathematical Association of America was held, by invitation, at Amherst College on Monday and Tuesday, September 3–4, 1928, in conjunction with the summer meeting and the colloquium of the American Mathematical Society. 183 persons were present, including 111 members of the Association:

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H. S. BROWN, Hamilton College.

W. D. CAIRNS, Oberlin College.
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F. F. DECKER, Syracuse University.
L. S. DEDERICK, Aberdeen Proving Ground.
L. L. DINES, University of Saskatchewan.
H. A. DOBELL, N. Y. State College for Teachers.
ELEANOR C. DOAK, Mount Holyoke College.
H. L. DORWART, Williams College.
ARNOLD DRESDEN, Swarthmore College.

T. C. ESTY, Amherst College.

W. B. FORD, University of Michigan.
TOMLINSON FORT, Lehigh University.
T. C. FRY, Bell Telephone Laboratories, New York, N. Y.

H. M. GEHMAN, Yale University.
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ELIZABETH W. WILSON, Cambridge, Mass.
W. A. WILSON, Yale University.

C. H. YEATON, Oberlin College.

The arrangements for housing and entertaining the mathematicians and those with them were unusually effective and enhanced decidedly the success of the meetings. The guests stayed for the most part in two college dormitories, and had their meals in the attractive Lord Jeffery Inn; a scale of special and very reasonable prices was arranged for the occasion. The session of Tuesday afternoon closed early enough to afford the opportunity for an excursion by autos or by bus to Mount Holyoke College, where the buildings were shown and tea was served in the Alumni House. President and Mrs. Pease gave a reception to the mathematicians on Tuesday evening at the Lord Jeffery Inn; and here, as throughout the meetings, the presence of Ex-President and Mrs. George D. Olds gave great pleasure to their many old-time friends. No session was held Wednesday afternoon; a group photograph was made early in the afternoon, after which a considerable portion of those in attendance took part in an excursion to the Sweetheart Tea Rooms at Shelburne Falls. There was full opportunity for tennis, swimming and golf. A motion offered by Professor Fort was unanimously adopted at the Tuesday session, instructing the secretary to express the appreciation and thanks of the members to the faculty, president, ex-president and trustees of Amherst College for their hospitality, and to Dean Esty and others of his committee on local arrangements for the most agreeable manner in which they provided for the comfort of the visitors.

The American Mathematical Society held its thirty-fourth summer meeting and twelfth colloquium from Tuesday to Friday, with lectures by Professor A. B. Coble on "The determination of the tri-tangent planes of the space sextic of genus four." There was a gratifying enrollment for the colloquium. Sessions were held for the reading of papers on Thursday morning and afternoon, and Friday morning.

About one hundred sixty attended the joint dinner on Wednesday evening at the Lord Jeffery Inn; following the dinner Professor Olds introduced the various speakers. President Pease, speaking as a classicist, welcomed the visitors and spoke of the great merits of mathematics and the classics. Professor Dines spoke of his pleasure in meeting with the Society and the Association, and expressed the hope that the organizations might meet in Canada, perhaps at Winnipeg. Professor Bliss praised highly an article on the mathematical basis of thermodynamics, a book on real function theory, and a presentation of the postulates of the theory of relativity in the Sitzungsberichte of the Berlin Academy by Carathéodory, who has spent the past year in this country. He urged also the need in our American education of paying special and adequate attention to students of unusual ability rather than devoting the very most of our time and effort to the great bulk of students of moderate or poor ability. While disclaiming anything in the nature of competition with recent meetings, Professor Kempner promised to those attending the mathematical meetings at Boulder, Colorado, August 26–30, 1929, beautiful mountain scenery with numerous excursions and the opportunity to bring mathematical inspiration to those who are so isolated from their mathematical colleagues. Professor Rietz supported the statements with regard to the beauties and the attractiveness of mountain climbing in the Estes Park region, and he bespoke also the interest of mathematicians in the meetings at Des Moines, along with the American Association next December; he referred to the great value in broadening our views in mathematics to be gained from the colloquium lectures given by men who have done notable work and the value derived from the extension of the sections of the Association.

The Mathematical Association held sessions on Monday and Tuesday afternoons, with President Ford in the chair, Vice-President Kempner presiding for a part of the Tuesday session. The program was arranged by a committee comprising Professors B. H. Camp (chairman), W. R. Ransom and J. K. Whittemore. Abstracts of the papers are given, numbered in accordance with the numbers of the papers.

FIRST SESSION OF THE ASSOCIATION

(1) "What we mean by postulates," by Professor A. A. BENNETT, Brown University.

(2) "Convergence and continuity," by Professor ARNOLD DRESDEN, Swarthmore College.

(3) "Higher forms of the icosahedron," by Mr. A. HARRY WHEELER, Lecturer, Wellesley College.

1. Professor Bennett made mention first of the notable address of Professor Pierpont on "Mathematical Rigor" that appeared this year in the Bulletin of the American Mathematical Society. Among other references was named particularly the readable and impartial treatise by Oskar Becker, Mathematische Existenz, Halle, 1927, which among other matters explains Hilbert's and Brouwer's positions with regard to infinite aggregates. The question of the role of formal logic in mathematics was discussed. In particular the matters of nonconstructive proofs and the traditional requirements of consistency, independence, categoricalness, etc., for postulational systems were examined. The essential difference in character between postulational systems for arithmetic, and those for other branches was emphasized.

2. In this paper Professor Dresden took up briefly some of the important extensions and developments of the concepts "continuity of a function" and "convergence of a sequence," which have become current in the mathematical literature of recent years. The first part of the paper dealt with absolute continuity (Vitali, 1905), its connections with the existence of a tangent to a curve and with rectifiability; with pseudo-intervals, quasi-continuity (Tonelli, 1921) and the related theory of the Lebesgue integral; with equicontinuity (Ascoli, 1884) of a sequence of functions, the existence of a limiting element of a set of continuous functions, and the applications of this concept. The second part was concerned with sub-uniform convergence (convergenza a tratti, Arzela, 1899), relative uniform convergence (E. H. Moore, 1906), approximate convergence (Weyl. 1909; F. Riesz, 1909; Tonelli, 1921) and with convergence in the mean (Fischer, 1907) and with indications of the principal applications of these concepts. The paper closed with a suggestion of a method by means of which the concepts of continuity of a function and convergence of a sequence could be unified.

3. In addition to the regular convex icosahedron there are other higher forms derived by extending the faces of a regular icosahedron. The complete icosahedron is a sixty-pointed star figure, and there are also twelve-pointed and twenty-pointed forms, and others consisting of groups of regular tetrahedra, and of regular octahedra. The regular twelve-pointed star icosahedron was discovered by Poinsot, and the concave wedge variety by Möbius.

There are several "double" varieties, each of whose complementary parts is a distinct icosahedron, and there is a triple icosahedron each of whose three parts is an icosahedron of higher species. In addition to these forms there are several "hollow," or "labyrinth" icosahedra, and several "discrete" varieties composed of separate disconnected elements, which in certain forms are double pyramids. There are two tetrahedral complexes of thirty tetrahedra each which are enantimorphs with reference to each other. At the International Mathematical Congress held at Toronto in 1924, Mr. Wheeler gave a table of twenty-two forms of the icosahedron, and to date this number has been increased to thirty-five different forms.

Beginning with any face of a regular convex icosahedron, we shall number the faces consecutively in counter-clockwise order, 1, 2, 3, $\cdots$, 20, and shall also designate these same faces in the same order by the letters, a, b, c, $\cdots$, r, s, t, respectively. Selecting the plane of one of these faces, as a, as a plane of reference, we will extend all other faces as, 2, 3, 4, $\cdots$, 17, 18, 19, not parallel to it, and designate their intersections with this plane, a, by means of subscripts to a, as a_1, a_2, a_3, $\cdots$, a_{19}. The plane of reference will thus contain a system of eighteen lines.

We shall use the notation $a_ia_ja_k$ to denote that we are to start with the line a_i and pass along it in counter-clockwise direction until we arrive at the intersection with the line a_j, along which we will travel in counter-clockwise direction to the line a_k, and thence to the starting point on the first line, a_i.

In the configuration of eighteen lines on the plane of reference, a, the face of the original icosahedron will thus be defined by its boundary as $a_2a_5a_8$.

The configuration of eighteen lines may be rotated about its geometric center, which is the geometric center of the triangle, $a_2a_5a_8$, through angles of $2\pi/3$, $4\pi/3$, and 2π, and made to coincide with itself. It follows that any figure formed by selecting lines, $a_ia_ja_k \cdots a_r$, from among the totality of lines $a_2a_3a_4 \cdots a_{19}$, may be transformed into a congruent figure by rotations of $2\pi/3$, $4\pi/3$ and 2π about the geometric center of the complete figure.

We shall now use $Fa_xa_ya_z \cdots$ to denote the plane figure formed by the lines $a_xa_ya_z \cdots$ taken in counter clockwise order to form a closed boundary. We shall call any such closed space, thus defined, an element, and shall denote it by $Ea_xa_ya_z \cdots$. By applying the cyclic substitutions, I, S, S^2, where $S = 2,5,8 \cdot 3,6,9 \cdot 4,7,10 \cdot 11,14,17 \cdot 12,15,18 \cdot 13,16,19$, we may transform any symbol $Ea_xa_ya_z \cdots$ into the symbol for a second element, and a third element, and these elements will constitute the "visible" parts of one face of an icosahedron of higher species.

In case there are elements of more than one type E, we may distinguish them by subscripts written to the letters, as E_1, E_2, E_3, etc. In such a case the

"visible" portions of one face of an icosahedron of higher species may be denoted by a group, G_a, of elements as follows:

$$G_a = \begin{cases} E_1, & E_2, & E_3, & E_4, & E_5, \\ SE_1, & SE_2, & SE_3, & SE_4, & SE_5, \\ S^2E_1, & S^2E_2, & S^2E_3, & S^2E_4, & S^2E_5. \end{cases}$$

With reference to the face a, the remaining faces of the icosahedron are given by groups of simply isomorphic substitutions.

The paper was illustrated by paper models chosen from the collection of the author.

SECOND SESSION OF THE ASSOCIATION

(4) "The past decade in cosmology," by Professor W. D. MACMILLAN, University of Chicago.

(5) "Some geometrical aspects of physics," by Professor L. D. EISENHART, Princeton University.

(6) "The development of the Russian mathematical school," by Professor J. D. TAMARKIN, Brown University.

4. For one hundred years the ideas of scientific men with respect to the origin of stars—and our own sun and planetary system in particular—were completely dominated by the nebular hypothesis of Laplace. That a vast cloud of hot gas, slowly rotating, should radiate its heat, contract and rotate more rapidly and result in a solar system such as our own seemed so simple and so obvious that almost no one doubted it. Such progress as was made served to strengthen it. The proof given by Helmholtz in 1854 that the process of contraction in itself would furnish enough heat to last twenty million years at the present rate of radiation gave the hypothesis great weight; and Lane's proof in 1870 that the temperature of monatomic gases would rise during the process added to its beauty.

The first serious hypothesis of a different character was put forward in 1900 by Professor T. C. Chamberlin of the University of Chicago. This hypothesis, known as the Planetesimal Hypothesis, was developed jointly by Professors Chamberlin and Moulton, who succeeded in convincing a large body of serious students that the origin of our planetary system was due to a close approach to the sun of another star. The birth of the planetary system was bi-parental, as Professor Chamberlin put it.

The first serious effort to account for the origin of the stars themselves and their abundant stores of energy by a different hypothesis was put forward in 1918 by Professor W. D. MacMillan, also of the University of Chicago, who regards matter as but one of the phases or states of energy. Energy is

conserved, but matter is not conserved. Matter comes into existence and passes out of existence, as do all organized structures. In the stars matter passes out of existence on account of the excessive gravitational stresses. Its energy, which is chiefly electrostatic potential energy, takes the radiant form, and is the source of the radiant energy of all the stars. Its amount, according to modern electrical theories, is sufficiently great to account for the phenomena observable in the skies and to form the basis of a rational cosmology. The radiant energy of the stars in its passage through space is, so to speak, condensed back into matter again, accounting not only for the blackness of the skies of night, but also for the vast clouds of tenuous gases which are visible there.

Energy is the one physical agent and its changes of state are manifested to us as physical phenomena.

5. Professor Eisenhart was not able to present his paper or to attend the meetings. The hope was repeatedly expressed by members that the Association may be favored by having this paper at its next meeting.

6. Professor Tamarkin gave a short review of what has been done in mathematics in Russia during the last 75 years. The reviewer's attention was concentrated mainly upon the three most outstanding figures of the Russian Mathematical School: Tchebyshev, Markov and Liapunov. A brief analysis of their work as well as of the work of some of their followers was presented.

GIFT TO THE CHAUVENET PRIZE FUND

A pleasant surprise, made known at the meeting of the Board of Trustees Monday evening and announced in the absence of the President at the session of the Association Tuesday afternoon, was the following communication from President Ford:

The Trustees of the Mathematical Association of America.

GENTLEMEN:

At the present time it is the custom of the Association to award a prize of $100 once in five years to the author of the best mathematical paper of expository character published during that period, the exact conditions of the award being as stated in the Monthly for October, 1925, page 439. It has occurred to me that this prize should, if possible, be awarded at more frequent intervals and it is my desire, therefore, to present to the Association at this time the sum of $500, the interest upon which combined with the income from the Association funds already allocated for this purpose will enable the prize to be given hereafter at an interval of every three years.

Very respectfully yours,

WALTER B. FORD.

The Trustees voted to accept the gift and to express to Professor Ford the appreciation and thanks of the Board for this admirable advance made possible in one of the outstanding activities of the Association. President Ford had

planned to make this gift at the close of his term of office, but determined to do so at this time inasmuch as he is now leaving for a year's absence to be spent in Europe.

MEETING OF THE BOARD OF TRUSTEES

Seven trustees were present at the session on Monday evening.

The following eleven persons were elected to membership, on application duly certified:

To Individual Membership

L. T. BLACK, A.M. (Michigan). Instr., Ashland College, Ashland, Ohio.

R. S. CHRISTIAN, A.B. (Westminster). Instr., Westminster College, Fulton, Mo.

W. M. DAVIS, M.S. (Iowa). Instr., Albion College, Albion, Mich.

EUGENE FEENBERG. Asst., Physics, University of Texas, Austin, Texas.

MAY N. HARWOOD, A.M. (Syracuse). Asst. Prof., Syracuse Univ., Syracuse, N. Y.

M. F. HOOPES, A.B. (Oberlin). Teacher, Southern State Normal, Springfield, S. Dak.

DENNIE MAY JONES, A.B. (North Texas State Teachers College). Prin., High School, Stephenville, Texas.

SALLIE E. PENCE, A.M. (Kentucky). Prof., Murray State Teachers College, Murray, Ky.

CAROLINE E. SEELY, Ph.D. (Columbia). 501 W. 116th St., New York, N. Y.

W. F. SMITH, A.B. (Western Kentucky State Teachers College.) Grad. Student, Univ. of Kentucky, Lexington, Ky.

DOROTHY H. VAN DEUSEN, B.S. in Eng. (Michigan). Senior Math. Teacher, High School, Battle Creek, Mich.

The Secretary-Treasurer reported that President Ford had appointed as the Association's delegates to the International Mathematical Congress Professors Daniel Buchanan, H. J. Ettlinger, E. B. Stouffer, J. W. Young and Dr. G. A. Plimpton.

Consideration was given to the proposal of an official affiliation between the Association and the National Council of Teachers of Mathematics, and of the appointment of a commission for the study of the redistribution of topics in plane and solid geometry. In view of the importance of these proposals and of a discussion by a full board of trustees, it was deemed necessary to postpone this until the December meeting. Such affiliation and study are entirely in line with the past policy of the Mathematical Association of America.

President Ford's gift of $500 to the Chauvenet Prize Fund has been described in an earlier part of this report.

Other business considered the greater effectiveness of the potentially useful Bureau of Information of Appointments conducted for the Association by Professor H. W. Kuhn, of Ohio State University, the distribution of the forthcoming Ahmes Papyrus, plans for a proposed international journal on the history of mathematics and measures for reducing the cost of printing the Monthly and for simplifying the estimation of printing costs.

W. D. CAIRNS, *Secretary-Treasurer.*

THE EARLY HISTORY OF PARTIAL DIFFERENTIAL EQUATIONS AND OF PARTIAL DIFFERENTIATION AND INTEGRATION

By FLORIAN CAJORI, University of California

The general events associated with the evolution of the fundamental concepts of fluxions and the calculus are so very absorbing, that the history of the very specialized topic of partial differential equations and of partial differentiation and integration has not received adequate attention for the early period preceding Leonhard Euler's momentous contributions to this subject. The pre-Eulerian history of the partial processes of the calculus is difficult to trace, for the reason that there existed at that time no recognized symbolism nor technical phraseology which would distinguish the partial processes from the ordinary ones. In consequence, historians have disagreed as to the interpretation of certain passages in early writers. As we shall see, meanings have been read into passages which the writers themselves perhaps never entertained. In connection with fluxions certain erroneous *a priori* conceptions of their theory were entertained by some historians which would have been corrected, had these historians taken the precaution of proceeding more empirically and checking their pre-conceived ideas by reference to the actual facts.

Partial Processes in the writings of Leibniz and his immediate followers

Partial differentiation and partial integration occur even in ordinary processes of the calculus where partial differential equations do not occur. The simplest example of partial differentiation is seen in differentiating the product xy, where one variable is for the moment assumed to be constant, then the other. Leibniz used partial processes, but did not explicitly employ partial differential equations. He actually used special symbols, in a letter[1] to de l'Hospital in 1694, when he wrote "δm" for the partial derivative $\partial m/\partial x$, and "ϑm" for $\partial m/\partial y$; De l'Hospital used "δm" in his reply of March 2, 1695. As stated in his letter, Leibniz considers the integration of $bdx+cdy$, where b and c involve x and y, and seeks an equation $m=0$ where m also involves x and y. Differentiating $m=0$ yields him $\delta mdx+\vartheta mdy=0$. We have here a total differential equation. It follows, he says, that $b:c=\delta m:\vartheta m$ or $b\vartheta m=c\delta m$. In the analysis which follows this statement, Leibniz says that this last equation is to be satisfied *identically*. It is clear that in deriving the above total differential equation Leibniz differentiates partially, taking first x as an independent variable, then y as an independent variable. That the *identity* which follows was recognized by him as a partial differential equation is not clear. Such recognition would demand in case of an identity an abstract view point hardly

[1] Leibnizens Mathematische Schriften, vol. II, Berlin (1850), pp. 261, 270.

attributable to writers in the prelude period of the history of partial differential equations.

A recent writer claims that Leibniz did use partial differential equations: "Auf partielle Differentialgleichungen kommt Leibniz durch ein geometrisches Problem."[1] Reference is made to Leibniz's article[2] of 1694 in which he finds the envelope of the circles $x^2+y^2+b^2=2bx+ab$. Differentiating with reference to b as a variable parameter, Leibniz obtains $2bdb=2xdb+adb$. Eliminating b between $2b=2x+a$ and the given equation, he obtains as the required envelope the parabola $ax+(a^2/4)=y^2$. After studying Leibniz's introductory remarks, we feel that, in differentiating the two sides of the equation, he consciously kept both x and y constant and took b as an independent variable. We feel this notwithstanding the fact that he does not state this relation explicitly when differentiating. Leibniz did not call the equation $2b=2x+a$ by any special name. It is not a differential equation, but the process of partial differentiation is involved in its derivation.

Nor can we accept the validity of the recent claim[3] that Jakob Hermann used partial differentiation and partial differential equations in 1717, in special solutions of the celebrated problem of orthogonal trajectories to plane curves.[4] That problem, as ordinarily treated, does not give rise to partial differential equations, nor even to partial differentiation, except perhaps in the differentiation of implicit functions. The process, as followed by Hermann, consists in finding the total derivative dy/dx, introducing $-dx/dy$ in its place, and eliminating a parameter. Hermann solves four special cases, but does not give the equations to be differentiated in the form requiring partial differentiation.

However, Hermann did use partial differentiation on another occasion. Leibniz,[5] in a letter to John Bernoulli, describes a procedure which Hermann is reported to have explained to Chr. Wolf, and which clearly involves this process.

Partial differential equations stand out clearly in six examples on trajectories published in 1719 by Nicolaus Bernoulli (1695–1726),[6] the twenty-four year old son of John. He takes the curve $y^m=a^{m-1}x$, "cujus differentialis com-

[1] Edmund Hoppe, Archiv für Geschichte der Mathematik, der Naturwissenschaften und der Technik, vol. 10 (1927), pp. 161, 162. Hoppe refers to articles and letters of Leibniz in Acta eruditorum (1694), p. 311; Leibnizens Mathematische Schriften, vol. II, p. 166, vol. III (1855), pp. 967, 969. But we have not been able to find that Leibniz gives or solves partial differential equations in these articles.

[2] Acta eruditorum, 1694, p. 311.

[3] E. Hoppe, *loc. cit.*, p. 163.

[4] J. Hermann, Acta eruditorum (1717), p. 349.

[5] Leibniz's letter to John Bernoulli, July 26, 1716, Leibnizens Mathematische Schriften, vol. III, pp. 967, 968.

[6] Acta eruditorum (1719), p. 298. Republished in Johannis Bernoullo Opera omnia, Lausannae et Genevae, vol. II (1742), p. 399.

pleta est $my^{m-1}dy=(m-1)a^{m-2}x\ da+a^{m-1}\ dx$; hic $p=my^{m-1}:a^{m-1}=mx:y$, et $q=(1-m)x:a$, . . . ". Here we have the "complete" differentiation, followed by the two partial differential equations, in which $p=\partial x/\partial y$ and $q=\partial x/\partial a$.

Argument over a differential equation of Newton involving three variables

In his *Method of Fluxions* Newton solves differential equations and gives one example, $2\dot{x}-\dot{z}+\dot{y}x=0$, of a differential equation involving three variables. The fluxion $\dot{x}$ signifies our time-derivative dx/dt. On the European continent, some writers have interpreted this equation as a partial differential equation, other writers, as a total differential equation. We begin by quoting the passage in Newton:[1]

"The Resolution of the Problem will soon be dispatch'd, when the Equation involves three or more Fluxions of Quantities. For between any two of those Quantities any Relation may be assumed, when it is not determined by the State of the Question, and the Relation of their Fluxions may be found from thence; so that either of them, together with its Fluxion, may be exterminated. Let the Equation proposed be $2\dot{x}-\dot{z}+\dot{y}x=0$; that I may obtain the Relation of the Quantities x, y, and z, whose Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$ are contained in the Equation; I form a Relation at pleasure between any two of them, as x and y, supposing that $x=y$ or $2y=a+z$, or $x=yy$ etc. But suppose at present $x=yy$ and therefore $\dot{x}=2\dot{y}y$. Therefore writing $2\dot{y}y$ for $\dot{x}$, and yy for x, the Equation proposed will be transform'd into this: $4\dot{y}y-\dot{z}+\dot{y}y^2=0$. And thence the relation between y and z will arise, $2yy+\frac{1}{3}y^3=z$. In which if x be written for yy and $x^{\frac{3}{2}}$ for y^3, we shall have $2x+\frac{1}{3}x^{\frac{3}{2}}=z$. So that among the infinite ways in which x, y, and z may be related to each other, one of them is here found, which is represented by these Equations, $x=yy$, $2y^2+\frac{1}{3}y^3=z$, and $2x+\frac{1}{3}x^{\frac{3}{2}}=z$."

The well known French writer on the calculus, Lacroix,[2] interpreted Newton's equation in three variables as a total differential equation.

On the other hand Weissenborn[3] says that Newton's problem was "nothing less than that of partial differential equations," in the treatment of which he was "not successful" since his solution is incorrect, as "one may see easily by trial." Weissenborn assigns no special reason for interpreting it as a partial differential equation any more than had Lacroix for calling it a total. That Newton's equation was "partial" was held also by the Swiss historian Heinrich Suter,[4] and the noted German historian Moritz Cantor[5] in the third volume of his

[1] Sir Isaac Newton, *The Method of Fluxions*, translated into English by John Colson, London (1736), p. 41.

[2] S. F. Lacroix, *Traité du calcul différentiel et du calcul intégral*, 2d. ed., vol. II (1814), p. 691.

[3] Hermann Weissenborn, *Die Principien der höheren Analysis*, Halle (1856), p. 39.

[4] H. Suter, *Geschichte der Mathematischen Wissenschaften*, Zürich, vol. II (1875), p. 74.

[5] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. III, 1898, p. 166.

Vorlesungen, but in the Preface Cantor retracts the statement, only to adhere to his original view in the second edition of that volume which appeared three years later.[1]

In direct opposition to Weissenborn's interpretation are E. Tischer[2] of Leipzig, Zeuthen[3] of Copenhagen, Eneström[4] of Stockholm and, very recently, Hoppe of Göttingen. Our own conclusion is in agreement with the last four writers; Newton's equation in three variables is a total differential equation. Newton treats this equation precisely as he does differential equations involving two variables, except that he assumes now a second relation $x=yy$ to exist, so that he can eliminate x and $\dot{x}$, and thereupon proceed as in case of two variables. Newton does not refer to any new principle involved in his equation in three variables. Moreover, Newton's solution of the equation is correct on the assumption that the differential equation is total, but incorrect on the assumption that it is partial. The conclusion is firmly established that neither Newton nor Leibniz in their published writings ever wrote down a partial differential equation and proceeded to solve it.

Erroneous conceptions about the theory of fluxions

While we agree with Tischer, Zeuthen and Hoppe that Newton's three-variable equation is a total differential equation, we do not agree at all with the reason which they assign for their conclusion. They base it on a preconceived erroneous conception according to which a partial differential equation is impossible on the Newtonian theory of fluxions, for the reason that Newton's fluxions are all time-derivatives and therefore exclude, as the critics state, any independent variable other than "time." Thus Tischer says on page 40 of his tract: "The concern here is not with a function of several independent variables, but with several functions of one and the same independent variable." Zeuthen states[5]: "Since the entire theory of fluxions rests upon the assumption of a single independent variable, he (Newton) remarks that in cases where an equation is given involving more than two variables with their fluxions, new relations between the variables may be introduced." We quote also Hoppe's statement: "When Newton has an equation in several fluents (we would say in variable magnitudes), that is, in x, y, z, u, etc., he derives the fluxions by assuming that

[1] *Op. cit.*, vol. 3, 2d. ed. (1901), p. 172.

[2] Ernst Tischer, *Ueber die Begründung der Infinitesimalrechnung durch Newton und Leibniz*, Jahresbericht des Nicolaigymnasiums in Leipzig, Leipzig (1896), pp. 37–39.

[3] H. G. Zeuthen, *Geschichte der Mathematik im XVI und XVII. Jahrhundert*, Deutsche Ausgabe, von R. Meyer, Leipzig (1903), p. 379; also Zeuthen in Bullet. de l'Acad. d. sc. de Danemark (1895), p. 263.

[4] G. Eneström, Bibliotheca mathematica, vol. IV (1903), p. 400; vol. 11 (1910–11), pp. 172–173.

[5] Zeuthen, *op. cit.*, p. 379. See also p. 358.

these variables are all functions of *one and the same variable magnitude*, for example, of the time. Then he marks the fluxion of x by $\dot{x}$, of y by $\dot{y}$, etc., so that $\dot{x}$ means in our notation dx/dt, $\dot{y}=dy/dt$, $\dot{z}=dz/dt$. Accordingly, if Newton had chosen to write his fluent equation briefly $f(x, y, z, \ldots)=u$, he would have meant by this, that $x, y, z, \ldots, u$ are functions of t, but not that u is a function of the independent variables, $x, y, z, \ldots$. The $x, y, z, \ldots, u$ were themselves functions of the variable t the time or of the temperature or some similar variable, and f signified only that an equation existed between these functions. If Newton wished to represent dy/dx he had to write $dy/dx=dy/dt \div dx/dt=\dot{y}\div\dot{x}=a:1$, where a is a magnitude measured according to the unit of the fluxions. . . . In this theory of fluxions no path was open to partial differential equations."

These writers are correct in stating that all fluxions are time-derivatives, but where in Newton and other writers on fluxions is it stated that all fluxions must result from *contemporaneous* fluents? Why is it not possible to consider the velocity (fluxion) of u when only x changes, or when only y or z changes? These continental writers assume all fluxions to be contemporaneous and do not go to the trouble to see what the practice of British writers really was in this respect. To show the error of this contention it is sufficient to quote from Newton and other writers on fluxions where partial processes freely enter. To present our case convincingly, we shall go into considerable detail and thereby hope to make a contribution to the history of partial processes in England during the time of Newton and the eighteenth century.

Partial differentiation and partial integration in Newton

In the following quotation[1] from the *Method of Fluxions*, Newton explains the differentiation of an implicit function in x and y:

"If the relation of the flowing Quantities x and y be $x^3-ax^2+axy-y^3=0$; first dispose the Terms according to x, and then according to y, and multiply them in the following manner.

Mult.	$x^3 - ax^2 + axy - y^3$	$- y^3 + axy - ax^2 + x^3$
by	$\frac{3\dot{x}}{x} \cdot \frac{2\dot{x}}{x} \cdot \frac{\dot{x}}{x} \cdot 0$	$\frac{3\dot{y}}{y} \cdot \frac{\dot{y}}{y} \cdot 0$
makes	$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$ *	$- 3\dot{y}y^2 + a\dot{y}x$ *

The sum of the Products is $3\dot{x}x^2-2a\dot{x}x+a\dot{x}y-3\dot{y}y^2+a\dot{y}x=0$, which Equation gives the Relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleasure, the Equation $x^3-ax^2+axy-y^3=0$ will give y. Which being determined, it will be $\dot{x}:\dot{y}::3y^2-ax:3x^2-2ax+ay$."

[1] Sir Isaac Newton, *Method of Fluxions*, edited by John Colson (1736), p. 21.

Here clearly Newton allows x to vary, while y remains constant, and *vice versa*. In modern symbols, if $z=x^3-ax^2+axy-y^3$, then $\partial z/\partial x=3x^2-2ax+ay$, $\partial z/\partial y=ax-3y^2$. Such a procedure is a violation of the theory of fluxions as understood by Tischer, Zeuthen and Hoppe.

In the process of partial integration, Newton's statement is equally clear. He considers the problem: A fluxional equation being given, to find a fluent equation. "As this Problem is the Converse of the foregoing, it must be solved by proceeding in a contrary manner."[1] He solves several examples. Thus Newton clearly and fully explained partial differentiation and partial integration, but nowhere does he give a partial fluxional or differential equation.

Newton worked on two problems for which he published only conclusions, namely the problem of the solid of least resistance,[2] and the problem of the path of a ray of light in a heterogeneous medium.[3] The modern general treatment of these problems involves partial differential equations. Whether or not Newton himself used such equations we cannot profitably discuss here.

Other British writers giving partial differentiation and partial integration

We have not been able to discover the explicit use of partial processes in Maclaurin, Taylor and Stirling. Apparently using partial fluxions consciously, John Turner,[4] a friend of Thomas Simpson, in 1748, maximized $v^4x^3y^2z$ when $v+x+y+z=b$. "Expunging z, $b-v-x-y=1/v^4x^3y^2$. In fluxions $-\dot{v}-\dot{x}-\dot{y}=-2\dot{y}/y^3x^3v^4-3\dot{x}/y^2x^4v^4-4\dot{v}/y^2x^3v^5$; whence $\dot{y}=2\dot{y}/y^3x^3v^4$, $\dot{x}=3\dot{x}/y^2x^4v^4$, $\dot{v}=4\dot{v}/y^2x^3v^5$.

And $1/v^4x^3y^2=y/2=x/3=v/4=b-v-x-y$." Thereupon each unknown is found in terms of b. In this process a total fluxional equation is found first; thereupon the partial fluxion obtained when y alone varies, on the left side of the fluent equation, is equaled to the partial fluxion with respect to y, on the right side of the equation. Similarly for the partial fluxions with respect to x, and v, respectively. Practically the same problem is solved in the same way by William Emerson.[5]

Using the fluxional notation, John Playfair performs partial differentiation and partial integration in finding solids of greatest attraction.[6] Partial processes occur, of course, in books employing partial fluxional equations.

[1] *Method of Fluxions*, p. 25.

[2] Newton's *Principia*, Bk. II, Prop. 34, Scholium. See a modern treatment in A. R. Forsyth, *Calculus of Variations*, Cambridge (1927), p. 340; O. Bolza, Bibliotheca Mathematica, vol. 13 (1912–13), p. 146.

[3] Newton's *Opticks*, Book II, Part III, Prop. 8. See a modern treatment of the problem in R. A. Herman, *Treasise on Geometrical Optics*, Cambridge (1900), p. 308.

[4] The Mathematical Questions proposed in the Ladies' Diary, by Thomas Leybourn, vol. 1, London (1817), p. 395.

[5] W. Emerson, *The Doctrine of Fluxions*, 3d. Ed., London (1768), pp. 174, 175.

[6] Transactions of the Royal Society of Edinburgh, vol. 6 (1812), pp. 195, 196.

Partial Differential Equations in Books on Fluxions

British writers on the history of the calculus have never claimed for themselves any share, however modest, in the development of partial differential equations before the nineteenth century. John Leslie[1] speaks of the "capital extension about the middle of the last century by what is termed the Calculus of Partial Differences, which applies with singular felicity to the solution of the most arduous and recondite physical problems.. . . The first specimen of this sort of Integration was given by Euler in 1734, but D'Alembert expanded the process in his Discourse on the *General Cause of the Winds*, which appeared in 1749.. . . " Similar statements are made by David Brewster in the *Edinburgh Encyclopaedia*, article "Fluxions."

Nevertheless, I have found a few occurrences of partial fluxional equations. In 1737, Thomas Simpson[2] derived the maximum of the expression $(b^3-x^3)(x^2z-z^3)\cdot(xy-y^2)$. He began: "First considering y as a variable, we have $x\dot{y}-2y\dot{y}=0$, or $y=\frac{1}{2}x \therefore xy-yy=\frac{xx}{4}$. By making z variable, we have $x^2\dot{z}-3z^2\dot{z}=0$, or $z=\frac{x}{\sqrt{3}}$. . . ." We see that, in obtaining the first differential equation, the fluxion of the given expression is found when y is the independent variable, z and x being taken to be constant; in obtaining the second equation, z is the independent variable, y and x being taken constant. If we introduce the letter u to represent the expression to be maximized, then, in modern symbols, the above analysis includes the process of finding $\partial u/\partial y=A(x-2y)$, $\partial u/\partial z=B(x^2-3y^2)$, where $A\equiv(b^3-x^3)(x^2y-z^3)$, and $B\equiv(b^3-x^3)(xy-y^2)$. There are thus obtained two simultaneous partial differential equations with two independent variables y and z, in which, for a maximum value of u, $\partial u/\partial y=0$ and $\partial u/\partial z=0$.

A similar problem, to find the minimum of an expression $xx+yy+zz$, when $ax+by+cz=d$, is answered in the *Ladies' Diary* for 1757–58 by Lionel Charlton of Whitley, and seems to indicate that partial processes were understood by the rank and file of British mathematicians. "Now seeing that any two of the quantities x, y, z may be varied independently of the other, we shall (by making x and y to flow, while z remains constant) have $a\dot{x}+b\dot{y}=0$ and $2x\dot{x}+2y\dot{y}=0$" "In the same manner" he lets x and z flow, while y remains constant. This amounts to taking y and z as independent variables and x as a dependent variable. Observing that $\dot{x}/\dot{y}=\partial x/\partial y$, there are derived in the solution of this problem the simultaneous partial differential equations in two independent variables $x\partial x/\partial y+y=0$, $x\partial x/\partial z+z=0$, $a\partial x/\partial y+b=0$ and $a\partial x/\partial z+c=0$.

[1] John Leslie, "On the Progress of Mathematical and Physical Science, chiefly during the eighteenth Century" in the eighth edition of the Encyclopaedia Britannica, p. 715.

[2] Thomas Simpson, *Treatise of Fluxions*, London (1737), p. 35.

A later writer, Vince,[1] solves the problem, given $x+y+z=a$ and xy^2z^3 a maximum, to find x, y, z. Vince says: "Let us suppose such a value of y to remain constant, whilst x and z vary till they answer the conditions, and then $\dot{x}+\dot{z}=0$ and $z^3\dot{x}+3xz^2\dot{z}=0$. . . . Now let us suppose the value of z to remain constant, and x and y to vary"

Partial differential equations are rarities in English articles of the eighteenth century and in English books (with the exception of Waring's). Rigorous conditions for maxima and minima in expressions of more than two variables were not attempted.

Edward Waring

He was the only eighteenth century Englishman who wrote on partial differential equations other than the simplest types of the first order. He was a Senior wrangler and was described as "one of the strongest compounds of vanity and modesty." According to David Brewster, "His writings are the only mathematical works published in this country, until late years, that have kept pace with the improvements made in this science on the continent." Waring admitted that he "never could hear of any reader in England, out of Cambridge, who took the pains to read and understand" his writings. In his *Meditationes analyticae* he devoted to partial differential equations twenty-four pages (pp. 231–254) in the first edition (1776), and seventeen pages in the second edition (1785). A persistent student may master this topic as presented in the first edition, but in the second edition hope of conquest vanishes. There are here fewer examples; the statements of processes have a brevity and generality never attempted by earlier writers nor probably by later ones. On this subject Waring displayed less originality than in the theory of equations, algebraic curves, and the theory of numbers. In the introduction to the first edition of his *Meditationes analyticae* he expresses indebtedness on this subject to Clairaut, Euler, D'Alembert and Condorcet; in the second edition, also to Fontaine, Lagrange and Laplace, but nowhere does he give specific bibliographical references. We find that nearly all partial differential equations given in the first edition are found in Euler's *Institvtiones calcvli integralis*, Petropoli, 1770. Waring's presentation is of interest in showing how this subject is treated in the fluxional notation. We give in translation from his Latin the following definition (1st ed., p. 231): "Let the quantity $\left(\frac{\dot{V}}{\dot{x}}\right)$ enclosed in parentheses denote the value of the fluxion of V, where x alone is variable, divided by $\dot{x}$; that is, if $\dot{V}=p\dot{x}+q\dot{y}$, then $\left(\frac{\dot{V}}{\dot{x}}\right)$ denotes the quantity p, and $\left(\frac{\dot{V}}{\dot{y}}\right)$ denotes the quantity q." "By $\left(\frac{\dot{V}}{\dot{y}\dot{x}}\right)$ is denoted the second fluxion of V, divided by $\dot{x}\dot{y}$, in the

[1] S. Vince, Principles of Fluxions, 1st American Edition, Philadelphia (1812), p. 19.

derivation of which at first x alone varies, then y alone varies." In our notation these derivations are $\partial V/\partial x$ and $\partial^2 V/\partial x \partial y$. The use of parentheses to distinguish partial derivatives from total derivatives was introduced by Euler. In the first edition, Waring treats briefly D'Alembert's equation arising in the famous problem of vibrating strings, $\left(\frac{\ddot{z}}{\dot{y}^2}\right) = a\left(\frac{\ddot{z}}{\dot{x}^2}\right)$, also thirteen other linear partial differential equations of the second order, two equations of the first order and one of the third. In the second edition the treatment is more cavalierly presented; the solutions of only eight equations of the second order are sketched, and three theorems are given relating to equations of the nth order, one for homogeneous equations with constant coefficients and x and y as independent variables, another for homogeneous equations with constant coefficients and any number of independent variables, and the third for linear partial differential equations. The following quotation (2nd ed., p. 299) exhibits the notation and Waring's style of presentation:

"Sit aequatio $P\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^n}\right)+Q\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^{n-1}\dot{y}}\right)+R\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^{n-2}\dot{y}^2}\right)+\text{etc.}+P'\left(\frac{\overset{n-1}{\dot{V}}}{\dot{x}^{n-1}}\right)+Q'\left(\frac{\overset{n-1}{\dot{V}}}{\dot{x}^{n-2}\dot{y}}\right)$ $+R'\left(\frac{\overset{n-1}{\dot{V}}}{\dot{x}\dot{y}^{n-2}}\right)+\text{etc.}+P''\left(\frac{\overset{n-2}{\dot{V}}}{\dot{x}^{n-2}}\right)+\text{etc.}+LV=0$; ubi in singulis terminis V vel ejus fluxio unam solummodo habet dimensionem; in hâc aequatione pro V & ejus fluxionibus scribantur $e^v \times u$ & ejus correspondentes fluxiones, ubi v & u sunt functiones quantitatum x & y: functiones v & u pendent e functionibus P, Q, R, etc., P', Q', etc., P'', etc. quo magis simplices sunt priores functiones, eo magis plerumque simplices erunt posteriores."

SOME TETRAHEDRAL COMPLEXES

By NATHAN ALTSHILLER-COURT, University of Oklahoma

1. Consider the lines s in space such that the feet of the perpendiculars dropped upon them from a fixed point A lie in a given plane α.

2. What is the configuration formed by the lines s which pass through a given point M? The foot U of the perpendicular AU from A upon s lies in the plane α and also on the sphere (AM) having the segment AM for diameter. Therefore the lines s passing through M project from M the circle of intersection of (AM) with the plane α. Thus: *The lines s which pass through a given point in space form, in general, a cone of second degree.*

3. The lines s which lie in a given plane μ may be obtained as follows. Take any point U on the line of intersection $\alpha\mu$ of the planes α, μ and erect, in μ, the perpendicular s to AU. Now if A' is the projection of A upon μ, $A'U$ is, by a well known theorem of elementary geometry, perpendicular to the line s. Thus the line s is a side of a right angle in the plane μ, the other side of which

passes through the fixed point A' and whose vertex U lies on a fixed line $\alpha\mu$ of μ. Hence: *All the lines s lying in a fixed plane envelope, in general, a parabola.*

The point A' is the focus of the parabola, and the line $\alpha\mu$ is the tangent at the vertex of the curve.

4. The preceding two paragraphs prove the proposition: *All the lines in space such that the feet of the perpendiculars dropped upon them from a fixed point lie in a given fixed plane form a tetrahedral complex.*[1]

5. Consider any two spheres (B), (B') whose centers B, B' are symmetric with respect to the point A (§1) and whose radical plane coincides with the plane α, but which do not lie one within the other.

Let s be a line of the tetrahedral complex (S) (§4), and C, D; C', D' the points of intersection of s with (B), (B') respectively. Let E, E' be the midpoints of the segments CD, $C'D'$, and let Q be the trace of s in α. The three planes perpendicular to s at the points E, Q, E' pass through the points B, A, B' respectively, hence

$$QE = QE'. \tag{1}$$

Since the point Q lies in the radical plane α of the two spheres, we have, both in magnitude and in sign,

$$QC \cdot QD = QC' \cdot QD'$$

or $$(QE + EC)(QE + ED) = (QE' + E'C')(QE' + E'D')$$

or $$(QE + EC)(QE - EC) = (QE' + E'C')(QE' - E'C')$$

or $$QE^2 - EC^2 = QE'^2 - E'C'^2.$$

Hence, considering formula (1), $EC^2 = E'C'^2$; that is, the chords CD, $C'D'$ are equal. Thus: *Any line of the complex of* §4 *determines equal chords in any two spheres whose centers are symmetrical with respect to the fixed point of the complex and whose radical plane coincides with the fixed plane of the complex.*

6. By reversing the steps in the above proof (§5) it may be shown that if a line determines equal chords in the two spheres (B), (B'), the line belongs to the complex of §4. Hence: *The lines on which two given spheres intercept two equal chords form a tetrahedral complex.*

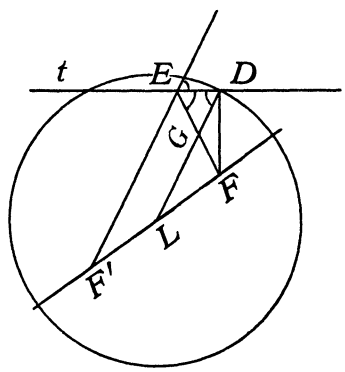
7. Consider all the lines t in space such that the feet of the perpendiculars dropped upon them from a fixed point A lie on a given sphere (C).

8. In order to find the configuration formed by the lines t (§7) which pass through a given point M we observe that the foot of the perpendicular dropped from A upon t lies on the sphere (AM) having the segment AM for diameter

[1] A tetrahedral complex is defined as the totality of straight lines which cut the four faces of a tetrahedron in four points having a constant anharmonic ratio. An example of a tetrahedral complex was discussed in this Monthly, vol. 21 (1914), pp. 287, 288, by D. N. Lehmer.

and also on the sphere (C), hence the lines t passing through M project from M the circle common to these two spheres. Thus: *The lines t (§7) passing through a given point form, in general, a cone of second degree.*

9. Let (L) be the circle of intersection of the sphere (C) with a given plane μ. If t (§7) lies in this plane, the line AD joining A to one of the points of intersection, say D, of t with (L) is perpendicular to t, hence t is also perpendicular to the line FD, where F is the projection of A upon the plane μ. What is the envelope of t?



Let F' be the symmetric of F with respect to the center L of (L), and E the trace on t of the parallel $F'E$ through F' to the radius LD. Since L is the mid-point of FF', the line LD bisects the segment EF, say, in G, so that in the right triangle DEF we have $DG=\frac{1}{2}EF$, and further $GL=\frac{1}{2}EF'$. Therefore

$$FE + F'E = 2DG + 2GL = 2LD.$$

Thus the sum of FE and $F'E$ is constant.

We also have angle $LDE=$angle GED; hence t is the external bisector of the angle FEF'.

In the above figure the point F falls inside the circle (L). If F falls outside of (L), instead of the sum, it will be the difference of the segments FE, $F'E$ that will be equal to the diameter of (L); and t will be the external bisector of the angle FEF'. Thus: *The lines t (§7) which lie in a given plane envelope, in general, a central conic. The circle common to the given sphere and the plane considered is the auxiliary circle of the conic. The projection of the given point upon the plane considered is a focus of the conic.*

10. The last two paragraphs (8, 9) prove the proposition: *All the lines in space such that the feet of the perpendiculars dropped upon them from a given point lie on a given sphere form a tetrahedral complex.*

11. Consider all the lines m in space from which two given points A, B may be projected by two orthogonal planes.

The planes mA, mB being orthogonal, the perpendiculars AC, BD from A and B to the line m are also perpendicular to the planes mB, mA, respectively.

(a). If C, D coincide, say, in W, then AW, BW are perpendicular to each other, and W is a point on the sphere (AB), having the given segment AB for diameter, the line m being the tangent to (AB) at W.

It is readily shown that, conversely, if a tangent m to (AB) at a point W is perpendicular to AW, it is also perpendicular to BW, and therefore the two planes mA, mB are orthogonal.

(b). If the points C, D are distinct, the line AC is perpendicular to the plane $BCD \equiv nB$, and therefore to the line BC; hence the point C lies on the sphere (AB). Similarly for D.

Conversely, suppose a line m' meets the sphere (AB) in two points C, D, and the line AC joining one of them, say, C, to the point A is perpendicular to m'. Then AC is perpendicular to both m' and BC, hence to the plane $BCD \equiv m'B$, and the two planes $m'A$, $m'B$ are orthogonal. It may further be shown that the line BD is perpendicular to m'.

It follows that the lines m considered are such that the feet of the perpendiculars dropped upon them from the point A (or B) lie on the sphere (AB), where A is a point on that sphere. Hence (§10): *The lines in space from which two fixed points may be projected by two orthogonal planes form a tetrahedral complex.*

12. The cone (P) formed by the lines of the complex (M) (§11) passes through the circles of intersection of the sphere (AB) with the two spheres having for their diameters the segments PA, PB (§8). The planes of these two circles are respectively perpendicular to the lines PB, PA. Thus the cone (P) is orthogonal and PAB is a plane of symmetry of this cone.[1]

13. Consider any two orthogonal spheres (A), (B) having the given points A, B (§11) for centers. Let U, U' be the points of intersection of the sphere (A) with a line m of the complex (M) (§11) and let (E) be the sphere having the segment UU' for diameter. The radical plane of the spheres (A), (E) passes through $m \equiv UU'$ and is perpendicular to the line AE joining the centers A, E of the two spheres; hence this plane is perpendicular to the plane mA, and therefore coincides with the plane mB. Thus the sphere (B) has its center in the radical plane of the spheres (A), (E), and is orthogonal to (A); hence (B) is also orthogonal to (E). Now m passes through the center of (E); hence the points of intersection V, V' of m with (B) are harmonically separated by the points U, U' common to m and (E), or, what is the same thing, to m and (A). Consequently: *Any line of the quadratic complex* (§11) *determines two harmonic*

[1] Theodor Reye, *Geometrie der Lage*, fifth edition, vol. 1, p. 119. See also this Monthly, December, 1916, p. 387.

chords in any two orthogonal spheres having for their centers the two fixed points of the complex.

14. Let us now consider the converse proposition. Suppose a line n meets two given orthogonal spheres (A), (B) in two pairs of harmonic points U, U'; V, V'. The sphere (E) having UU' for diameter is orthogonal to (B), for the points U, U' are inverse with respect to (E). Thus (B) is orthogonal to both (A) and (E), hence the center B of (B) lies in the radical plane of the two spheres (A), (E). Now this radical plane $UU'E \equiv nB$ is perpendicular at the mid-point E of UU' to the line of centers EA of these two spheres; hence any plane passing through EA, and in particular the plane (n, EA) is perpendicular to nB, i.e., the planes which project from n the centers A, B of the two spheres are mutually orthogonal. Thus (§11): *The lines on which two given orthogonal spheres determine pairs of harmonic chords form a tetrahedral complex.*

15. The line $m \equiv CD$ (§11) is perpendicular to both lines AC, BD; hence m is the line of their shortest distance. Furthermore the lines AC, BD are perpendicular to each other. Hence the proposition (§11) may also be stated as follows: *The line of shortest distance of two perpendicular lines which pass each through a fixed point forms a tetrahedral complex.*

By making use of the imaginary circle at infinity the reader may state this proposition in a projective form.

THE OSCILLATION OF AN INFINITESIMAL BODY ATTRACTED BY THREE FINITE MASSES

By W. L. DUREN, JR., Tulane University

1. Introduction. It is well known that three spheres placed at the vertices of an equilateral triangle with proper initial velocities will revolve in circles about their common center of gravity. We propose to investigate the motion of an infinitesimal[1] body placed at the center of gravity of such a system and given an initial velocity, v_0, in a direction perpendicular to the plane of the motion of the finite bodies. In order to simplify the problem let us consider only the case of three equal spheres homogeneous in concentric layers. Then the three finite masses rotate about their common center of gravity in a circle whose radius may be taken as unity by a suitable choice of the unit of length.[2] Let the coordinates be referred to a set of axes so that the three bodies lie in the (ξ, η) plane with the origin at the center of gravity. And let these axes be rotating about the ζ axis with the same angular velocity as that of the three

[1] For the definition of an infinitesimal body, see F. R. Moulton, *An Introduction to Celestial Mechanics* (1923), p. 277.

[2] Ibid., p. 311.

finite bodies so that the three bodies retain fixed positions with respect to the ξ and η axes. It is evident then that the motion of the infinitesimal body will be restricted to the ζ axis.

2. The differential equations of the motion. Let the mass of each finite body be M. Then the differential equations of the motion of the body reduce to a single one:

$$d^2\zeta/dt^2 = -a^2\zeta(1+\zeta^2)^{-3/2}; \qquad a^2=3M. \tag{1}$$

We wish to integrate this equation subject to the initial conditions, $\zeta=0$, $d\zeta/dt=v_0$, $t=0$. The right hand member of (1) can be expanded into a power series in ζ of the form:

$$\sum_{k=0}^{\infty}A_{2k+1}\zeta^{2k+1}, \text{ where } A_{2k+1}=\frac{(-1)^k 1\cdot 3\cdot 5\cdots(2k+1)}{2^k k!}, \qquad k=0,1,2\cdots;$$

so that equation (1) may be written as

$$d^2\zeta/dt^2 = -a^2\sum_{k=0}^{\infty}A_{2k+1}\zeta^{2k+1}, \tag{2}$$

in which the right hand member converges for $|\zeta|\leqq r<1$.

3. Existence of power series solutions of this equation. Let us make the substitutions[1] $\zeta=\epsilon' z$, $t=(1+\delta)\tau$.

Equation (2) becomes:

$$d^2z/d\tau^2 = -(1+\delta)^2a^2\sum_{k=0}^{\infty}A_{2k+1}\epsilon'^{2k}z^{2k+1}.$$

Now let us replace ϵ' by a parameter ϵ which takes on small values. Then we consider the solution of the equation

$$d^2z/d\tau^2 = -(1+\delta)^2a^2\sum_{k=0}^{\infty}A_{2k+1}\epsilon^{2k}z^{2k+1}, \tag{3}$$

which, however, has a physical significance only when we replace ϵ by ϵ'. We propose to integrate this equation subject to the initial conditions,

$$z=0, \quad z'=\alpha+\gamma, \quad \tau=0, \tag{4}$$

where α is a constant and γ is a variable parameter. Equations (3) and (4) are equivalent to the following set of equations:

$$dz/d\tau = v+\alpha+\gamma, \tag{5}$$

$$dv/d\tau = -a^2z-\delta a^2(1+\delta)z-\epsilon^2a^2(1+\delta)^2\sum_{k=1}^{\infty}A_{2k+1}\epsilon^{2k-2}z^{2k+1}, \tag{6}$$

[1] See F. R. Moulton, *Periodic Orbits*; Carnegie Institute of Washington, Publication No. 161 (1920), p. 155.

and the initial conditions $z=0$, $v=0$, $\tau=0$. We seek solutions of this equation as power series in ϵ, δ, and γ which vanish identically in ϵ, δ, and γ when $\tau=0$. The right hand members of (5) and (6) are regular for $\epsilon=\delta=\gamma=z=v=\tau=0$. Hence (5) and (6) may be solved as power series in ϵ^2, δ, and γ for z and v.[1] These solutions will be in the form

$$z = P_1(\epsilon^2,\delta,\gamma\ ;\tau), \qquad v = P_2(\epsilon^2,\delta,\gamma\ ;\tau),$$

where P_1 and P_2 denote power series and τ does not occur explicitly. With the help of (5) these may be written in the form:

$$z = P_1(\epsilon^2,\delta,\gamma\ ;\tau), \tag{7}$$

$$z' = P_2(\epsilon^2,\delta,\gamma\ ;\tau) + \alpha + \gamma, \tag{8}$$

where z' denotes $dz/d\tau$.

4. Determination of δ to make these solutions periodic. If the solutions (7) and (8) are to be periodic in τ with period T, it is necessary and sufficient that[2]

$$z(T+\tau) - z(\tau) \underset{\tau}{\equiv} 0, \tag{9}$$

$$z'(T+\tau) - z'(\tau) \underset{\tau}{\equiv} 0. \tag{10}$$

In this special case we can show that these equations are not independent but that (10) is a consequence of (9). Substituting the result (7) in (9) we get

$$P_1(\epsilon^2,\delta,\gamma\ ;T+\tau) - P_1(\epsilon^2,\delta,\gamma\ ;\tau) \underset{\tau}{\equiv} 0.$$

Since this is an identity in τ, when we differentiate with respect to τ we get, by means of (5) and (8),

$$P_2(\epsilon^2,\delta,\gamma\ ;T+\tau) - P_2(\epsilon^2,\delta,\gamma\ ;\tau) \underset{\tau}{\equiv} 0.$$

This equation is equivalent to (10).[3]

Substituting the power series solution (7) in equation (3) and equating the terms independent of ϵ, γ, and δ, we get

$$d^2z_0/d\tau^2 = -(1+\delta)^2a^2z_0,$$

where z_0 is the term of P_1 which is independent of ϵ, δ, and γ. The general solution of this equation is

$$z_0 = B_1 \sin(1+\delta)a\tau + B_2 \cos(1+\delta)a\tau.$$

Upon applying the initial conditions $z_0=0$, $z_0'=\alpha+\gamma$, $\tau=0$ this becomes

$$z_0 = (\alpha+\gamma)(1+\delta)^{-1}a^{-1} \sin(1+\delta)a\tau. \tag{11}$$

[1] This is a set of equations of type II according to F. R. Moulton, *Periodic Orbits*, p. 15.

[2] The symbol $\underset{\tau}{\equiv}$ is here used to mean, "equal identically in τ."

[3] In a similar problem, D. Buchanan depresses the order of the system by a particular integral. See F. R. Moulton, *Periodic Orbits*, p. 326.

We have shown that the first periodicity equation (9) alone is necessary and sufficient. Hence substituting (7) in equation (9) we obtain with (11) the equation

$$0 = (\alpha + \gamma)(1 + \delta)^{-1}a^{-1} \sin 2\pi(1 + \delta) + \epsilon^2 Q_1(\epsilon^2, \delta, \gamma), \tag{12}$$

where Q_1 is a power series in ϵ^2, δ, and γ, from which equation we seek to determine the δ and γ as power series in ϵ^2. No solution of this equation giving γ as a power series in ϵ^2 for an arbitrary value of δ exists. For if δ is not zero or an integer, $\gamma = \epsilon^2 = 0$ is not a solution as required by (7). If δ is zero or an integer then for every value of γ the linear term of (12) vanishes.

We then turn to solving (12) for δ as a power series in ϵ^2, setting $\gamma = 0$ arbitrarily. A particular solution of this equation is $\epsilon^2 = \delta = 0$. Moreover the partial derivative of the right hand member with respect to δ is

$$\alpha a^{-1}[2\pi(1 + \delta) \cos 2\pi(1 + \delta) - \sin 2\pi(1 + \delta)](1 + \delta)^{-2} + \epsilon^2 Q_{1\delta}(\epsilon^2, \delta, \gamma).$$

This does not vanish at $\epsilon^2 = \delta = 0$ except for the trivial case $\alpha a^{-1} = 0$. Hence we may solve equation (9) for δ as a power series in ϵ^2, vanishing for $\epsilon^2 = 0$ and convergent for sufficiently small values of ϵ^2.[1]

That is, the solution is of the form

$$\delta = \epsilon^2 p(\epsilon^2), \tag{13}$$

where $p(\epsilon^2)$ is a power series. Moreover this solution is unique. Then for $\gamma = 0$ and for the value of δ from (13) we get a solution of (3) as a power series in ϵ^2 which is convergent in the neighborhood of $\epsilon = 0$. And each term of this expression is separately periodic in τ with period $2\pi/a$. This solution also satisfies the initial conditions,

$$z = 0, \qquad z' = \alpha, \qquad \tau = 0. \tag{14}$$

5. Direct construction of the solution. We have shown that the equations (3) and (4) can be satisfied by relations of the form

$$z = \sum_{j=0}^{\infty} z_{2j}\epsilon^{2j}, \qquad \delta = \sum_{j=1}^{\infty} \delta_{2j}\epsilon^{2j}. \tag{15}$$

Substituting these in (3) we get

$$\sum_{j=0}^{\infty} \epsilon^{2j} \frac{d^2 z_{2j}}{d\tau^2} = -a^2(1 + \delta_2\epsilon^2 + \cdots)^2\{(z_0 + z_2\epsilon^2 + \cdots) + A_3\epsilon^2(z_0 + z_2\epsilon^2 + \cdots)^3 + \cdots\}. \tag{16}$$

[1] F. R. Moulton, *Periodic Orbits*, p. 326.

We seek to determine the z_{2j} and so to determine the δ_{2j} that the z_{2j} will be periodic functions of τ. From the explicit form of (15) we see that the initial conditions (14) are equivalent to conditions of the form,

$$(17) \qquad \begin{cases} z_0 = 0, & z_0' = \alpha, \quad \tau = 0, \\ z_{2j} = 0, & z_{2j}' = 0, \quad \tau = 0, \quad j = 1,2,3, \cdots. \end{cases}$$

Equating the coefficients of ϵ^0 in (16) we get $d^2z_0/d\tau^2 = -a^2z_0$. The solution of this equation satisfying (17) is $z_0 = \alpha a^{-1} \sin a\tau$. Equating the coefficients of ϵ^2 we get

$$d^2z_2/d\tau^2 = -a^2z_2 - 2a^2\delta_2 z_0 - a^2A_3z_0^3 ;$$

and expressing z_0^3 as a trigonometric sum in sines of odd multiples of $a\tau$, we get

$$d^2z_2/d\tau^2 = -a^2z_2 - (2a\alpha\delta_2 + \tfrac{3}{4}\alpha^3a^{-1}A_3) \sin a\tau + \tfrac{1}{4}\alpha^3a^{-1}A_3 \sin 3a\tau.$$

In order that the solution of this equation be periodic, it is necessary and sufficient that the coefficient of $\sin a\tau$ vanish identically, namely that $\delta_2 = -\frac{3}{8}\alpha^2a^{-2}A_3$. Then a solution which satisfies the initial conditions and is periodic is

$$z_2 = 2^{-5}\alpha^3a^{-3}A_3(3 \sin a\tau - \sin 3a\tau).$$

Similarly δ_4 and z_4 can be determined and it is found that

$$\delta_4 = -303 \cdot 2^{-10}\alpha^4a^{-4},$$

$$z_4 = -\frac{1}{2^7}\frac{\alpha^5}{a^5}\left\{\left(\frac{41}{8}A_3^2 - \frac{40}{3}A_5\right)\sin a\tau - \frac{1}{8}\left(\frac{3}{2}A_3^2 - 5A_5\right)\sin 3a\tau - \frac{1}{24}\left(\frac{3}{8}A_3^2 + A_5\right)\sin 5a\tau\right\}.$$

It is necessary to complete this method by showing by an induction that as many of the δ_{2r} and the z_{2r} may be calculated by this method as is desired. This will be done in the next paragraph.

The solution of (3) can now be written out to three terms as follows:

$$z = \alpha a^{-1} \sin a\tau - 3 \cdot 2^{-6}\alpha^3a^{-3}(3 \sin a\tau - \sin 3a\tau)\epsilon^2 - 2^{-12}\alpha^5a^{-5}(821 \sin a\tau - 192 \sin 3a\tau - 29 \sin 5a\tau)\epsilon^4 \cdots ;$$

and δ is expressed to two terms by the following series:

$$\delta = 9 \cdot 2^{-4}\alpha^2a^{-2}\epsilon^2 - 303 \cdot 2^{-10}\alpha^4a^{-4}\epsilon^4 \cdots .$$

Now if we again set $\zeta = \epsilon'z$, $\epsilon = \epsilon'$, $a\tau = a(1+\delta)^{-1}t = ct$, we get a solution having a physical significance once more. And differentiating with respect to t we see that

$$d\zeta/dt = \epsilon'(dz/d\tau) = \epsilon'(dz/d\tau)(d\tau/dt) = \epsilon' ca^{-1}(dz/d\tau).$$

But when $t=0$, also $\tau=0$, $d\zeta/dt=v_0$, $dz/d\tau=\alpha$, so that we have $v_0=\alpha\epsilon' c/a$; and if we let $\alpha\epsilon'/a=v_0/c=h$, we get a solution of (3) in the form

$$\zeta = h\sin ct - 3\cdot 2^{-6}h^3(3\sin ct - \sin 3ct) - 2^{-12}h^5(821\sin ct - 192\sin 3ct - 29\sin 5ct)\cdots; \tag{18}$$

and c is given by the expression:

$$c = 3^{1/2}M^{1/2}/[1 + (9/16)h^2 - (303/1024)h^4\cdots].$$

The period of the solution (18) is $T=2\pi/c$.

6. The induction on the general terms of the solution. Let us assume that z_{2j} and δ_{2j} have been determined up to and including z_{2r-2} and δ_{2r-2}, that is for $j=1, 2, \cdots, (r-1)$, and let us further assume that they have the following properties:

(i) The δ_{2j} are not all identically zero and depend upon α, a, and A_{2j-1}.

(ii) The z_{2j} are not all identically zero and are functions of τ as trigonometric sums of sines of odd multiples of $a\tau$, the highest multiple being $(2r-1)\,a\tau$.

We propose to show that the δ_{2r} and z_{2r} can then be determined and that z_{2r} is a trigonometric sum of sines of odd multiples of $a\tau$, the highest multiple being $(2r+1)a\tau$. By equating the coefficients of ϵ^{2r} in (16) we see that δ_{2r} and z_{2r} must satisfy an equation of the form

$$d^2z_{2r}/d\tau^2 = -a^2z_{2r} - 2a^2\delta_{2r}z_0 - a^2R_{2r}(z_{2j},\delta_{2j}), \quad j = 1,2,\cdots,(r-1). \tag{19}$$

Now since the expansion in the right member of (3) proceeds in odd powers of z we see that $R_{2r}(z_{2j}, \delta_{2j})$ is a polynomial in z_{2j} of degree not greater than $2r+1$ and having each term of odd degree in z_{2j}; and by property (ii) R_{2r} may be regarded as a polynomial in $\sin a\tau$ with coefficients involving α, a, and A_{2j+1}. Since such a polynomial can always be expressed as a trigonometric sum of sines of odd multiples of $a\tau$ of order not greater than $2r+1$, equation (19) takes the form

$$d^2z_{2r}/d\tau^2 = -a^2z_{2r} - \left(2a^2\delta_{2r} - a^2\sum_{\lambda\mu}c_{\lambda\mu}\delta_{2\lambda}\delta_{2\mu} - a^2E_{2r}\right)\sin a\tau + T_{2r+1}[\sin(2k+1)a\tau,\ \delta_{2j}], \tag{20}$$

wherein $\lambda+\mu=r$, $c_{\lambda\mu}=2$ if $\lambda\neq\mu$ and $c_{\lambda\mu}=1$ if $\lambda=\mu$, and E_{2r} is the constant coefficient of $\sin a\tau$ in the development of R_{2r} in a trigonometric sum. T_{2r+1} is a trigonometric sum of order $2r+1$ having coefficients involving the δ_{2j} and in which $k=1,\cdots,r$. If the solution of (20) is to be periodic it is necessary and sufficient that the coefficient of $\sin a\tau$ vanish, that is that the following equation holds:

$$2\delta_{2r} + \sum_{\lambda\mu}c_{\lambda\mu}\delta_{2\lambda}\delta_{2\mu} + E_{2r} = 0. \tag{21}$$

This equation is linear in δ_{2r}, and δ_{2r} can always be determined uniquely from it by property (i). With equation (21) satisfied, the differential equation (20) can always be solved subject to the initial conditions given. Moreover the solution will be periodic and z_{2r} will be expressed as a trigonometric sum in $\sin(2j+1)a\tau$, where $j=1, 2, \cdots, r$ at most. Hence the induction is complete and the construction may be carried as far as is desired, each term having the properties mentioned above.

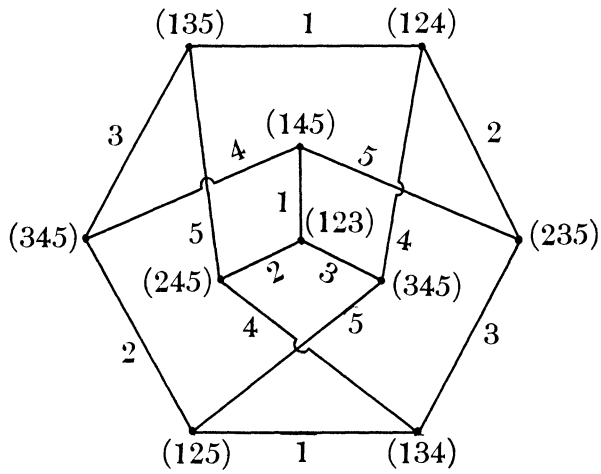
QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTION

57. A reader of this Monthly desires information as to the literature and some discussion of a certain configuration in S_3 arising out of a "complete pentrahedral" and, of course, dualizable for a complete pentagon. When the planes 1, 2, 3, 4, 5 of a complete pentahedral meet by threes in space, in ten points (ijk), there is formed in each plane a complete quadrilateral by the lines



in which it meets the other planes. The configuration referred to is formed by the 15 diagonals, three in each plane, of these quadrilaterals. Their connections are shown in the appended diagram, in which accidental crossings of lines not corresponding to intersections in space are marked in the electric engineer's way. The skew polygons of the least number of sides meeting in each vertex

are six pentagons, and there are, in all, twelve pentagons. The number of vertices and lines is half that of a dodecahedron and one is tempted to use the latter as a diagram, with certain instructions as to the pairing of vertices and edges. The lines ("diagonals") are indicated in triplets by the numeral of the plane in which they lie. In the actual configuration, and as it happens, in two cases out of three in the diagram, each pair of diagonals of the same numeral divides the third harmonically. What about the group theory?

DISCUSSION

A Note on Bertrand Curves

By Harold L. Dorwart, Williams College

Almost everyone who has done any reading in differential geometry knows of the interesting class of curves whose principal normals are the principal normals of other curves. These are commonly known as Bertrand curves from the study of them which M. J. Bertrand has made in a part of his *Mémoire sur la théorie des corbes à double courbure* in Liouville's Journal de Mathématiques, volume 15 (1850). It is the purpose of this note to point out that the names of two other men might also be associated with these curves.

It was Saint-Venant who first raised the question of the existence of these curves in his *Mémoire sur les lignes courbes non planes* in the Journal de l'Ecole Polytechnique, volume 18 (1845). In a footnote on page 48 he asks: "Sur la surface gauche formée par l'ensemble des rayons de courbure d'une courbe donnée, peut-on tracer une seconde courbe dont les génératrices de la surface soient aussi les rayons de courbure?"

The principal result that M. Bertrand found in his *Mémoire*, published in 1850, was that the curvature and torsion of these curves satisfy a linear relation

$$(a/\rho) - (C/R) = 1,$$

to use his notation (page 348, line 10), where a and C are constants, ρ is the radius of first curvature, and R is the radius of second curvature or torsion. This formula, however, or rather it's equivalent, had already been published by M. O. Bonnet in his *Mémoire sur la théorie générale des surfaces* in the Journal de l'Ecole Polytechnique, volume 19 (1848). On page 136, line 2, one finds

$$\tan \phi = \pm arR^{-1}(r - a)^{-1},$$

which can be easily written

$$ar^{-1} \pm (a/\tan \phi)R^{-1} = 1.$$

On page 135, the angle ϕ, which is the angle between the osculating planes of a Bertrand curve-pair, is shown to be constant; a denotes the constant distance between the two curves, r the radius of first curvature, and R the radius of

second curvature. Hence, placing $a/\tan\phi = \mp C$, this formula is identical with the formula of M. Bertrand.

That M. Bonnet felt there was a question of priority here is shown in a portion of one of his letters to M. A. Mannheim which M. Mannheim has reprinted in his *Note sur Géométrie* in Comptes Rendus,[1] volume 85 (1877), page 788.

Vous avez annoncé l'intention de revenir, dans vos intéressantes études relatives aux surfaces réglées, sur les propriétés de la surface lieu des normales principales communes à deux courbes. Vous m'obligeriez beaucoup, si vous mettez votre projet à exécution, de vouloir bien signaler quelques réclamations de priorité que le manque d'occasion favorable m'a jusqu'ici empêché de produire.

La formule que vous attribuez[1] à M. Bertrand se trouve dans mon premier Mémoire publié en 1848, ...

Of the well known writers on differential geometry, Scheffers is the only one who mentions Saint-Venant, and Darboux is the only one who mentions Bonnet in connection with the Bertrand curves.

In conclusion it might be well to state that the study of these curves, as well as the study of many curves more or less closely connected with them, has been greatly simplified by the introduction of the moving trihedral.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE JUNIOR MATHEMATICS CLUB OF THE UNIVERSITY OF WISCONSIN, Madison, Wisconsin.

The officers of the Junior Mathematics Club of the University of Wisconsin for the year 1926–1927 were: President, Karl Jansky; vice-president, Bertha Furminger; secretary-treasurer, Peter Finstad.

The following programs were given:

October 7, 1926. Business Meeting. Plans for the year were discussed.

October 12. Annual fall picnic.

October 21. "American mathematicians" by Professor E. B. Skinner.

November 4. Business meeting and reception of new members.

November 18. Illustrated lecture "The history of mathematics" by Professor Arnold Dresden.

December 2. "Crinkly curves" by Professor L. W. Dowling.

December 16. First annual banquet. "Unilateral and bilateral surfaces" by Professor E. B. Van Vleck who was the guest of honor.

February 17, 1927. "The effect of the thermal coefficient on the piezo-electric effect in quartz" by Mr. Harold Osterberg.

March 3. "Solid geometry" by Professor W. W. Hart.

March 14. "Theoretical study of quartz electric oscillators" by Professor E. M. Terry of the physical department.

April 21. "Unicursal quartics" by Miss Sara Zinder.

May 5. "Some classified problems in applied mathematics" by Professor H. W. March.

May 19. Election of officers and social hour. Professor Arnold Dresden was the honor guest.

June 1. Joint picnic with Senior Mathematics Club.

(Report by Professor Skinner)

[1] Comptes Rendus, volume 85 (1877), page 213. (The reference here is to the formula of Bertrand given above.)

THE MATHEMATICS CLUB OF SAINT STEPHEN'S COLLEGE, Annandale-on-the-Hudson, New York.

On Thursday evening, February 9, 1928, a mathematics club was formed at Saint Stephen's College under the supervision of Dr. Phalen. Twenty-two men were enrolled in the organization as charter members. The following officers were elected: President, Harry Dillin ('28); Vice-President, Charles Miller ('30); secretary, Walter Lemley ('30); treasurer, John Hagen ('30). Dr. Phalen explained the purpose of the Club and a program committee was appointed to draw plans for future activities. The first regular meeting was held in the Students' recreation hall. The following program was given:

1. "Pythagoras and his school" by Harry Dillin ('28).
2. "Duplication of the cube by the cissoid of Diocles" by Charles Miller ('30).
3. "Mechanism and use of the star clock" by William Meissner ('31).

(Report by Walter H. Lemley ('30))

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City.

Officers of the Club at the Main Building: Antoinette Finocchi, president; Rose Klein, vice-president; Lillian Abramowitz, treasurer; Rosemary Tighe, secretary; Margaret Hosey, publicity manager; Dr. Lester S. Hill, faculty adviser.

For the first time in the history of the club, all meetings of a year were devoted to the systematic discussion of a single topic. Operations in finite algebraic fields were studied, and the resulting geometry of finite sets was developed to a considerable extent. Fields were treated in which the number of elements was any positive integral power of a prime positive integer. The principal object of the symposium was to emphasize that projective geometry can be regarded as the mere picturization of the relations of algebraic forms. Those elementary non-metric theorems of ordinary familiar projective geometry in the plane which are direct consequences of the properties of linear and quadratic forms were derived and illustrated with particular care in the finite domain. Students presented the papers at every meeting and the program worked out with marked success. Two prizes were offered for the solution of the problems, one of which was related to the topic under discussion during the year. These prizes were won by Rose Klein and Florence Simon.

Each semester, there has been a large and enthusiastic audience to greet an outside speaker. On December 2, Mr. Frank C. Higgins addressed the club on the topic: *The 47th Problem of Euclid, its Influence on Religion and Philosophy.* On February 24, Dr. R. C. Archibald of Brown University inspired us with a talk on *Egyptian Mathematics.* The social activities have included a party at the beginning of the year for all club members, and for the freshmen, a theater party and several informal teas.

Hunter College now has three centers outside the Main Building and in each of these there has been an active club with a member of the faculty as adviser. The topics have been varied and have been presented by the faculty, students and outside speakers, including Mr. John A. Swenson of Wadleigh High School, and a representative of the Keuffel & Esser Co. A special feature of this club work has been the assigning of a problem for solution at each meeting.

(Report by Professor Lao G. Simons)

THE MATHEMATICS CLUB OF WESLEYAN COLLEGE, Macon, Georgia.

The officers for the year 1927–1928 were: Mary Brooks Lester, president; Geraldine Wheeler, vice-president; Essie Mae Cobb, secretary; Margaret Newton, treasurer.

The programs for the club for 1927–1928 were as follows:

September 26, 1927. Election of officers. "Suggestive topics for discussion by the club" by Miss Edna Day.

October 24. "Multiplication methods" by Mae Ainsworth. "Magic squares" by Florimel Williams.

December 5. "The three unsolvable problems" by Geraldine Wheeler.

February 23, 1928. "Short cuts in arithmetic" by Miss Edna Day.

March 12. "Civic values in the study of mathematics" by Vivian Majors.

"Geometric fallacies" by Edith Partin. "Calculating prodigies" by Estelle Perkins.

April 26. Picnic at Brown's Mountain.

The following officers were elected for the year 1928–1929: Geraldine Wheeler, president; Mary Brooks Lester, vice-president; Agnes Kelly, secretary; Estelle Perkins, treasurer.

(Report by Miss Essie Mae Cobb)

The Mathematical Club of Harvard University, Cambridge, Massachusetts.

During the year 1927–1928, the officers of the club were: Mr. C. I. Lubin, president; Mr. G. Bailey Price, secretary-treasurer; Professor Marston Morse, faculty adviser.

The programs of the club for 1927–1928 were as follows:

October 19, 1927. "Some aids in the study of mathematics" by Professor W. F. Osgood.

November 2. "Discontinuities of surface potential" by Mr. J. J. L. Hinrichsen.

November 16. "Mathematical impressions in Italy and France" by Professor J. L. Coolidge.

November 30. "Bicircular quartics" by Mr. Morris Marden.

December 14. "The 27 straight lines in the general cubic surface" by Mr. A. E. Currier.

January 4, 1928. "Integral curves through singular points of differential equations" by Mr. A. B. Brown.

January 18. "Function space" by Mr. S. S. Cairns.

February 15. "Linear diophantine equations" by Mr. J. K. Peterson.

February 29. "Bounded analytic functions" by Professor C. Caratheodory, visiting lecturer from the University of Munich.

March 14. "Poincaré's last geometric theorem" by Mr. C. N. Liu.

March 28. "The mathematics of wave mechanics" by Professor J. C. Slater.

April 25. "Transcendental numbers" by Mr. Charles Wexler.

May 9. "Weierstrass' continuous non-differentiable function" by Mr. O. J. Farrell.

May 23. "A problem in maxima and minima" by Professor Norman Miller of Queen's University, Kingston, Ontario. The winners of the Robert Fletcher Rogers Prizes were announced as follows: Mr. O. J. Farrell, first prize; Mr. S. S. Cairns, second prize, Mr. A. E. Currier, honorable mention.

The following officers were elected for the year 1928–1929: Mr. G. Bailey Price, president; Mr. G. A. Hedlund, secretary-treasurer; Professor M. H. Stone, faculty adviser.

(Report by G. Bailey Price)

The Mathematics Club of the North Carolina College for Women, Greensboro, North Carolina.

The officers of the club for the year 1927–1928 were: Elizabeth LeRoy ('28), president; Doris Hanvey ('28), vice-president; Tivila Mae Darden ('30), secretary-treasurer; Dallie Smith ('28), chairman of the program committee.

The club held monthly meetings with the program as follows:

October, 1927. "The history and calculation of π" by Doris Hanvey ('28).

"Two geometrical fallacies" by Margaret Redwine ('30).

November. "The historical development of logarithms and their computation" by Ethel Spruill ('29).

"On a circle attached to a collapsible 4-bar" by Dr. H. Barton.

"Two arithmetical tricks" by Virginia Tucker ('30).

December. "The life and works of Archimedes" by Ethel Spruill ('29).

"Magic squares" by Sallie Smith ('28).

February 1928. Initiation of new members. Social.

March. "Some old books on arithmetic" by Sallie Spratt ('29).

"The slide rule" by Bertha Barnwell ('29).

April. Illustrated lecture on Astronomy by Professor C. Strong.

May. Election of officers. The officers for 1928–1929 are: Sallie Spratt ('29), president; Louise Leary ('30), vice-president; Olive Renfroe ('31), secretary-treasurer.

(Report by Professor H. Barton)

The Napierian Club, De Pauw University, Greencastle, Indiana.

The officers for the year 1927–1928 were: President, Hugh N. Spohn ('28); vice-president, Mary L. Thomas ('28); secretary, Mary E. Cline ('28); treasurer, Ralph Ring ('28).

The following programs were given at the regular monthly meetings:

October 13, 1927. Selection of new members. Grades were considered.
"Different transformations" by Professor W. C. Arnold.

November 3. "The history of the Napierian Club" by Mary E. Cline. "The life and works of John Napier" by Mertion A. Lish.

January 5, 1928. "History of mathematics" by Charles La Hue. "Number scales" by Professor H. E. H. Greenleaf.

January 19. "Note on knots" by Geneva B. Annis.

March 8. "The making of maps—military style" by John C. Gregory.

April 5. Book review of "Flatland" by Christina E. Scull. "Mathematics in manufacturing" by Everett C. Tranbarger, given by Alfred L. Vaughan.

May 3. "Fourth dimension" by Wendell H. Furry. "Mathematical absurdities" by Karl L. Flanigan.

May 24. Election of officers.

(Report by Mary E. Cline)

Mathematics Club of Northwestern University, Evanston, Illionis.

During the year 1927–1928 the officers of the club were: President, Violet M. Andrews; vice-president, John O. Chellevold; secretary, Evelyn Frank; treasurer, Pearl Bierman; faculty adviser, Dr. Lois Griffiths. Meetings were held fortnightly and refreshments were served during a social gathering before each program.

The program for the year 1927–1928 was as follows:

November 16, 1927. "Maxima and minima" by Professor D. R. Curtiss.

December 1. "Synthetic discussion of conic sections" by Violet M. Andrews.

December 15. "Simple inequalities relative to a circle" by Professor H. A. Simmons.

January 19, 1928. "Pythagorean triads" by Edwin Comfort.

February 16. "Approximation to a power series by means of continued fractions" by Dr. Wall.

March 1. "Trisection of angles" by D. R. Clark.

March 15. "Types of indeterminate equations" by Elizabeth Williams.

March 29. "Congruences" by Albert May.

April 19. "Hyperbolic functions" by John Chellevold.

May 3. "A particular solution of a three-body problem" by H. Glenn Peebles.

May 24. The annual Mathematics Club picnic at Cook County Forest Preserve.

(Report by Violet M. Andrews)

The Mathematics Club of the Eastern Illinois State Teachers College, Charleston, Illinois.

Early in the school year the mathematics students of the college met and organized a mathematics club. The club had a membership of fifty students and teachers and held ten meetings throughout the school year with an average attendance of forty-one.

The program for the year 1927–1928 was as follows:

December 16, 1927. Organization meeting.

December 21. Business meeting with election of officers for the year 1927–1928. President, Cyril Reed; vice-president, Dwight Reed; secretary-treasurer, Lola Redden; program committee, three above officers, Lowell Story, Mildred Lacey; faculty member, Mr. E. H. Taylor.

January 11, 1928. "Mathematics in the German schools" by Mildred Lacey. "Fourth dimension" by Cyril Reed. "Mathematical recreation" by Samuel Mitchell.

January 18. "Perpetual calendar" by Ernest Ballard. "Non-Euclidean geometry" by Lola Redden. "Geometrical inconsistencies" by Thelma Quicksall.

February 8. "Pythagoras and the Pythagorean Club" by Rex Murray. "Introduction to infinity" by Hubert Schmidt. "Mathematical wrinkles" by Nolan Sims.

February 22. An old time ciphering contest between teams from the seventh and eighth grades and from each class in high school and college.

Winning team—Lowell Story, Dwight Reed—College Seniors
Individual Honor—Lucille Vanatti—High School Sophomore

March 21. (By the faculty members) "One hundred eighty six thousand" by Mr. Spicer. "Application of mathematics to business" by Mr. E. H. Taylor. "A paradox party" by Miss Alice Daniels.

April 16. "History of geometry" by Helen Sheehan "History of algebra" by Walter Van Dyke. "Mathematical recreation" by Dwight Reed.

May 3. "Ethical end of mathematics" by Burnis Hostetler. Election of officers for the year 1928–1929: President, Nolan Sims; vice-president, Burnis Hostetler; secretary-treasurer, Hazel Hicks.

May 28. Club picnic.

(Report by Lola Redden)

The Mathematics Club of George Washington University, Washington, D. C.

The club was organized on October 6, the following officers being elected for the academic year 1927–1928: Mr. Michael Goldberg, president; Mr. B. Z. McLeroy, secretary.

The program for the year 1927–1928 was the following:

October 6, 1927. "The Platonic and Archimedean solids" by Mr. Michael Goldberg.

October 20. "Mathematical puzzles" by Professor P. J. Federico of G. W. U.

November 2. "Pythagorean numbers" by Mr. B. Z. McLeroy.

November 16. "Restricted constructions" by Mr. Michael Goldberg.

November 30. "The operational calculus" by Professor C. L. Frederick of G. W. U.

December 14. "The principle of virtual work" by W. E. Deming of the Fixed Nitrogen Laboratory, American University, Washington, D. C.

January 11, 1928. "The geometry of plane transformations" by Dr. Tobias Dantzig of the University of Maryland.

February 1. "Summable series" by Professor W. J. Berry.

February 15. "Newton's classification of cubics" by Professor P. J. Federico.

February 29. "Some simple finite groups" by Professor F. E. Johnston.

March 14. "Poles and polars" by Mr. M. C. Simmons. "Nim, a mathematical game" by Mr. Michael Goldberg.

March 29. "Hyperbolic functions" by Professor J. T. Erwin.

April 18. "Fermat's last theorem" by Mrs. Lenora H. Jones.

May 9. "Matrices" by Dr. Tobias Dantzig of the University of Maryland.

(Report by Michael Goldberg)

Irrational Club, University of Wyoming, Laramie, Wyoming.

The Mathematics Club, known as the Irrational Club, of the University of Wyoming presented a varied program during the year 1927–1928. The officers for the year were: Jack Stenberg, president; Ruth Williams, vice-president; Chauncey Griswold, secretary-treasurer.

The program for the year was as follows:

October 12, 1927. The election of officers, followed by social hour.

October 27. "History and purposes of the club" by Jack Stenberg. "Frederick the Great and mathematics" by Chauncey Griswold.

November 17. "Space navigation" by Philip Pepoon. Algebraic ciphering match.

December 8. "Slide rules and their uses" by Professor J. C. Fitterer.

January 20, 1928. "The fourth dimension" by Professor O. H. Rechard.

February 3. "Mathematics and chemistry" by Mr. Kemp.

February 14. Social evening; mathematical recreations.

March 1. "Hyperbolic functions" by Miss Condit. "The nine-point circle" by Mr. Morgan.
April 12. "Mathematics as applied to military tactics" by Captain R. R. Ring.
April 26. "Napier's bones" by Mr. Ikuno. "The derivation of Pi" by Miss Morrow. "Life history of Sir Isaac Newton" by Mr. Achenbach.
May 15. "Some solutions of the cubic equation" by Miss Neubauer. "The measurement of the diameter of a star" by Mr. Pfeifer. "Continued fractions" by Mr. Taylor.
May 29. A beef-steak fry in the mountains.

(Report by Chauncey Griswold, Jr.)

THE PHI CHI MU CLUB OF WASHINGTON AND JEFFERSON COLLEGE, Washington, Pennsylvania.

It has been the custom of this club to hold one meeting each month and to have only one paper presented at each meeting, followed by a free for all discussion, which is generally finite but unbounded. The following papers have been presented by undergraduates since last report:

1. "An elementary exposition of statistics" by John H. Miller.
2. "The place of chemistry in medicine" by Harry Lachman.
3. "A new system of coordinates, and its value in geometry of inversion" by J. Knox Long.
4. "Isotopes" by W. H. Kunkelman.
5. "Speed of reaction in catalysis" by Patsy Briola.
6. "Progress in chemistry" by Perry Volpitto.
7. "The duodecimal system" by Walter A. Hawkinson.
8. "Progress of mathematics in America" by Fred K. Voehl.

The last meeting of the club was held May 15. The officers for the year 1928–1929 are: A. B. Bowden, president; Paul Jose, secretary-treasurer.

(Report by Professor C. S. Atchison)

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N.Y., to whom books and communications should be sent.

REVIEWS

The Mathematics of Statistics. By ROBERT WILBUR BURGESS. Boston, Houghton Mifflin Company, 1927. viii+304 pages.

The special aim of this book as stated in the preface is "to bring together the practical and the mathematical methods of approach to statistical analysis." It is fairly obvious that previous authors have tried to do this, but the author of the present book seems peculiarly qualified to do so because of his successful experience both as a teacher of mathematical statistics and as a practical statistician. If the special aim is carried out more effectively in this book than elsewhere, the publication seems worth while in spite of the recent large scale production of elementary textbooks on statistics.

A general elementary treatment is given with illustrations from many fields, such as biometry, education, economics, and business administration. The feature of the book which impresses the reviewer most favorably is the method of

presentation of regression and trend lines preliminary to correlation. This procedure tends to emphasize functional relationships in advance of the correlation coefficient. The treatment of regression includes, besides the linear case, the determination of the constants for parabolic regression $y=ax^2+bx+c$, with auxiliary tables to facilitate the computation in the case of historical series. Various devices for the treatment of seasonal variation are given with suggestive comments on the merits of the devices. In sampling theory, the elementary ideas are discussed but the treatment lacks precise criteria for simple sampling. The standard deviation of the binomial distribution and the probable error of the arithmetic mean are derived.

Chapters II and III do not come up to expectations of the reviewer, especially from the standpoint of presentation for beginners. It is his judgment that Chapter II on "Ratio or Percentage Analysis" would be much improved by putting to the front the type of concrete data to be analyzed instead of putting ratios or percentages to the front. Thus, if we should start with the four typical situations indicated near the top of p. 40, the common sense methods of analysis by the use of relative numbers, i.e., ratios and percentages, would suggest themselves very naturally. Again, the usefulness of the chapter would not be impaired by the omission of certain advice about fairly obvious matters. For example, it seems unnecessary to tell us that (p. 16) "an investigator should not find the ratio of a to b unless he thinks the meaning of the facts can be brought out more clearly in that way."

In Chapter III on "The Combination of Ratios" the beginner is likely to accept the treatment as being more general than it is because the limitations on the process of combination developed are not given sufficient prominence.

From the teaching standpoint, the usefulness of the book would be increased by giving the data on more of the problems instead of requiring a search for the data. For example, we find (Prob. 2, p. 40) such problems as the following: Analyze the expenditure of the United States Government over a series of years. Searching for such data without references will probably take more time than can be wisely given to the collection of data in a first course on Mathematical Statistics. This need for data could be supplied by a small supplementary problem book.

H. L. RIETZ

Advanced Algebra. By HERBERT E. HAWKES. Revised Edition. Boston, Ginn and Company, 1928. xxii+299 pages.

There are many changes in the revised edition of Hawkes' *Advanced Algebra.* There has been considerable rearrangement of material. The chapter on variation has been combined with that on ratio and proportion; the chapter on graph-

ical representation has been distributed among several chapters; a condensed treatment of the fundamental concepts and the paragraphs on linear equations have been inserted in the chapter on equations; and nearly all of the part on quadratic equations together with some of the theorems from the chapter on quadratics form a new chapter entitled "Properties and Graphs of Quadratic Equations." Logarithms have been moved from the latter part of the book to a place after the theory of indices. Combinations and permutations, with probability, form a chapter which follows the theory of equations. These are all changes which lend greater naturalness to the order of topics.

The parts omitted are largely those which we have found it desirable to pass over or touch lightly in using the text in a half-year college course. Equivalence of pairs of quadratic equations, several proofs of theorems by induction, a paragraph on solving equations when two roots are nearly equal, the evaluation of a determinant by factoring, and the chapter on inequalities will not be missed. But we wonder at the omission of polar representation of complex numbers and all that goes with it.

One might expect the problems in a book written about twenty-five years ago to be old-fashioned, but this is not particularly true in this case. The author has however deemed it advisable to write entirely new but similar sets of exercises. Bicyclists become automobilists, and motion pictures and airplanes have a place. Also a few sets of oral exercises are introduced. Trigonometric functions are defined in the chapter on logarithms, and a table of logarithms of trigonometric functions is added. Systems of linear equations whose determinants are zero are given a paragraph in the proper chapter. The difficulties of "variable" and "unknown" are smoothed by careful differentiation and correct use of these terms. There are other minor changes, too numerous to record here.

It is not a very different book; but a successful text on this subject, which was carefully written in the first place, should not need much revision. Those who have the old edition do not need to replace it, but the new one is a better, more teachable book.

G. A. HEDLUND

The Early Mathematical Sciences in North and South America. By FLORIAN CAJORI. Boston, Richard G. Badger, The Gorham Press, 1928. 156 pages.

In the history of education in the American colonies and earlier settlements on American soil, there has been, of recent years, a keen interest. Many studies of these educational aspects have appeared in magazine articles, bulletins of the Bureau of Education, and essays within the covers of a book. In these studies, the mathematical sciences have come in for a share of attention; but the work under review is the first output of a serious nature bearing directly on these

early efforts. It shows that much of this work was the product of European talent, in some instances that of men sent over on a special mission, although some of it was achieved by men born and educated in America. The amount of so-called pure mathematics was negligible.

The author characterizes his style in this work in the words: "snapshots of practices and achievements, here and there." Each chapter starts with certain introductory remarks which give an explanation of the subject matter and cite some historical references, as with "orreries." And this introduction is followed by an account of the beginnings of the subject on American soil. In passing, when the doctors disagree, it is open to question whether an author should state that "a set of data · · · point unmistakably to the conclusion that our $ or $" is this or that. The justification of such statements lies in the evidence the book offers of the most painstaking examination of a great number of secondary sources. There has hereby been made available an immense amount of valuable and interesting material not generally accessible, at any rate to the teacher of the history of mathematics. The work is interesting to the lay reader as well as to the reader more directly concerned with mathematical affairs.

The contents are: Mathematics, Practical Astronomy and Surveying, Survey and Maps, Meridian Measurements of the Earth, Transit of Venus, 1761 and 1769, Comets, Almanacs, Orreries, Earliest Permanent Observatory in America, Physics, Societies, Academies and Journals. The work has some thirty-two illustrations covering portraits, instruments, maps, and original documents. At the end of each chapter, there is an excellent bibliography which constitutes the sources from which the material has been drawn.

LAO G. SIMONS

Graphical Mathematics. By T. R. Running. New York, John Wiley and Sons, 1927. 89 pages. Price $1.75.

The present book is an introduction to the use of graphical methods, particularly in performing integrations and finding derivatives and in the solutions of differential equations. This principal part of the treatment may be considered as supplementing the first course in calculus, in much the same way that the usual chapter on empirical equations in our analytic geometry texts supplements the standard treatment of the straight line, circle, and conics. Graphical integration and differentiation is developed chiefly as a tool for the use of problems arising from experiment or observation.

The method of presentation is wholly that of working out in detail many illustrations covering the methods to be used. The illustrative examples are well chosen and clearly set forth, so that the problems at the end of each chapter may be solved by the models given in the text as a pattern.

Chapters II, III, IV deal with elementary analytic geometry methods for constructing charts, by means of which one may read off the results of substitution into formulas, or whereby one may readily obtain numerical solutions, real or imaginary, of the quadratic, cubic, or biquadratic equations. Chapter I gives a point-by-point method of plotting the general equation of the second degree. The method is that of finding the slope of the curve, dy/dx, as a fraction whose numerator and denominator are linear in x and y. This yields immediately a pair of conjugate diameters for the central conics, from which the center may be found, and then the axes as the pair of perpendicular diameters which are conjugate. By "reflecting" in these diameters, one may start with the X and the Y intercepts of the curve and without further numerical work obtain a considerable number of other points. For the parabola, the above method yields the axis of the parabola. For the hyperbola, after the center is found, the asymptotes are obtained by using dy/dx as a slope. For this chapter, the only preparation necessary is that of the knowledge of differentiating a positive integral power. Chapters V, VI, VII are essentially based on "counting squares." This is accomplished by using step functions as approximations. The natural and historical procedure of treating integration first is followed. Simpson's rule is used in connection with the solution of differential equations.

The reviewer has emphasized the elementary character of the author's treatment to exhibit the fact that these methods are available for freshmen and sophomores as well as juniors, assuming that in some institutions an elementary course in calculus comes at the close of the first year. The book is to be recommended for use where graphical methods are desired to supplement the usual calculus course. In his preface, the author emphasizes very properly the importance of "allowable error" and "close approximation." Would it not therefore be desirable to include at each stage some expression for the "limit of error"?

H. J. ETTLINGER

Spherical Harmonics. By T. M. MACROBERT. New York, E. P. Dutton and Company, 1927. xii+302 pages.

In considering a book like MacRobert's *Spherical Harmonics*, it is necessary to keep in mind the author's expressed or apparent aim in writing it. Professor MacRobert defines his purpose in the opening sentence of his preface: "The writing of this book was undertaken with the object of providing a text-book on the elements of the theory of the Spherical Harmonics, with applications to mathematical physics, so far as this could be done without employing the method of contour integration." The author has given us, as a consequence of this point of view, a companion to Gray and Matthews' well-known text on

Bessel functions, in the second edition of which he collaborated; and, in the field of spherical harmonics, will thus render a service comparable to theirs in the realm of Bessel functions.

The study of the spherical harmonics and their applications is wisely introduced by a brief account of the theory of Fourier series together with applications to problems of heat flow and elastic vibrations; the theory of the transverse vibrations of a stretched elastic string certainly affords the clearest insight into the mathematical facts and methods which are to be encountered in later chapters. With the fourth chapter the consideration of the main subject matter begins. The functions of Legendre are discussed at length and are then applied to the various kinds of spherical harmonics occurring in problems of gravitational and electrostatic potential theory. The material covered is classical. At the close of the book, there are three chapters on Bessel functions. An admirable feature is the introduction of a series of problems at the end of each chapter.

Professor MacRobert has assumed on the part of his reader a thorough knowledge of the use of power series, an elementary knowledge of the theory of functions of a complex variable, and a sound critical faculty. There are occasional points at which the reader must be on the alert to supply the reasoning necessary to justify assumptions of uniqueness, interchanges of order of integration, and so on. Now and then statements are made which are false unless amplified or interpreted in the light of facts not placed at the reader's disposal. Let us cite the sentence at the top of page 142. After the chapter on the convergence of Fourier series, there is no attempt to prove convergence and representation theorems for the formal series determined as solutions of physical problems, because of the fact that in no case have simple direct methods been published until recently. The great number of formulas crowded into the text increases its usefulness for reference purposes but tends to obscure the mathematical principles underlying the whole development. A good index makes the material readily available to any one using the book for reference.

I believe that Professor MacRobert could have performed a greater service by replacing his aim by either of two others. If, instead of excommunicating the methods of contour integration, he had proceeded to write a book of the same character as Watson's remarkable treatise on Bessel functions, he would have made a contribution which could scarcely be superseded in one or two generations. On the other hand, if he had written a clear introductory text dealing in an elementary manner with the convergence and applications of developments in spherical harmonics, without so much attention to formal properties of the Legendre functions, he would have earned the lasting gratitude of students approaching this branch of mathematical physics for the first time.

Since he has chosen neither of these aims, the merit of his work cannot, in my my opinion, be permanent. It is a useful book, especially at this time; for it appears at a peculiarly opportune moment, when physicists are taking renewed interest in spherical harmonics and similar functions because of their applications to some of the problems of wave-mechanics.

M. H. STONE

Zehn Vorlesungen über die Grundlegung der Mengenlehre. By A. FRAENKEL. Leipzig-Berlin, B. G. Teubner, 1927. 182 pages.

There are indeed few mathematicians who are not interested to a greater or less extent in the underlying structure of the subject. Consequently, a book such as the one under review, aside from its possibilities as a text book, holds a considerable amount of interest for almost any mathematician who might care to read it. The reviewer does not feel that it is a book written primarily, as might be inferred from the title, for the student or specialist in the so-called "Mengenlehre." It seems rather to make its greatest appeal to the person interested more particularly in mathematical logic and the foundations of mathematics. The book abounds in discussions of various mathematical paradoxes and antinomies, such as those due to Russell, Burali-Forti, Richard, and others. The author bases the whole treatment on as few undefined concepts as possible. Definitions are given for almost every term used—even for such terms as "set," "null set," "existence of a set," etc. In fact, it seems to the reviewer that too many concepts are defined; so many that it makes one suspect that a conflict is inevitable.

The book is conveniently divided into five parts of two lessons each, under the following titles: (I) Umrisse der Cantorschen Mengenlehre. Die Antinomien der Mengenlehre und ihre Wirkung; (II) Die nichtprädikativen Begriffsbildungen. Der Intuitionismus; (III) Die Axiome der Mengenlehre; (IV) Verschärfung des Aussonderungsaxioms. Allgemeines und Historisches zum Axiomensystem. Theorie der Äquivalenz; (V) Theorie der Ordnung. Die endlichen Mengen. Über die Vollständigkeit, Widerspruchsfreiheit und Unabhängigkeit des Axiomensystems.

The book is written in a very pleasing style, the lessons being addressed by the author to his class. The very sparing use of symbolic language makes the book quite readable and enhances its value as a reference work. The reviewer believes that this volume could be used quite successfully as a text or as a source book for a course in mathematical logic; and that it would be particularly adaptable where the instructor wishes to confine the main theme of his course to questions in the theory of sets.

G. T. WHYBURN

A Survey of Methods for the Inversion of Integrals of Volterra Type. By HAROLD T. DAVIS, Bloomington, Indiana University Studies, Nos. 76, 77, 1927. 72 pages. Price $1.00.

This pamphlet is the second of a series of monographs by the author on the general subject of integral equations. It is devoted to the inversion of the integral

$$f(x) = \int_a^x K(x,t)u(t)dt, \tag{1}$$

which is the form in which the subject of integral equations first acquired a systematic treatment by Volterra, and includes the famous equation of Abel, with singular kernel,

$$f(x) = \int_0^x \frac{u(t)dt^\alpha}{(x-t)^\alpha}.$$

The treatment includes a brief exposition of methods suitable for this type of equation, with singular and with non-singular kernel, and an appropriate bibliography selected from that already given in the author's earlier study on *The Present Status of Integral Equations.*[1]

Part I consists of the classical methods of solution by reduction to an equation of the second kind, by means of differentiation, if $K(x, x) \neq 0$, and reduction to a Fuchsian equation if the kernel vanishes suitably when $t=x$, with some consideration of the irregular singular point; there are included also equations, which are generalizations of Abel's equation, with kernel of the form $G(x, t)/(x-a)^\alpha$, where $G(x, x) \neq 0$ and $\alpha < 1$, the differential equations of infinite order equivalent to (1), and a brief examination of (1) again where $K(x, x) \neq 0$ and does not change sign, but where the integrals are Lebesque integrals. The exposition of this last section might well be more complete, since it requires considerable reconstruction on the part of the reader. The central method in this part is that of successive approximations.

In part II the author considers especially the kernels of the form $K(x-t)$ and shows the connection of particular forms of these kernels with the method of fractional differentiation—an exposition which will be of considerable benefit to the prospective reader. The possible connection is immediately perceived if one recalls the formula

$$ {}_cD_x^{-a}u(x) = \int_c^x \frac{(x-t)^{a-1}}{\Gamma(a)} u(t)dt, \quad a > 0,$$

$$ {}_cD_x^{m+a}u(x) = \frac{d^{m+1}}{dx^{m+1}} \int_c^x \frac{(x-t)^{-a}}{\Gamma(1-a)} u(t)dt, \quad \begin{cases} m = 0,1,2, \cdots, \\ 0 < a < 1, \end{cases}$$

[1] Indiana University Studies, No. 70, 1926.

and the form, for instance, of Abel's equation. Other special methods are also given, such as Whittaker's, where the kernel is represented by an interpolation formula.

The classification and selection of material for studies such as this is necessarily difficult and to some extent arbitrary. The reviewer would have liked in Part II to see a clear-cut exposition of the relation of the theory to the complex variable, but is quite ready to admit that such a discussion can appear equally well in the treatment of the applications. Further contributions to this series of studies will be awaited with interest.

G. C. EVANS

Probability and Its Engineering Uses. By THORNTON C. FRY, Ph.D. New York, D. Van Nostrand Company, 1928. Price $7.50.

This book, written by a member of the technical staff of the Bell Telephone Laboratories, is cordially recommended to the student and teacher of statistical methods, and to all who are interested in the mathematical theory of probability as a basis for the interpretation and control of experimental or observational work. Although it is not an uncommon practice in the making of tests and treatises to read nine books and then write a more or less conglomerate tenth, Dr. Fry's book departs widely from this procedure. It rings with sincerity in the sense of offering a smooth and unified exposition of all the material dealt with, an exposition based upon an initmate acquaintance with theory and upon a wide range of experience with practical problems.

Fundamental considerations of *a priori* probability theory are presented with admirable clearness. The theorems of Bernoulli and Bayes are adequately discussed and illustrated; and the treatment of the binomial, multinomial, and Poisson laws leaves little to be desired. The sections devoted to the Poisson law are especially noteworthy. In numerous places the author gives striking evidence of resourcefulness as an expositor and teacher. Let us indicate the manipulation of just one topic; in that way we can illustrate the painstaking attention devoted to problems of clear presentation. By what is in effect a brief and informal study of changes of variables in definite integrals, and of the rôle of the Jacobian in such changes, Dr. Fry contrives to present more clearly that this reviewer has seen presented elsewhere the precise meaning of a relative frequency function—which could very suitably be termed a "frequency density" or a "probability density" function, although it is not thus labeled by the author—and of a frequency curve or surface. Moreover there is applied at this point the happy thought of driving Maxwell's formula for the distribution function of the velocities of gas molecules, and of transforming subsequently to spherical coördinates. It is difficult to see how even the dullest schoolboy

could fail, upon reading these sections, to grasp the exact meaning of a frequency distribution function in any system of variables.

The second half of the book opens with a brief account of the normal law of distribution, and of the Pearson and Gram-Charlier methods in curve fitting. These methods are then carefully illustrated in the full discussion, from many standpoints, of an interesting problem. The author, with a judgment tempered by experience and always eminently sound, does not overstate the theoretical importance of the probability speculations commonly proposed in justification of accepted types of frequency curves—such curves often furnishing, under the circumstances of their application, hardly more than convenient graduation formulas. The last hundred pages of the book are devoted to practical applications of the theory previously developed. In particular, engineering problems arising in the telephone industry, and important problems of theoretical physics, are studied in detail by the methods of the probability calculus. The idea of "statistical equilibruim," largely neglected in English texts, is not overlooked here.

The reviewer feels that the early chapters of the book develop elementary mathematical notions concerning limits, etc., rather more fully than is necessary. It may also be observed that in matters of mathematical notation and convention these chapters show an occasional tendency to diverge abruptly from the customary. But it would be absurd to criticize a really excellent work on such a basis. After all, paragraphs like that which begins at the bottom of page 90 show quite unmistakably that the author fully understands the niceties of his mathematical analysis and that he can apply it with discrimination. The paragraph to which reference is here made directs attention to an important point which fails, in many another text, to receive appropriate emphasis. Dr. Fry strongly voices his insistence that underlying the definition of a statistical probability there is absolutely no assumption of the existence of a limit for the statistical ratio; that the variable which approaches a limit (in fact, the limit zero) is, in accordance with the Bernoulli theorem, the *a priori* probability that the statistical ratio will, as the number of trials increases indefinitely, diverge from a certain number by more than a preassigned constant. This is one of many touches for which the book deserves high commendation.

While several important methods of probability and statistical theory are altogether omitted or merely touched upon notably correlation methods, space has thereby been reserved for an exceptionally full and clear analysis of considerations of outstanding importance which have been found immediately capable of productive engineering application. Emphatically, this is a book which is worth while. The list price is stated to be $7.50; but, everything considered, the price does not appear to be unreasonable.

Bound with the book are forty-four pages of useful tables, some of which are a little difficult of access elsewhere, and a chart to facilitate the application of the commoner distribution laws. The typography is excellent throughout.

LESTER S. HILL

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS AND SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

PROBLEMS FOR SOLUTION

3349. Proposed by S. A. Corey, Des Moines, Iowa.

If $|lmn|$ denote the determinant whose columns are the lth, mth and nth columns of the array

$$\begin{Vmatrix} y & x & -av & au \\ u & bv & x & -by \\ v & -cu & cy & x \end{Vmatrix}$$

prove that

$$(x^2+bcy^2+acu^2+abv^2)^3 = |234|^2+bc\,|134|^2+ac\,|124|^2+ab\,|123|^2.$$

3350. Proposed by Frank Irwin, University of California.

Find the sum of all products of n factors each, the factors being chosen from the numbers $2,3,4,\cdots,2+t$. Repetitions of factors are allowed; order is not considered; and two products such as $2\cdot 6$ and $3\cdot 4$ are regarded as distinct, though numerically equal.

3351. Proposed by C. D. Smith, Louisiana College.

Assume $AOBC$ to be the octant of a sphere of radius R and centre O. Find the radius of the inscribed sphere. Also find the radius of a sphere which will be tangent to the two spheres and the planes AOC and BOC.

SOLUTIONS

263 [1917, 177]. Proposed by J. L. Riley.

To find values, positive integral, which verify the equation $x^3+2=y^2$.

SOLUTION AND NOTE BY ALFRED BRAUER, University of Berlin

The question for the integral solutions of the equation $x^3+2=y^2$ has been discussed several times. For a reference to the literature, see L. E. Dickson, *History of the Theory of Numbers*, vol. 2 (Washington, 1920), p. 533–539, and L. Aubry, *Réponse* 5085, L'intermédiaire des Mathématiciens (2), vol. 2 (1923), p. 107.

In my paper[1] *Ueber einige spezielle diophantische Gleichungen* I have shown that the above equation is verified only by the values $x=-1$, $y=\pm 1$. I get this result as a special case from the following theorem:

If $a=2b^3>0$, if $6b$ is free of square factors and the number of the classes of ideals of the field $P(\sqrt{-6b})$ is relatively prime to 3, then it is possible to indicate all integral solutions x, y of the equation $x^3+a=y^2$.

This theorem is proved by applying a lemma about imaginary quadratic fields.

In a quite different way and without any reference to me, B. Delaunay showed in 1922, in a Russian paper, that the equation $x^3+2=y^2$ has only the solutions $x=-1$, $y=\pm 1$. In his proof he makes use of some theorems about cubic fields.[2] The above result about equations of the form $x^3+2b^3=y^2$ is not obtained by this method which was unknown to me when I issued my paper.

2939 [1921, 468]. Proposed by C. F. Gummer, Queen's University.

Show that the determinant

$$\begin{vmatrix} a_1 & a_2 \cdots a_n & -b_1 & 0 & \cdots & 0 \\ 0 & a_1 \cdots a_{n-1} & -b_2 & -b_1 & \cdots & 0 \\ \cdot & \cdots\cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 \cdots a_1 & -b_n & -b_{n-1} & \cdots & -b_1 \\ b_1 & b_2 \cdots b_n & a_1 & 0 & \cdots & 0 \\ 0 & b_1 \cdots b_{n-1} & a_2 & a_1 & \cdots & 0 \\ \cdot & \cdots\cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 \cdots b_1 & a_n & a_{n-1} & \cdots & a_1 \end{vmatrix}$$

has the property that every n-rowed determinant of the n left-hand columns is equal to its cofactor.

Solution by the Proposer

Let m be the minor determinant obtained by taking the n left-hand columns in their original order and associating them with rows number $i_1, i_2, \cdots, i_n$, in the order of naming, of the given determinant D. The cofactor M of m in D may be written as a $2n$-rowed determinant whose n right-hand columns are the same as those of D, while the elements in its n left-hand columns are all zeros except those forming the leading diagonal of m; that is, those in row i_1 column 1; row i_2, column 2; etc.; and these elements are each unity. Representing M in this way, multiply it by the determinant (equal to a_1^n) whose left half is identical with that of D, and whose right half has all its elements zero except those in the leading diagonal of the lower half of it, which are each unity. On carrying out the multiplication in such a way that the element in row p and column q of the product is the result of combining column p of the determinant representing M with column q of the determinant representing a_1^n, we get a determinant whose upper left-hand quarter is precisely m, whose lower right quarter is the same as the upper left of D, while the lower left quarter consists of zeros. Hence $a_1^n m = a_1^n M$. Therefore, when a_1 is not zero, $M=m$; and this can only be an algebraic identity, true for all cases.

The theorem is more interesting than it looks, especially when a number of a's and b's at the end of the series vanish. Thus the determinant

$$d = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 & 0 \\ 0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & 0 & 0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 \end{vmatrix}$$

[1] Mathematische Zeitschrift, vol. 25 (1926), pp. 499–505.

[2] See also Comptes Rendues (1920) and Journal de la Société Phys.-Math. de Léningrad t. 1, p. 265.

is built up by a variation of Sylvester's dialytic method which shows that it is divisible by the resultant

$$R = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & 0 & 0 \\ 0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \end{vmatrix}$$

of the polynomials $a_1x^3+a_2x^2+a_3x+a_4$ and $b_1x^2+b_2x+b_3$; but the other factor is not evident. On recognizing d as a case of m, with $n=7$, $a_5=a_6=a_7=0$, $b_4=b_5=b_6=b_7=0$, and applying the present theorem it becomes clear enough that $d=(b_2^2-b_1b_3)R$.

2961 [1922, 129]. Proposed by J. L. Riley, Stephenville, Texas.

Being given a triangle ABC, to determine the two points P, P' such that the angles PBC, PCA, PAB are equal; and such that also the angles $P'CB$, $P'BA$, $P'AC$ are equal. Find also the equation of the circle which passes through the points P, P' and through the center of the circle circumscribing the triangle ABC.

Solution by Nathan Altshiller-Court, University of Oklahoma

The same question was proposed in the *Nouvelles Annales de Mathématiques* in 1875 (p. 192) by H. Brocard. The points are known as the Brocard points of the triangle. These two points together with the symmedian point constitute the backbone of the Modern Geometry of the triangle. The circle determined by the Brocard points and the circumcenter of the triangle is known as the Brocard circle of the triangle. See J. L. Coolidge, *A treatise on the circle and the sphere*, Oxford, 1916. William Gallatly, *Modern geometry of the triangle*. Francis Hodgson, London, 1910. *College Geometry*, Altshiller-Court, pp. 243 etc., *Traité de Géométrie*, Rouché et Comberousse, 8th ed., vol. 1, pp. 473 etc.

2962. [1922, 129]. Proposed by R. M. Mathews.

To construct a triangle similar to a given triangle with its vertices lying on: (*a*) any three coplanar lines; (*b*) any three lines in space. (See Problem **2895, [1921, 184.]**

Solution by Lawrence Hamilton, University of Oklahoma.

Let l, m, n be the three fixed straight lines which in case (*a*) lie in the same plane, and let α, β, γ be the angles of the given triangle. Suppose that a variable triangle ABC has these angles at the corresponding vertices and that A is fixed on l while B describes m. Then, as shown in *College Geometry* by N. A. Court, the vertex C describes a straight line p inclined at the angle α to m and passing through C_1, where C_1 is the position of C when BC lies on m. If the order of A, B, C is reversed there is a second locus p' such that p and p' are symmetrically situated with respect to the perpendicular from A to m. If now C moves on m we shall have in general two more lines parallel to the first as loci of B. By considering also the angles β and γ at the fixed point on l, we shall have in all twelve lines as the locus of the third vertex. The intersections of these lines with n determine in general twelve constructions for the selected point on l.

In case (*b*) for each position of A on l and B on m the locus of C is a circle in space cutting p and p'. As B moves on m these circles determine a surface through the lines p and p'. If on the other hand we let C move on m we have in general another such surface as the locus of B. There are in general two more pairs of such surfaces for the selected point on l for which the angles at this point are β and γ. The intersections of these six surfaces with n determine the constructions for one vertex at the selected point of l.

2968. Proposed by Malcom Foster, Yale University.

A curve C is the directrix of a ruled surface and g is a ruling. Relative to the trihedral of C at the point of intersection with g, the direction cosines of g are α, β, γ, expressed as functions of the arc of C. Prove that the distance t along g from C to the line of striction is given by

$$t = \frac{\beta\rho^{-1}-\alpha'}{\sum\alpha'^2 + 2\rho^{-1}(\alpha\beta' - \alpha'\beta) + 2\tau^{-1}(\beta'\gamma - \beta\gamma') + \rho^{-2}(1-\gamma^2) + \tau^{-2}(1-\alpha^2) + 2\rho^{-1}\tau^{-1}\alpha\gamma}$$

where ρ and τ are the radii of first and second curvature of C.

SOLUTION BY J. H. NEELLEY, Carnegie Institute of Technology.

A ruled surface referred to rectangular axes has the equations

$$x = x_0 + lt, \qquad y = y_0 + mt, \qquad z = z_0 + nt, \tag{1}$$

where (x_0, y_0, z_0) is the intersection of a ruling and the directrix C; where l, m, n are the direction cosines of the ruling; and where t is the distance along the ruling from C to any point of the surface. We assume these quantities expressed as functions of the arc of the directrix. Then the distance t along the ruling from C to the line of striction as given in Eisenhart's *Differential Geometry*, pp. 241–243, is

$$t = -\frac{(l'x_0' + m'y_0' + n'z_0')}{l'^2 + m'^2 + n'^2}. \tag{2}$$

The primes represent the absolute rates of change of the quantities as to the arc of C. Now suppose the trihedral of C coincident with the x, y, z-axes. Then the relations between the absolute and relative rates of change of the direction cosines of the ruling[1] are given by

$$l' = \alpha' - \beta\rho^{-1}, \qquad m' = \beta' + \alpha\rho^{-1} + \gamma\tau^{-1}, \qquad n' = \gamma' - \beta\tau^{-1} \tag{3}$$

and

$$x_0' = 1, \qquad y_0' = 0, \qquad z_0' = 0. \tag{4}$$

The substitution of relations (3) and (4) in (2) gives the distance t in the form requested if we make use of the fact that $\sum\alpha^2 = 1$. (See Eisenhart, l. c., §16).

3276 [1927, 381]. Proposed by L. L. Silverman and J. Tamarkin, Dartmouth College.

Prove that if ν is an integer greater than or equal to 1, then

$$\int_0^\infty \frac{(1+z)^{-\nu}dz}{\log^2 z + \pi^2} = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu}dt, \text{ where } \binom{t}{\nu} = \frac{t(t-1)\cdots(t-\nu+1)}{\nu!}.$$

SOLUTION BY J. TAMARKIN, Brown University.

We use the method of "Generating Functions." Let

$$a_0 = -1, \quad a_\nu = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu}dt \qquad (\nu = 1, 2, \cdots)$$

and let x be any positive number less than 1 $(0<x<1)$. Then

$$f(x) \equiv \sum_{\nu=0}^{\infty} a_\nu x^\nu = -\int_0^1 dt \sum_{\nu=0}^{\infty} \binom{t}{\nu}(-x)^\nu = -\int_0^1 (1-x)^t dt = \frac{x}{\log(1-x)},$$

the term by term integration being permissible because of the uniform convergence of the series under the sign of integral. We can compute $f(x)$ by using Cauchy's integral. For that purpose we take the complex z-plane which is cut along the segment of the axis of reals from 1 to ∞. In the remaining part the function $\log(1-z)$ is single-valued and analytic, and we can set

$$\log(1-z) = \log|1-z| + \omega i,$$

where

$$\omega = \arg(1-z) \text{ and } -\pi < \omega < \pi.$$

In the same part of the plane the function $f(z)$ is also single-valued and analytic.

Now, let (C) be any contour which has no points in common with the cut $1\to\infty$ and which contains both points $z=0$ and $z=x$ in its interior. The Cauchy's theorem applied to the function $f(z)$ shows then:

[1] (0, 0, 0) of trihedral of C is (x_0, y_0, z_0) of x, y, z-axes.

$$f(x) = \frac{1}{2\pi i}\int_{(C)} \frac{f(z)}{z-x}dz = \frac{1}{2\pi i}\int_{(C)} \frac{zdz}{(z-x)\log(1-z)}$$

$$= \frac{1}{2\pi i}\int_{(C)} \frac{dz}{\log(1-z)} + \frac{x}{2\pi i}\int_{(C)} \frac{dz}{(z-x)\log(1-z)}$$

$$= -1 + \frac{x}{2\pi i}\int_{(C)} \frac{dz}{(z-x)\log(1-z)} = -1 + x\phi(x),$$

the function $1/\log(1-z)$ being analytic in (C) except for a simple pole at $z=0$, with the residue (-1).

Now, let R, ρ, h, be arbitrary positive numbers of which $\rho<1$, $R>1+e$, and $h<\rho$. Set $z=\xi+i\eta$ and draw the circles

$$(C_R)\,|z| = R\,; \quad (C_\rho)\,|1-z| = \rho$$

and the half-lines

$$(D_h) \qquad \eta = \pm h,\ \xi > 1.$$

Obviously, we can take for (C) the boundary of the region between (C_ρ) and (C_R) and outside the strip (D_h). Denote by (C_R'), (C_ρ') and (D_h') respectively the circular and the rectilinear parts of (C). Then

$$\phi(x) = \frac{1}{2\pi i}\left[\int_{(C'_R)} - \int_{(C'_\rho)} + \int_{(D'_h)}\right]\frac{dz}{(z-x)\log(1-z)}$$

$$= \frac{1}{2\pi i}\left[\int_{(C'_R)} - \int_{(C'_\rho)}\right]\frac{dz}{(z-x)\log(1-z)} + \lim_{h\to 0}\frac{1}{2\pi i}\int_{(D'_h)}\frac{dx}{(z-x)\log(1-z)}.$$

The rectilinear part (D'_h) of (C) consists of two segments parallel and symmetrical with respect to the axis of reals, and described in opposite directions. Hence, it is readily seen that

$$\lim_{h\to 0}\frac{1}{2\pi i}\int_{(D'_h)}\frac{dz}{(z-x)\log(1-z)} = \frac{1}{2\pi i}\int_{1+\rho}^{R}\left[\frac{1}{\log^+(1-\xi)} - \frac{1}{\log^-(1-\xi)}\right]\frac{d\xi}{\xi-x},$$

where $\log^+(1-\xi)$ and $\log^-(1-\xi)$ denote, respectively, the limiting values of $\log(1-z)$ as z approaches the point ξ on the cut $1\to\infty$, from above or from below. On account of our agreement above concerning the function $\log(1-z)$, we have

$$\log^+(1-\xi) = \log(\xi-1) - \pi i\,; \quad \log^-(1-\xi) = \log(\xi-1) + \pi i\,;$$

$$\frac{1}{\log^+(1-\xi)} - \frac{1}{\log^-(1-\xi)} = \frac{2\pi i}{\log^2(\xi-1)+\pi^2}\,;$$

and

$$\lim_{h\to 0}\frac{1}{2\pi i}\int_{(D'_h)} = \int_{1+\rho}^{R}\frac{d\xi}{(\xi-x)[\log^2(\xi-1)+\pi^2]},$$

$$\phi(x) = \frac{1}{2\pi i}\left[\int_{(C_R)} - \int_{(C_\rho)}\right]\frac{dz}{(z-x)\log(1-z)} + \int_{1+\rho}^{R}\frac{d\xi}{(\xi-x)[\log^2(\xi-1)+\pi^2]}.$$

Now, we can make $R\to\infty$, $\rho\to 0$. If ϵ is any positive number, then for R sufficiently large, we have on (C_R)

$$|1/\log(z-1)| < \epsilon.$$

Hence,

$$\left|\frac{1}{2\pi i}\int_{(C_R)}\right| < \frac{\epsilon}{2\pi}\int_0^{2\pi}\frac{Rd\theta}{R-1} \to 0 \text{ as } \epsilon\to 0\ (R\to\infty).$$

Since

$$\frac{1}{\log(1-z)} \to 0 \text{ as } z\to 1,$$

we have on (C_ρ), for ρ sufficiently small:

$$|1/\log(1-z)| < \epsilon.$$

Hence,

$$\left|\frac{1}{2\pi i}\int_{(C_\rho)}\right| < \frac{\epsilon}{2\pi}\int_0^{2\pi}\frac{\rho d\theta}{1-\rho-x} \to 0 \text{ as } \epsilon \to 0\ (\rho \to 0).$$

On collecting these results we get finally

$$\phi(x) = \lim_{\rho\to 0, R\to\infty}\int_{1+\rho}^{R}\frac{d\xi}{(\xi-x)[\log^2(\xi-1)+\pi^2]} = \int_1^\infty \frac{d\xi}{(\xi-x)[\log^2(\xi-1)+\pi^2]}$$

and the substitution $\xi=\zeta+1$ brings this to the form

$$\phi(x) = \int_0^\infty \frac{d\zeta}{(\zeta+1-x)(\log^2\zeta+\pi^2)}.$$

If we expand the integrand in ascending powers of x and assume the possibility of term by term integration, we see immediately

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu = -1 + x\phi(x) = -1 + \sum_{\nu=1} x^\nu \int_0^\infty \frac{(\zeta+1)^{-\nu}d\zeta}{\log^2\zeta+\pi^2}.$$

Therefore

$$(\star) \qquad a_\nu = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu} dt = \int_0^\infty \frac{(\zeta+1)^{-\nu}d\zeta}{\log^2\zeta+\pi^2},$$

which is the desired result. To justify the term by term integration, set

$$\frac{1}{\zeta+1-x} - \frac{1}{\zeta+1}\left[1 + \frac{x}{\zeta+1} + \cdots + \frac{x^{n-1}}{(\zeta+1)^{n-1}}\right] = \frac{x^n}{(\zeta+1)^n(\zeta+1-x)} = r_n(\zeta,x),$$

$$\phi(x) - \sum_{\nu=0}^{n-1} x^\nu \int_0^\infty \frac{d\zeta}{(\zeta+1)^{\nu+1}(\log^2\zeta+\pi^2)} = R_n(x) = \int_0^\infty \frac{r_n(\zeta,x)}{\log^2\zeta+\pi^2}d\zeta.$$

Hence,

$$0 < R_n(x) < x^n\int_0^\infty \frac{d\zeta}{(\zeta+1-x)[\log^2\zeta+\pi^2]} \to 0 \text{ as } n\to\infty$$

and

$$\phi(x) = \lim_{n\to\infty}\sum_{\nu=0}^{n-1} x^\nu \int_0^\infty \frac{d\zeta}{(\zeta+1)^{\nu+1}[\log^2\zeta+\pi^2]}, \qquad \text{Q.E.D.}$$

The formula $(\star)$ has been proved for all *positive integral* values of ν. We can, however, consider the left- and right-hand members of this formula as analytic functions of an *unrestricted* variable ν. For, we may set

$$(-1)^{\nu-1}\binom{t}{\nu} \equiv \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)\Gamma(\nu+1)}e^{\pi i(\nu-1)},$$

thus obtaining the function

$$F(\nu) \equiv \int_0^1 e^{\pi i(\nu-1)}\Gamma(t+1)/[\Gamma(t-\nu+1)\Gamma(\nu+1)]\,dt$$

which is analytic for all values of ν. On the other hand, the function

$$\Phi(\nu) \equiv \int_0^\infty \frac{(\zeta+1)^{-\nu}d\zeta}{\log^2\zeta+\pi^2}$$

surely is analytic if the real part of ν is greater than 1. The fact that these functions are equal for

infinitely many values of $\nu=1, 2, \cdots$ may suggest that $\Phi(\nu)$ and $F(\nu)$ represent the same analytic function. This is not true, however, as can be proved as follows:

The function $F(\nu)$ is analytic at $\nu=1$. Now, we shall prove that $\Phi(\nu)$ is not analytic at $\nu=1$. On setting $\nu-1=s$, $\zeta+1=e^t$ we have

$$\Phi(\nu) = \Psi(s) = \int_0^\infty \frac{e^{-st}dt}{\log^2(e^t-1)+\pi^2} = \left[\int_0^a + \int_a^\infty\right] \frac{e^{-st}dt}{\log^2(e^t-1)+\pi^2},$$

where a is a fixed positive number, >1. The first term of the right-hand member is analytic for all values of s (entire transcendental function). To transform the second term we observe that, on the interval (a, ∞),

$$\log(e^t-1) = \log(e^t) + \log(1-e^{-t}) = t - \theta e^{-t}$$

with $0<\theta<e^a/(e^a-1)<2$.
Hence

$$\frac{1}{\log^2(e^t-1)+\pi^2} - \frac{1}{t^2+\pi^2} = \frac{2t\theta e^{-t} - \theta^2 e^{-2t}}{(t^2+\pi^2)[(t+\theta e^{-t})^2+\pi^2]} = t^{-3}e^{-t}P(t),$$

where $P(t)$ is bounded on (a, ∞). On substituting, we find

$$\int_a^\infty \frac{e^{-st}dt}{\log^2(e^t-1)+\pi^2} = \int_a^\infty \frac{e^{-st}dt}{t^2+\pi^2} + \int_a^\infty \frac{e^{-(s+1)t}P(t)}{t^2}dt.$$

The second term of the right-hand member is analytic if the real part of s is greater than -1. Denote the first term by

$$(\star\star) \qquad \psi(s) = \int_a^\infty \frac{e^{-st}dt}{t^2+\pi^2}.$$

Let $s=\sigma+i\tau$, where τ is arbitrary and $\sigma>0$. We have right to differentiate $(\star\star)$ under the sign of integral, because of the uniform convergence of the resulting integral, whence

$$\psi''(s) = \int_a^\infty \frac{e^{-st}t^2dt}{t^2+\pi^2} = \int_a^\infty e^{-st}dt - \pi^2\int_a^\infty \frac{e^{-st}dt}{t^2+\pi^2}.$$

Now, as $s\to 0$ (remaining in the right half-plane), the integral

$$\int_a^\infty \frac{e^{-st}}{(t^2+\pi^2)}dt$$

remains uniformly bounded in absolute value:

$$\left|\int_a^\infty \frac{e^{-st}dt}{t^2+\pi^2}\right| \leqq \int_0^\infty \frac{dt}{t^2+\pi^2} = \frac{1}{2}.$$

On the contrary, the term $\int_a^\infty e^{-st}dt = e^{-as}/s \to \infty$ as $s\to 0$, which shows that $\psi''(s)$ is not bounded as $s\to 0$, hence $\psi(s)$ is not analytic at $s=0$. The same is true, then, of the functions $\Psi(s)$ at $s=0$ and $\Phi(\nu)$ at $\nu=1$. Hence $F(\nu)$ and $\Phi(\nu)$ are two distinct analytic functions whose values coincide at $\nu=1, 2, 3, \cdots$. The question arises now whether the values of these functions may coincide at the points of another infinite set $\{\nu_n\}$ $(n=1, 2, 3, \cdots)$. It is readily proved, however, that this is not the case, provided the set $\{\nu_n\}$ admits of at least one finite limiting value whose real part is greater than 1. For, if we assume (without loss of generality) that $\nu_n\to\nu_0$ as $n\to\infty$, and that the real part of ν is greater than 1, then the function $\Omega(\nu)\equiv F(\nu)-\Phi(\nu)$ is analytic at $\nu=\nu_0$ and has infinitely many roots in the vicinity of $\nu=\nu_0$, which implies $\Omega(\nu)\equiv 0$. Of course the set $\{\nu_n=n\}$ does not satisfy the requirement above.

To conclude we may say that on substituting in the expression of $F(\nu)$ above the known asymptotic formulas for the Gamma-functions involved, it is readily proved that $a_\nu = O(1/\nu\log^2\nu)$. (Cf. M. Riesz, *Sur l'équivalence de certaines méthodes de sommation*, Proceedings of the London Math. Soc. (2) 22, 1923–1924, 412–419, esp. p. 416).

3286 [1927, 438]. Proposed by V. M. Spunar, Chicago, Illinois.

A doubly infinite system of similar conics in parallel planes have their centers collinear and their corresponding axes parallel. Show that they can be cut orthogonally by a family of surfaces only if the line of centers is perpendicular to their planes.

SOLUTION BY THEODORE BENNETT, University of Illinois.

The equation of the family of conics can be written:

$$ax^2 + 2b(y - cv)x + d(y - cv)^2 = u, \quad z = v, \tag{1}$$

where a, b, c, d are constants, and u, v are the parameters of the system. If the surface $f(x, y, z) = k$ cuts these curves in the plane $z = v$ orthogonally, then it is obvious that $f = k$ must be a cylinder with elements parallel to the z-axis. Hence f does not contain z, and the curve of intersection of $f = k$ with $z = v$ is determined by

$$(ax + by - cvb)dy - (bx + dy - cvd)dx = 0.$$

Two cases are to be considered. If $c \neq 0$ then we must have

$$\begin{vmatrix} ax + by, & bx + dy \\ b & d \end{vmatrix} = 0,$$

or $b^2 - ad = 0$. In this case the conics are parallel straight lines which we may discard as trivial. Hence in the remaining case $c = 0$, and the line of centers is the z axis, which was to proved.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus Ohio.

What can one say for mathematics in two minutes?

The Fox Film Corporation is inviting representatives of various specialties to give two-minute talks for the "Fox Movietone News." At the University of Chicago, Professors A. A. Michelson and H. E. Slaught were invited to speak for physics and mathematics, respectively. The following is what Professor Slaught said:

I am asked to speak for two minutes about the human significance of mathematics.

Most people, I take it, think of mathematics, aside from commercial arithmetic, as a mere plaything of a few specialists—something quite devoid of any general human interest.

On the contrary it is easy to show that mathematics underlies our present-day civilization in much the same fundamental way as sunshine forms the source of all life and activity on the earth. We do not need to know the constitution of the sun in order to enjoy its light and heat. Likewise we do not need to be mathematicians in order to enjoy the benefits accruing to us from mathematical science.

It was mathematics that unlocked the mysteries of the heavens and dispelled the ignorance and superstition which prevailed when men believed that

the earth was flat and that the sun and the stars revolved around it. By our knowledge of the heavens gained through mathematics we regulate our chronometers and guide our ships on the high seas.

It is mathematics that enables us to visualize the contour of the earth's surface. Without this aid no accurate map or chart of a coast line or of a mountainous region could possibly be made.

This is the age of electricity, but it is not difficult to show that the present marvelous development of this mysterious force known as electricity rests fundamentally upon mathematics. This is also the age of the gasoline engine but we shall find that the automobile, the tractor, the air-plane, the zeppelin, and scores of other products of the twentieth century made possible by the gasoline engine have one and all been obliged to wait upon the engineer and the draughtsman whose laboratories were bristling with mathematics.

If we consider the whole range of the physical sciences such as mechanics, engineering, ballistics, physics, and much of chemistry and geology; also certain phases of the biological sciences, including important researches in physiology and biometry; likewise all subjects involving statistical study of any kind, such as insurance, annuities, etc.—in fact every form of quantitative investigation whatsoever—we shall see that these one and all rest fundamentally upon mathematics.

Even certain phases of beauty have a mathematical basis. For example, symmetry which is beauty of form, harmony which is beauty of tone, rhythm which is beauty of motion, are all terms having mathematical content.

We see, then, that mathematics has a very close relationship to our every day lives and hence is of far-reaching human significance.

Some readers of the Monthly may be interested in the following announcement from the United States Civil Service Commission: To fill a vacancy on the staff of the United States Naval Observatory, applications for the position of associate librarian will be received from now until December 12. The duties are having charge of the highly specialized library of the U. S. Naval Observatory, consisting of works on astronomy, mathematics, and physics, and including numerous collections of the transactions of the principal scientific societies of the world; maintaining a reference service in the field of astronomy, mathematics, and allied sciences, including the compilation of comprehensive and selective bibliographies; cataloging and classifying the material, which is in approximately 27 languages, requiring an extensive knowledge of French and German, and a working knowledge of some of the other languages.

Competitors will not be required to report for examination at any place, but will be rated on their education, experience, and fitness, and a thesis or publication to be filed by the applicant.

A man is desired for the position. The entrance salary is $3,200 a year. Higher-salaried positions are filled through promotion. Full information may be obtained from the United States Civil Service Commission, Washington, D. C.

At the International Congress of Mathematicians held recently at Bologna, Professor EDWARD KASNER of Columbia University presented papers on "Geometrie des fonctions polygene" and "The higher derivatives of polygenic functions."

Dr. GREGORY BREIT, mathematical physicist of the department of terrestial magnetism of the Carnegie Institution, has been assigned to carry out work in atomic physics in various laboratories and universities of Europe for the present year.

Professor E. R. HEDRICK, chairman of the department of mathematics of the University of California at Los Angeles and editor-in-chief of the Bulletin of the American Mathematical Society, has been nominated for the presidency of the Society by the Council.

Associate Professor C. C. BRAMBLE has been promoted to a professorship of mathematics in the post graduate school of the United States Naval Academy.

Mr. H. C. CHRISTOFFERSON has been appointed professor of mathematics at Miami University.

Associate Professor G. R. CLEMENTS, of the United States Naval Academy, has been promoted to a professorship of mathematics.

Associate Professor ALBERT E. COOPER, of the mathematics department of the University of Texas, has been elected a director of the Republic Bank and Trust Company, of Austin, Texas.

Dr. W. L. CRUM, professor of statistics, Graduate School of Business, Stanford University, has been elected editor of the Review of Economic Statistics, published by the Harvard Economic Society, Inc.

Associate Professor ALEXANDER DILLINGHAM has been promoted to a professorship in mathematics at the United States Naval Academy.

At the University of Texas, Professor H. J. ETTLINGER has been appointed director of intercollegiate athletics for the year beginning Dec. 1, 1928. The appointment is temporary, to meet an emergency. He will continue his work as professor of mathematics, teaching a full schedule. Professor Ettlinger played football, baseball, and basketball at Washington University from 1907 to 1910.

Dr. ORRIN FRINK has been appointed assistant professor of mathematics at the Pennsylvania State College.

Dr. MARY G. HASEMAN has been appointed professor of mathematics at Hartwick College, Oneonta, N. Y.

Associate Professor O. H. RECHARD, of the University of Wyoming, has been promoted to a professorship of mathematics.

Dr. L. S. SHIVELY, of Mount Morris College, has been appointed associate professor of mathematics at Ball Teachers College, Muncie, Indiana.

Dr. H. W. STAGER, of Los Gatos, California, has been appointed head of the department of mathematics at Salinas Junior College.

Dr. J. M. STETSON has been appointed professor of mathematics at the College of William and Mary.

Assistant Professor W. J. WEBBER, of the University of Toronto, has been promoted to an associate professorship of mathematics.

Professor R. M. WINGER, of the University of Washington, has an article entitled *The equi-anharmonic cubic and its group* in the current volume of the Tôhoku Mathematical Journal (vol. 29, pp. 376–400).

EDGAR W. WOOLARD, assistant meteorologist of the U. S. Weather Bureau, has been appointed instructor in mathematics at George Washington University.

The following appointments to instructorships in mathematics are announced:

Hunter College, Dr. MARGUERITE D. DARKOW, Dr. ROSA L. JACKSON; Miss R. LUCILE ANDERSON.

Mississippi A. and M. College, Mr. G. B. DRUMMOND.

United States Naval Academy, Messrs. F. J. BAIER, J. R. BLANK, W. F. KERN, G. A. LYLE, H. C. STOTZ, H. J. WINSLOW.

Wittenberg College, Mr. GUY S. HARRIS.

Brigadier-General W. H. BIXBY, retired, formerly chief of engineers of the United States Army, died on September 29, 1928. He was a charter member of the Mathematical Association, and regularly attended the national and sectional meetings in Washington and vicinity.

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BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.

Thirteenth Annual Meeting, New York City, December 28-29, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 31.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, April 28.

OHIO, Columbus, Ohio, April 5.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Dunham, N. C., March.

SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

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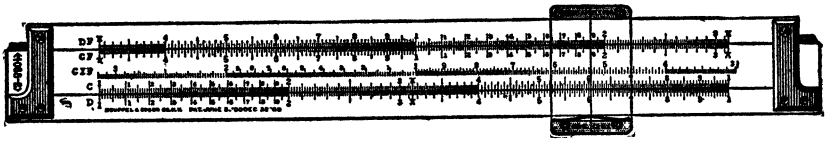
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P. 37, first line, replace MIAMI by OXFORD.
P. 97, 21st line, replace C by G.
P. 97, 2nd line, from bottom, replace F. by W. F.
P. 114, 8th line, replace D. H. by D. N.
P. 127, 5th line from bottom, replace "it" by "its".
P. 134, 24th line, replace $f(t)dt(t-z)$ by $f(t)dt/(t-z)$.
P. 217, replace Thomson by Thompson.
P. 279, 2nd line, replace "bi-orthoganal" by "bi-orthogonal."
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P. 389, 3rd line, for "sets" read "nets."
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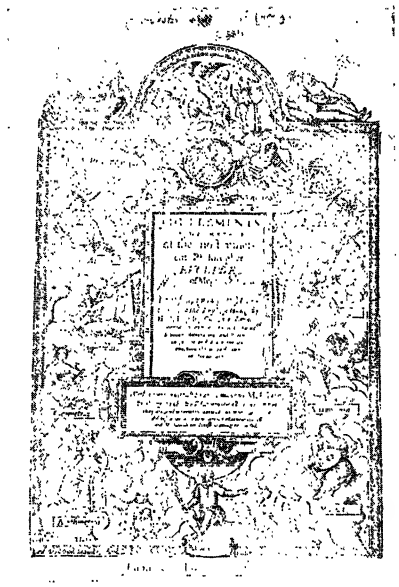


FIGURE 1

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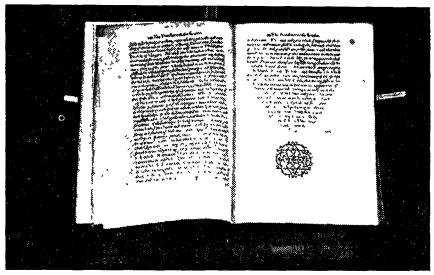


FIGURE 2

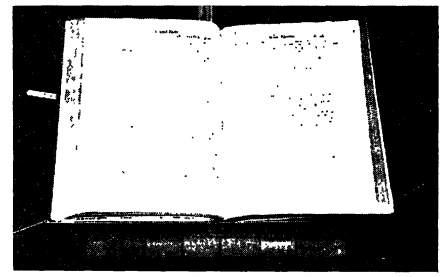


FIGURE 3

"THE TRANSLATOR TO THE READER."

"There is (gentle reader) nothing (the Word of God onely set apart) which so much beautifieth and adorneth the soule and minde of mã, as doth the knowledge of good artes and sciences: as the knowledge of naturall and morall Philosophie. The one setteth before our eyes, the creatures of God, both in the heauens aboue, and in the earth beneath: in which as in a glasse, we behold the exceding maiestie and wisedom of God, in adorning and beautifying them as we see: in geuing vnto them such wonderfull and manifolde proprieties, and natural workinges, and that so diuersely and in such varietie: farther in maintaining and conseruing them continually, whereby to praise and adore him, as by S. Paule we are taught. The other teacheth vs rules and preceptes of vertue, how, in common life amongest men, we ought to walke uprightly: what dueties pertaine to our selues, what pertaine to the gouernment or good order both of an housholde, and also of a citie or common wealth. The reading likewise of histories, conduceth not a litle, to the adorning of the soule and minde of man, a study of all men cõmended: by it are seene and knowen the artes and doinges of infinite wise men gone before vs. In histories are contained infinite examples of heroicall vertues to be of vs followed, and horrible examples of vices to be of vs eschewed. Many other artes also there are which beautifie the minde of man: but of all other none do more garnishe and beautifie it, than those artes which are called Mathematicall. Vnto the knowledge of which no man can attaine, without the perfect knowledge and instruction in the principles, groundes, and Elementes of Geometrie. But perfectly to be instructed in them, requireth diligent studie and reading of olde auncient authors. Amongst which, none for a beginner is to be preferred before the most auncient Philosopher Euclide

of Megara. For of all others, he hath in a true methode and iuste order, gathered tigether whatsoeuer any before him had of these Elementes written: inuenting also and adding many thinges of his owne: whereby he hath in due form accomplished the arte: first geuing definitions, principles, and groundes, whereof he deduceth his Propositions or conclusions, in such wonderfull wise, that that which goeth before, is of necessitie required to the proufe of that which followeth. So that without the diligent studie of Euclides Elementes, it is impossible to attaine vnto the perfect knowledge of Geometrie, and consequently of any of the other Mathematicall sciences. Wherefore considering the want and lacke of such good authors hitherto in our Englishe tongue, lamenting also the negligence, and lacke of zeale to their countrey in those of our nation, to whom God hath geuen both knowledge and also abilitie to translate into our tounge, and to publishe abroade such good authors, and bookes (the chiefe instrumentes of all learninges): seing moreouer that many good wittes both of gentlemen and of others of all degrees, much desirous and studious of these artes, and seeking for them as much as they can, sparing no paines and yet frustrate of their intent, by no meanes attaining to that which they seeke: I haue for their sakes, with some charge and great trauail, faithfully translated into our vulgare toũge, and set abroad in Print, this booke of Euclide. Whereunto I haue added easie and plaine declarations and examples by figures, of the definitions. In which book also ye shall in due place finde manifolde additions, Scholies, Annotations, and Inuentions: which I haue gathered out of many of the most famous and chiefe Mathematiciẽs, both of olde time, and in our age: as by diligent reading it in course, ye shall well perceaue. The fruite and gaine which I require for these my paines and trauaile, shall be nothing els, but onely that thou gentle reader, will gratefully accept the same: and that thou mayest thereby receaue some profite: and moreouer to excite and stirre vp others learned, to do the like, and to take pains in that behalfe. By meanes wherof, our Englishe tounge shall no lesse be enriched with good Authors, then are other straunge tounges: as the Dutch, French, Italian, and Spanish: in which are red all good authors in a maner, found amongest the Grekes or Latines. Which is the chiefest cause, that amongest thẽ do florishe so many cunning and skilfull men, in the inuentions of straunge and wonderfull thinges, as in these our daies we see there do. Which fruite and gaine if I attaine vnto, it shall encourage me hereafter, in such like sort to translate, and set abroad some other good authors, both pertaining to religion (as partly I haue already done) and also pertaining to the Mathematicall Artes. Thus gentle reader farewell."

Matters of very considerable interest are contained in the fruitful preface by John Dee, of London, who seems to have been a rather learned man of his day in Mathematics and Science, but who for some reason or other was not in particularly good repute with some of his contemporaries. His preface is addressed "To the Vnfained Lovers of truthe, and constant Studentes of Noble Sciences, John Dee of London, hartily wisheth grace from heauen, and most prosperous successe in all their honest attemptes and exercise." In this article Mr. Dee sets himself to the task of naming and defining all the various subjects which have a mathematical background and then presents a tabular summary the size of four folio pages in which all these are listed in relation to each other. It is a matter of sincere regret that I have been unable to get a satisfactory photograph of this table. Not the least interesting bit of this extensive preface—it covers forty-six folio pages—is a paragraph of invective against those who do not look kindly upon his work, from which I quote in part as an exceedingly good use of the English language in invective:

"A Digression Apologeticall

And for these, and such like maruelous Actes and Feates, Naturally, Mathematically, and Mechanically, wrought and contriued: ought any honest Student, and Modest Christian Philosopher, be counted & called a CONIURER? Shall the folly of Idiotes, and the Mallice of the Scornfull, so much preuaile, that He, who seeketh no worldly gaine or glory at their handes: But onely, of God, the treasor of heauenly wisedome, & knowledge of pure veritie: Shall he (I say) in the meane space, be robbed and spoiled of his honest name and fame? Will they provoke him, by worde and Printe, likewise, to Note their Names to the World: with their particular deuises, fables, beastly Imaginations, and vnchristenlike slaunders? Well: Well. O (you such) my unkinde Country men. O vnnaturall Countrey men. O vnthankfull Countrey men. O Brainsicke, Rashe, Spitefull, and Disdainfull Countrey men. Why oppresse you me, thus violently, with your slaundering of me: Contrary to Veritie: and contrary to your owne Consciences?"

After considerable more of this delicious raving, he continues with his summary, and finally closes with these humble words: "If Hast, hath caused my poore pen, anywhere, to stumble: You will, (I am sure) in part of recompence, (for my earnest and sincere good will to pleasure you), consider the rockish huge mountaines, and the perilous vnbeaten wayes, which (both night and day, for the while) it hath toyled and labored through, to bryng you this good Newes, and comfortable profe, of Vertues frute. So, I commit you vnto Gods Mercyfull direction, for the rest: hartely besechyng hym, to prosper your

Studyes, and honest Intentes: to his Glory, & the Commodity of our Countrey. Amen.

Written at my poore House
At Mortlake.
Anno. 1570. February 9."

The first volume contains the first nine books of Euclid, printed with beautiful initials, extensive proofs, and many additional notes by Pelitarius, Orontius, Flussates, Campane, Dee, and others. Figure 3 shows the book open at folio 58, displaying on the right the figure and proof of the Pythagorean Theorem essentially as it is given now in practically all modern texts. A note acknowledging Pythagorus as the inventor says: "This most excellent and notable Theoreme was first inuented of the greate philosopher Pithagoras, who for the exceeding joye conceiued by the inuention thereof, offered in sacrifice an Oxe, as recorde Hierone, Proclus, Lycius, and Vitruutus. And it hath bene commõly called of barbarous writers of later time Dulcarnon."

The second volume contains the tenth to fifteenth books of Euclid, with a sixteenth book added by Flussas. The wood-cuts illustrating the solid figures are most beautiful, but one of the most interesting features of the eleventh book is that many of the figures are made of paper and so pasted in the book that they may be opened up to make actual models of the space figures. Figure 4 shows the page of definitions of pyramids, and illustrates with actual models, pyramids with triangular, quadrilateral, and pentagonal bases.

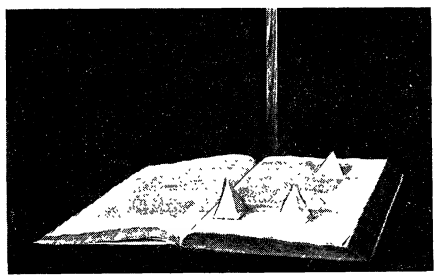


FIGURE 4

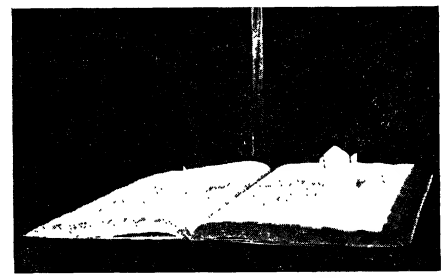


FIGURE 5

Figure 5 shows similar figures to illustrate the proofs of "From a point geuen on high, to draw vnto a ground plaine superficies a perpendicular right line." (Model on Left Page) and "Vnto a playne superficies geuen, and from a poynt in it geuen, to rayse up a perpendicular line." The models to illustrate "If two right lines touching the one the other be parallels to other right lines touching also the one the other and not being in the self same plaine superficies with the two first: the plaine superficies extended by those right lines, are also paral-

LIEFE.IS.DEATH. AND DEATH IS.LIEFE:AETATIS.SVAE: XXXX."

In company with this text which brought into our mother tongue the elements of Geometry which today play such a large part in our business and professional life, the library which American University has received from Artemas Martin is rich in other very early mathematical publications. It also contains a collection of the arithmetics published in America from the very earliest dates up to the present century, as well as other historical mathematical material. It will be a great pleasure to make these volumes available to the members of the Mathematical Association to view and use for such research purposes as they might suggest.

THE PHASE RULE

By J. E. TREVOR, Cornell University

For the case of a body constituted of a single component substance and consisting of n homogeneous masses or "phases" coexisting in thermodynamic equilibrium, the famous phase rule of J. Willard Gibbs may be expressed by the statement:

The pressure and temperature of the body, and the specific volumes and specific entropies of its several phases, form a set of $2n+2$ variables of which $3-n$ are independent.

It has long been recognized that this "rule" is merely a conclusion from the circumstance that the $2n+2$ variables are connected by $3n-1$ relations, and that it is not clear whether an arbitrary choice of $3-n$ variables, of which the remaining variables are then functions, can be made. The purpose of the present note is to examine the functional dependence involved.

The necessary data may be assembled as follows. The volumes, entropies, and masses of the several phases are variables,

$$V_i, \quad S_i, \quad M_i, \quad i = 1,2, \cdots, n,$$

which are independent save as M_1 is defined by the equation

$$c = M_1 + M_2 + \cdots + M_n,$$

where the constant c is the mass of the body. The energies of the phases are continuous functions,

$$E_i(V_i,S_i,M_i), \quad i = 1,2, \cdots, n.$$

The volume V, the entropy S, and the energy E of the body are functions defined by the equations,

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The volume V, the entropy S, and the energy E of the body are functions defined by the equations,

$$(1) \qquad V = \sum V_i, \qquad S = \sum S_i, \qquad E = \sum E_i ;$$

and it is required to find necessary and sufficient conditions for the functions V, S, E to satisfy a relation of the form

$$(2) \qquad E = F(V,S).$$

By (2), $dE-(\partial F/\partial V)\,dV-(\partial F/\partial S)dS=0$. When this equation is expanded by making use of the definitions (1), whereupon the dependent differential $dM_1=-dM_2-\cdots-dM_n$ is eliminated, necessary conditions are found to be:

$$(3) \qquad \begin{aligned} (\partial F/\partial V) &= (\partial E_1/\partial V_1) = (\partial E_2/\partial V_2) = \cdots = (\partial E_n/\partial V_n), \\ (\partial F/\partial S) &= (\partial E_1/\partial S_1) = (\partial E_2/\partial S_2) = \cdots = (\partial E_n/\partial S_n), \\ (\partial E_1/\partial M_1) &= (\partial E_2/\partial M_2) = \cdots = (\partial E_n/\partial M_n). \end{aligned}$$

I am not aware that the sufficiency of these conditions has ever been discussed. The last of the definitions (1) is

$$(4) \qquad E = E_1(V_1,S_1,M_1) + E_2(V_2,S_2,M_2) + \cdots + E_n(V_n,S_n,M_n);$$

Here, by the remaining equations (1) and the definition of M_1, let the variables V_1, S_1, M_1 be the functions

$$V_1 = V - V_2 - \cdots - V_n, \quad S_1 = S - S_2 - \cdots - S_n, \quad M_1 = c - M_2 - \cdots - M_n.$$

Then, by differentiating (4) with regard to the subscripted variables, we find

$$\begin{aligned} (\partial E/\partial V_i) &= -(\partial E_1/\partial V_1) + (\partial E_i/\partial V_i), & i &= 2, 3, \cdots, n, \\ (\partial E/\partial S_i) &= -(\partial E_1/\partial S_1) + (\partial E_i/\partial S_i), & i &= 2, 3, \cdots, n, \\ (\partial E/\partial M_i) &= -(\partial E_1/\partial M_1) + (\partial E_i/\partial M_i), & i &= 2, 3, \cdots, n. \end{aligned}$$

By the conditions (3), when the members $\partial F/\partial V$ and $\partial F/\partial S$ are omitted, the second members of these equ ations are equal to zero. So E contains none of the subscripted variables, and hence is a fun ction of V and S alone. The $3(n-1)$ conditions (3) employed are therefore sufficient.

It is given, further, that each of the n functions E_i is homogeneous of degre one in its variables. The first derivatives of these functions are therefore homogeneous of degree zero, and thus are functions of the respective ratios,

$$(5) \qquad v_i = V_i/M_i, \qquad s_i = S_i/M_i, \quad i = 1,2, \cdots, n,$$

which are the specific volumes and the specific entropies of the several phases. Now the pressure p and the absolute temperature θ under which the state of the body is a state of thermodynamic equilibrium are continuous functions of V, S, and are defined by the equations $-p=\partial F/\partial V$, $\theta=\partial F/\partial S$. So we have that the conditions (3) subject the $2n+2$ quantities p, θ, v_i, s_i, to $3n-1$

relations. When (3) are satisfied the number of these quantities remaining independent is $(2n+2)-(3n-1)=3-n$.

It is improbable that the conditions (3) can be satisfied when n exceeds 3. When $n=1$ it is desired to know what sets of two of the variables p, θ, v_1, s_1, are functions of the two remaining variables. When $n=2$ it is desired to know what sets of five of the variables $p, \theta, v_1, s_1, v_2, s_2$, are functions of the remaining variable. When $n=3$ it is desired to know whether the quantities p, θ, v_i, s_i are all constants.

The Case $n=1$

We must begin by expressing the conditions (3) in terms of p, θ and the "specific" variables v_i, s_i. Let t be any positive number. Then, the function E_i being homogeneous of degree one, we have

$$E_i(tV_i, tS_i, tM_i) = t \cdot E_i(V_i, S_i, M_i) ;$$

or, putting $t=1/M_i$ and using the definitions (5),

$$E_i = M_i \cdot e_i(v_i, s_i). \tag{6}$$

When expressed by means of this formulation, the conditions (3), for $n=1$, are

$$-p = \partial e_1/\partial v_1, \quad \theta = \partial e_1/\partial s_1, \tag{7}$$

which shall be written

$$f_1 = p + (\partial e_1/\partial v_1) = 0, \quad f_2 = \theta - (\partial e_1/\partial s_1) = 0.$$

When the body is in a one-phase state of thermodynamic equilibrium these equations are satisfied by a set of values of the variables p, θ, v_1, s_1. And, in the neighborhood of this set of values, the functions f_1, f_2 are continuous and possess continuous first derivatives with regard to all their variables. Now, to determine whether, in this neighborhood, a selected two, u, v, of the four variables are single-valued continuous functions of the remaining two variables x, y, we require to determine whether the jacobian $\partial(f_1, f_2)/\partial(u, v)$ is non-vanishing at the point. The jacobians of f_1, f_2 with regard to the elements of each of the six possible pairs of variables are as follows:

$$\begin{aligned}
&\partial(f_1,f_2)/\partial(p,\theta) = +1, && \partial(f_1,f_2)/\partial(p,s_1) = -\partial^2 e_1/\partial s_1^2,\\
&\partial(f_1,f_2)/\partial(v_1,\theta) = +\partial^2 e_1/\partial v_1^2, && \partial(f_1,f_2)/\partial(v_1,s_1) = -\Delta_1,\\
&\partial(f_1,f_2)/\partial(\theta,s_1) = -\partial^2 e_1/\partial v_1 \partial s_1, && \partial(f_1,f_2)/\partial(p,v_1) = -\partial^2 e_1/\partial s_1 \partial v_1,
\end{aligned}$$

where Δ_1 is the hessian of the function $e_1(v_1, s_1)$. When the thermodynamic equilibrium of the one-phase state of the body is stable or metastable, and hence realizable at all, conditions of stability,

$$\partial^2 e_1/\partial v_1^2 > 0, \quad \partial^2 e_1/\partial s_1^2 > 0, \quad \Delta_1 > 0,$$

must be satisfied. It follows then that, in the neighborhood of any one-phase state of equilibrium,

$$\begin{array}{l} p \text{ and } \theta \text{ are continuous functions of } v_1, s_1, \\ p \text{ " } s_1 \text{ " " " " } v_1, \theta, \\ v_1 \text{ " } \theta \text{ " " " " } p, s_1, \\ v_1 \text{ " } s_1 \text{ " " " " } p, \theta\ ; \end{array} \tag{8}$$

and that

$$\begin{array}{l} \theta \text{ and} s_1 \text{ are continuous functions of } p, v_1, \\ p \text{ and } v_1 \text{ are continuous functions of } \theta, s_1, \end{array} \tag{9}$$

in the neighborhood of any point of any subregion throughout which $\partial^2 e_1/\partial v_1 \partial s_1$ differs from zero. The first result, that p and θ are functions of v_1, s_1, is of course initially given by the conditions of equilibrium themselves.

To interpret the condition $\partial^2 e_1/\partial v_1 \partial s_1 = 0$, we may differentiate the conditions (7). Writing e_{11}, e_{12}, e_{22} for the second derivatives of $e_1(v_1, s_1)$,

$$-dp = e_{11}dv_1 + e_{12}ds_1, \quad d\theta = e_{12}dv_1 + e_{22}ds_1,$$

and eliminating ds_1,

$$\Delta_1 dv_1 = -e_{22}dp - e_{12}d\theta,$$

whence follows

$$(\partial v_1/\partial\theta)_p = -e_{12}/\Delta_1.$$

Since $\Delta_1 > 0$, we have that the condition $e_{12} = 0$ is the equation of the locus representing states of maximum or minimum density at constant pressure. Such a locus exists in the field of states of thermodynamic equilibrium of bodies of liquid water.

Conclusions concerning single-valuedness of the functions (8) and (9) can be deduced from the theorem that any two states (p, θ, v_1, s_1) and ($p+\delta p$, $\theta+\delta\theta$, $v_1+\delta v_1$, $s_1+\delta s_1$) of a given region of one-phase states are in stable equilibrium only when

$$-\delta p\delta v_1 + \delta\theta\delta s_1 > 0. \tag{10}$$

Thus, for given values of v_1 and s_1, the function p of v_1, s_1 cannot have two values differing by $\delta p \neq 0$, since the values $\delta v_1 = \delta s_1 = 0$ and $\delta p \neq 0$ would not satisfy (10). In this manner we find that the eight functions (8) are single-valued when the equilibrium is stable. On the other hand it is not excluded that, for given values of p and v_1, the function θ of p_1, v_1 can have two values differing by $\delta\theta \neq 0$. For the values $\delta p = \delta v_1 = 0$ and $\delta\theta \neq 0$ would reduce (10) to $\delta\theta\delta s_1 > 0$, which merely requires that corresponding changes of θ and s_1 at unchanged p, v_1 must have

the same sign. In this manner we find that the four functions (9) are not necessarily single-valued. They are, in fact, multiple-valued for certain ranges of the independent variables in the field of states of a body of liquid water.

SUMMARY: *Let p be the pressure, θ the absolute temperature, v_1 the specific volume, and s_1 the specific entropy, of any one-component body in any region of realizable one-phase states of thermodynamic equilibrium. Any two of these variables are continuous functions of the other two. The eight functions of the several pairs of "work-heat variables"*

$$v_1, s_1, \quad v_1, \theta, \quad p, s_1, \quad p, \theta,$$

are single-valued; but the four functions of the "work variables" p, v_1 and the "heat variables" θ, s_1 may be multiple-valued.

The Case $n=2$

When expressed by means of (6), the conditions (3), for $n=2$, are:

$$-p = \frac{\partial e_1}{\partial v_1} = \frac{\partial e_2}{\partial v_2}, \quad \theta = \frac{\partial e_1}{\partial s_1} = \frac{\partial e_2}{\partial s_2},$$

$$e_1 - \frac{\partial e_1}{\partial v_1} v_1 - \frac{\partial e_1}{\partial s_1} s_1 = e_2 - \frac{\partial e_2}{\partial v_2} v_2 - \frac{\partial e_2}{\partial s_2} s_2,$$

which shall be written:

$$f_1 = p + \frac{\partial e_1}{\partial v_1} = 0, \qquad f_2 = p + \frac{\partial e_2}{\partial v_2} = 0,$$

$$f_3 = \theta - \frac{\partial e_1}{\partial s_1} = 0, \qquad f_4 = \theta - \frac{\partial e_2}{\partial s_2} = 0,$$

$$f_5 = e_1 - (\partial e_1/\partial v_1)v_1 - (\partial e_1/\partial s_1)s_1 - e_2 + (\partial e_2/\partial v_2)v_2 + (\partial e_2/\partial s_2)s_2 = 0.$$

When the body is in a two-phase state of thermodynamic equilibrium, these equations are satisfied by a set of values of the variables p, θ, v_1, s_1, v_2, s_2. And, in the neighborhood of this set of values, the five f_i are continuous and possess continuous first derivatives with regard to all their variables. Now, to determine whether θ, v_1, s_1, v_2, s_2, in this neighborhood, are single-valued continuous functions of the pressure p taken independent, we require to determine whether the jacobian of the five f_i with regard to these variables is non-vanishing at the point. This jacobian, and the corresponding jacobians when $\theta, v_1, s_1, v_2, s_2$ are successively taken independent, are as follows:

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(\theta,v_1,s_1,v_2,s_2)} = -\Delta_1\Delta_2(s_2 - s_1), \tag{11}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,v_1,s_1,v_2,s_2)} = +\Delta_1\Delta_2(v_2 - v_1), \tag{12}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,s_1,v_2,s_2)} = +\Delta_1\Delta_2\psi_1, \tag{13}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,v_1,v_2,s_2)} = +\Delta_1\Delta_2\phi_1, \tag{14}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,v_1,s_1,s_2)} = +\Delta_1\Delta_2\psi_2, \tag{15}$$

$$\frac{\partial(f_1,f_2,f_3,f_4,f_5)}{\partial(p,\theta,v_1,s_1,v_2)} = +\Delta_1\Delta_2\phi_2, \tag{16}$$

where Δ_i is the hessian of the function $e_i(v_i, s_i)$, and ϕ_i, ψ_i are the expressions'

$$\phi_i = \frac{1}{\Delta_i}\left(\frac{\partial^2 e_i}{\partial v_i^2}(v_2 - v_1) + \frac{\partial^2 e_i}{\partial v_i \partial s_i}(s_2 - s_1)\right), \qquad i = 1,2,$$

$$\psi_i = \frac{1}{\Delta_i}\left(\frac{\partial^2 e_i}{\partial v_i \partial s_i}(v_2 - v_1) + \frac{\partial^2 e_i}{\partial s_i^2}(s_2 - s_1)\right), \qquad i = 1,2.$$

When the thermodynamic equilibrium of two coexisting phases is stable or metastable, and hence realizable at all, conditions of stability $\Delta_1 > 0$ and $\Delta_2 > 0$ are satisfied. If we now assume, from physical experience, that coexisting phases have distinct specific volumes and specific entropies, and thus that an isothermal transformation of either phase into the other always involves a change of volume and an absorption or development of heat, we have

$$v_2 - v_1 \neq 0, \quad s_2 - s_1 \neq 0. \tag{17}$$

At a "critical point" it is indeed true that $v_2 = v_1$ and $s_2 = s_1$; but a critical state is not a two-phase state.

Because of these data it follows from (11) and (12) that, in the neighborhood of any two-phase state of equilibrium, the quantities θ, v_1, s_1, v_2, s_2, are continuous functions of p, and that the quantities p, v_1, s_1, v_2, s_2 are continuous functions of θ. To formulate the first derivatives of these functions, we divide each of the equations (12) to (16) by (11), and divide each of (11) and (13) to (16) by (12), member by member, and reduce by the theorem on the jacobians of implicit functions, thereby obtaining:

$$(18)\qquad \begin{aligned} \frac{d\theta}{dp} &= \frac{v_2 - v_1}{s_2 - s_1}, & \frac{dv_i}{dp} &= -\frac{\psi_i}{s_2 - s_1}, & \frac{ds_i}{dp} &= \frac{\phi_i}{s_2 - s_1}, \\ \frac{dp}{d\theta} &= \frac{s_2 - s_1}{v_2 - v_1}, & \frac{dv_i}{d\theta} &= -\frac{\psi_i}{v_2 - v_1}, & \frac{ds_i}{d\theta} &= \frac{\phi_i}{v_2 - v_1}. \end{aligned}$$

The equation for $dp/d\theta$ is the "Clapeyron-Clausius equation."

The dependence of p and θ on each other is represented in the p, θ plane by curve segments which intersect in points representing regions of three-phase states of equilibrium. By (17) and the Clapeyron-Clausius equation, none of these segments can have a horizontal or a vertical tangent. On each segment, therefore, p and θ are single-valued functions of each other.

A similar conclusion concerning v_i, s_i, regarded as functions of p or of θ, can be drawn. Within a given field of two-phase states of stable equilibrium take any two states having equal pressures, and hence equal temperatures. Let the quantity v_1 have the respective distinct values v_1 and $v_1+\delta v_1$ in the two states. Then the states of the first phase are states of the field of stable one-phase states for this phase, and hence are subject to the criterion of stability (10). But this inequality is not satisfied, since $\delta p = \delta\theta = 0$. Hence the two states of the first phase are not distinct—a transformation of either into the other is not a physical change of state. The quantity v_1 is therefore a *single-valued* function of p or of θ. If for v_1 we read s_1 or v_2 or s_2, the reasoning is the same. So we conclude that, along any continuous segments of the curves representing these quantities as functions of p or of θ, these functions are single-valued.

There remain for consideration the equations (13) to (16), where we require to examine the possible vanishing of the factors ϕ_i, ψ_i. By (18), $\psi_i = 0$ when and only when dv_i/dp and $dv_i/d\theta$ vanish—when the density of the ith phase is a maximum or minimum. A minimum occurs, for example, at the temperature of maximum density of liquid water which is "saturated" with respect to a phase of water vapor. When this example is represented in the V, p plane, or in the V, θ plane, or in the V, S plane, it becomes obvious that p, θ, s_1, v_2, s_2 are two-valued functions of the specific volume v_1 of the liquid, for a range of values of this variable. When the density of the ith phase is neither a maximum nor a minimum the first members of (13) and (15) differ from zero, wherefore p, θ, s_i, and the specific volume and entropy of the other phase are continuous functions of v_i. At a point of maximum or minimum v_i two branches of each function reach a common value.

Again, by (18) $\phi_i = 0$ when and only when ds_i/dp and $ds_i/d\theta$ vanish—when the specific entropy of the ith phase is a maximum or a minimum. Such a maximum often occurs, and even minima are observed. When either is represented in the S, p plane, in the S, θ plane, or in the S,V plane, it becomes obvious that p, θ, v_1, v_2, s_2 are two-valued, and sometimes even three-valued.

we find, for example,

$$\begin{vmatrix} v_1 & v_2 & v_3 \\ s_1 & s_2 & s_3 \\ 1 & 1 & 1 \end{vmatrix} M_1 = \begin{vmatrix} V & v_2 & v_3 \\ S & s_2 & s_3 \\ c & 1 & 1 \end{vmatrix}.$$

If the coefficient of M_1 should vanish, the second member of this equation would vanish also, and the equations (20) would be satisfied by an infinite number of values of M_1, M_2, M_3, while v_i, s_i, V, S remained constant. But this is impossible, since the state of equilibrium of the body cannot be altered at constant V, S. So the coefficient of M_1 is different from zero. Further, by conditions of stability, the hessians Δ_1, Δ_2, Δ_3 in (19) are positive for realizable states of equilibrium. It follows that the eight variables are connected by eight independent relations, and hence are constants.

THE SCIENCE OF CALCULATION BY THE BOARD

By BIBHUTIBHUSAN DATTA, University of Calcutta.

In a previous issue of this Monthly[1] it has been observed by Professors David Eugene Smith and Salih Mourad that the word *takht* appears in the titles of several treatises on arithmetic by the eastern Moslem scholars of the tenth century A.D. and later. They have shown that the Arabic word *takht* is derived from the Persian word *takhta* and that it means "the board" or "the wood," not "the method" or "the table" as supposed by previous writers. The so-called ghobar-numerals (*Haruf al-ghobar*) of the western Arabs are ordinarily believed to have derived their name from the ancient practice of writing them on a wooden board covered with a thin layer of dust or sand (the Arabic *ghobar*). This writing of the learned authors reminds me of a term in the Hindu mathematics, which has an identical significance with *takht*, but which occupies a position of far greater importance. A comparative study of the two terms is bound to disclose how much deeper was the relation between the cultures of the two neighboring civilizations, so far at least as mathematics was concerned. It forms, indeed, a very interesting and instructive study from the point of view of the historian of mathematics. I propose in this paper to lay before the readers of this Monthly a brief statement of this relation.

The term in view is *pāṭi* or *pāṭī*. The modern Sanskrit word *pāṭi* is of doubtful origin. It does not appear to be purely Sanskrit, for it is not found in

[1] Vol. 33 (1927), pp. 258–300.

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the ancient writings and kindred literatures of India,[1] nor does it occur in the earlier Sanskrit lexicons such as the *Nirukta* and the *Amarakoṣa*. We miss it even in the *Abhidhānaratnamālā* of Halāyudha in the eleventh century. According to Gaṇeśa (1545), the eminent mathematician and commentator of Bhāskarācārya, the term *pāṭi* means "the express order of sequence of addition, etc."; and hence it denotes "arithmetic." Another commentator, Munīśvara (born 1603) takes it as having the same significance as *paripāṭi*.[2] This latter is a pure Sanskrit word and is contained in the *Amarakoṣa*. It means "succession," "method" or "the arrangement." These interpretations of the word *pāṭi* have found place in all the later Sanskrit lexicons, but the exact derivation of the word is not found in any dictionary. In the popular tongues of India, the word *pāṭi* means "the board," usually "the writing board." Hence it seems originally to have been a corrupt form of the pure Sanskrit word *paṭṭa*, which means "the slab," "the tablet,"[3] and which frequently occurs in the earlier Indian literatures. The still more ancient and common word for the board was *phalaka*.[4]

The use of the writing board has been common in India since at least five centuries before the Christian era. There is mention of it in the Buddhist *Vinaya Piṭaka*,[5] the *Jātaka*,[6] and the *Lalita-vistara*,[7] and in the other Sanskrit and kindred literatures of India.[8] The wooden writing board is still found to be in use in village elementary schools in some parts of India, though in other parts it is being gradually replaced by the modern slate boards. The students used to write on the board in early days, as at present, with the *varṇaka* (literally coloring rod, meaning the style or the pen), or the *pāṇḍu-lekha* or the *śveta-varṇi*, that is, with a piece of yellow or white soapstone.[9] Among the astronomers the more common practice seems to have been to cover the board with a thin layer of dust and then to do computing with a style. It is probably for this reason that the astronomical computation is called the *Dhūli-karma*

[1] The word *pāṭi* occurs in the *Vimāna-vattu* commentary of the Buddhists but its significance is not clear. Compare the *Pāli-English Dictionary*, by T. W. Rhys Davids and William Stede.

[2] See the commentary on Bhāskara's *Līlāvatī* by Ganeśa and by Munīśvara; also H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhâscara*, London, 1817, p. 1 ff.

[3] The word *paripāṭi* itself was possibly derived originally from the word *pāṭi* with the addition of the prefix *pari*.

[4] The word *phalaka*, meaning "the board," "the plank," "the leaf," occurs in the *Brāhmanas* and the *Grhya-* and *Srauta-sūtras* and also in the post-Vedic literatures of India. The word *paṭṭa* occurs in the *Mahābhārata*, *Harivamsa*, and the Buddhist *Nikāyas* and *Jātakas*.

[5] *Vinaya Piṭaka*.

[6] *Jātaka*, Fausboll, I, p. 451; Robert Chalmer's translation of the Jatakas, I, p. 275.

[7] *Lalitavistara*, ed. Rajendralal Mitra, Calcutta, p. 143.

[8] Compare Bühler's *Indian Palaeography*, English translation by J. F. Fleet, p. 5 and §37, C.

[9] At present, chalk is also used.

(literally the "dust-work") by Brahmagupta and Bhāskara.[1] It is strange that the practice continued to be in vogue among the Hindu astronomers in later times even when improved and better materials for writing were known and used by the people.[2] The celebrated Arabian traveller Al-Bîrunî (*c.* 1030) informs us that the practice of writing on the dust in his time was prevalent in Kashmir.[3]

The above practice of doing numerical computations on the dust-board must have been carried to Arabia by the Hindu astronomers or others; and I presume that the Arabic terms, *Ḥisāb al-ghobar* for Indian arithmetic and *Haruf al-ghobar* for the Indian numeral figures must have originated in that connection.[4] In fact the term *Ḥisāb al-ghobar* is nothing but the Arabic version of the Sanskrit term *Dhūri-karma.*

In later times the term *pāṭi* came to be very closely associated with the science of mathematics; so that arithmetic (including geometry) is called *Pāṭi-gaṇita,* which literally means "the science of calculation on the *pāṭi.*" This is exactly the title of a treatise on arithmetic by Sinên ibn al-Fath (c. 940), a native of Harran, *Ilm hisāb al-takht* ("The science of calculation of the *takht*").[5] The Arabic word *hisāb* is equivalent to the Sanskrit word *gaṇita.*

In India, arithmetic was some times called the *Pāṭi-śāstra* ("The book of knowledge of the *pāṭi*" or "The book of the *pāṭi*") or briefly *pāṭi.*[6] Similarly, we get in Arabia *Kitab al-takht* ("The book of the *takht*") as the title of a work by ar-Râzî. It has been stated before that the expression *kitâb al-takht* or *hisâb al-takht* occurs in the title of many Arabic works on elementary mathematics.

According to the celebrated Hindu mathematician Bhāskarācārya (born 1114), there are two principal branches of the *Gaṇita,* namely the *Paṭi-gaṇita* and the *Vīja-gaṇita.* The former branch corresponds to modern arithmetic (including geometry and mensuration) and the latter to algebra. They are also called respectively the *Vyakta-gaṇita* ("The science of calculation with the known") and the *Avyakta-gaṇita* ("The science of calculation with the unknown").[7]

The title *Pāṭi-gaṇita* was also given to some of the earlier Hindu works on

[1] *Brāhma-sphuṭa-siddhānta,* x. 62, 66, 67. *Siddhānta-siromani, Yantrādhyāya,* verse 24 (*vāsanā*). I am indebted to Mr. Probodhchandra Sengupta for this information.

[2] It may be noted that the astrologers in Bengal, even now, begin nativity calculations on a wooden board covered with dust.

[3] *Alberuni's India,* English translation by E. C. Sachau, London, 1878, vol. 1, p. 174.

[4] A commentary on the *Sēfer Yesīrāh,* composed about 950, speaks of the Indian arithmetic known under the name of *ghobar* calculation. (Smith and Karpinski, *Hindu-Arabic Numerals,* Boston, 1911, pp. 67–8).

[5] The *Fihrist,* pp. 37, 70; Arabic text, p. 281.

[6] *Mahāsiddhānta,* i. 1, XV. I. Also compare the colophon of Śrīdhara's *Triśatikā.*

[7] *Siddhanta-śiromani.*

arithmetic. Thus we have *Pāṭi gaṇita-sāra* ("The essence of calculation of the board") as the title of the arithmetical treatise of Śrīdhara (c. 750). This work is more commonly known as *Triśatikā* from the fact that it contains 300 couplets. The author states in the opening verse of the book that it is a compendium of his larger work on the *Pāṭi-gaṇita*.[1] This latter title was probably given to the twelvth chapter of Brahmagupta's larger astronomical treatise, *Brāhma-sphuṭa-siddhānta*.[2] We have also the *Pāṭi-gaṇita* by Âryabhaṭa the Younger (c. 950),[3] the *Pāṭi-gaṇita-kaumudi* by Nārāyaṇa (1357), and the *Pāṭi sāra* of Munīśvara. In modern times, in Sanskrit and in some of the important vernaculars of India, *pāṭi-gaṇita* is the common name for a treatise on arithmetic.[4]

It is certainly rather strange that such a term as *pāṭi* or "the board" came to be so closely connected and almost identified with the Hindu arithmetic. Sankar Balkrisna Dikshit thinks that the name *pāṭi-gaṇita* for arithmetic has its origin in the practice of the Hindu astrologers in using a wooden plank for calculation.[5] He has not clearly explained, however, how far and in what important respect the board is related to arithmetic; whether its presence in the Hindu name for arithmetic refers simply to the ancient and more common use of the board as the writing material for the arithmetical operations; or whether it goes still deeper and implies the use of the board as the "abacus," or an instrument of calculation which was so essential for expressing the place value before the invention of the zero. Bayley[6] and Fleet[7] are of the latter opinion. They suspect that the origin of the term *paṭi* in the name of the Hindu arithmetic lies in the use of the board as an abacus in India at some distant date. That this conjecture is without foundation is evident for the following reasons: First, it is very doubtful if the abacus, in any form, ever existed in India. No such trace of it, whether literary or palaeographic, has as

[1] Compare Śrīdhara's text *pāṭyā ganitasya* with the Arabic *hisâb al-takht*.

[2] We say "probably" because, as has been observed by Sudhakara Dvivedi (*Brāhma-sphuṭa-siddhānta*, Benares, 1902), the full title *Pāṭi-ganitādhyāya* ("the chapter on the *Pāṭi-ganita*") appears in some of the available manuscripts, while in others the word *pāṭi* is dropped.

[3] This is the title of the 15th chapter of the *Mahā siddhānta*.

[4] In southern and western parts of India arithmetic is called *anka-ganita*, while in northern and eastern India it is called *pāṭi-ganita*.

[5] *Indian Antiquary*, vol. 20, 1891, p. 54, footnote 7. R. C. Temple tells of a similar custom being prevalent among the Burmans even in the nineteenth century. "The Burman does his calculations either on the ground in the dust or on black *parabaik*. *Parabaik* is a thick course indigenous paper with a smooth greasy surface on which characters are written with a soft soapstone style." (*Ibid.* p. 54.) As stated in the first footnote on p. 522, the astrologers in Bengal, even now, begin nativity calculations on a wooden board covered with dust.

[6] Journal of the Royal Asiatic Society, N. S., vol. 14 (1882), p 369, footnote 3.

[7] J. F. Fleet, *The use of the abacus in India*, Journal of the Royal Asiatic Society (1911), p. 519.

yet been discovered.[1] Second, the term *pāṭi-gaṇita* does not appear so far as is known, before the seventh century A.D., but the abacus, if it had ever existed in India, must have disappeared long before that time, for we now know definitely that the place value and the zero existed in India even before 200 B.C.[2] Third, if the board abacus, before its complete disappearance from the country, had in reality exerted any influence on the coining of a name for Hindu arithmetic, the earliest name for the latter would have been *phalaka-gaṇita* or *paṭṭa-gaṇita*; for as we have aready pointed out, the oldest Sanskrit term for the board was *phalaka* or *paṭṭa*, not *pāṭi*. The total absence of a term like this from the earliest Indian literatures is highly significant.[3]

Professor Jogeshchandra Roy is of opinion that the term *pāṭi* as occurring in the name of the Hindu arithmetic has no connection whatsoever with the board abacus. It is used only to distinguish one branch of arithmetic from the other. He says that a branch, rather the higher branch of the science of the *gaṇita*, came to be called by the name *pāṭi-gaṇita* on account of the indispensable need of using the *pāṭi*, as merely the writing material, for performing all the necessary operations in it. It was thus separated from the lower or (more) elementary branch in which all the operations could be carried on mentally, without the aid of any writing material or of any other external agency.[4] This seems to be the most correct view of the origin of the name *pāṭi-gaṇita*.[5] In modern times in India it is customary to distinguish between the *pāṭi-gaṇita* and the *mānasānka* (or "the mental arithmetic").

This practice of differentiating between the elementary and the higher classes of arithmetic seems to be a very ancient one. In the earlier Buddhist

[1] It has been pointed out elsewhere (this Monthly, vol. 33 (1926), p. 450, footnote 5) that the instance quoted by Fleet in support of the use of the abacus in India implies something quite different.

[2] Bibhutibhusan Datta, *Early literary evidence of the use of the zero in India*, this Monthly, vol. 33 (1926), pp. 449–454.

[3] In the earlier literatures of India, we occasionally meet with such terms as *lipi-phalaka* (writing board), *citra-phalaka* (drawing or painting board), *śāri-phalaka* (gaming board), *sphatika-phalaka* (slab at the base of the pedestal), etc. Some of these terms, especially the compounds with the parts of the body (*aṁśa-*, *Jānusroni-*, etc.) are very ancient and occur in the *Brāhmanas*. Similarly, we have *citra-paṭṭa*, *dvāra-paṭṭa*, *ayo-paṭṭa*, *tāmra-paṭṭa*, etc. But never do we meet with any such term as *gananā-phalaka* or *gananā-paṭṭa* (counting board). There would have remained a trace of the counting board in the literatures, if it had ever existed in India.

[4] This opinion was given by Professor Roy in a personal letter to the writer in approving of and elaborating upon a similar suggestion of his own.

[5] This hypothesis is supported by the following remark of Keśava (c. 1450). In stating the object of writing a handbook of astronomy, *Grahakautuka*, Keśava observes that there are indeed many manuals which give the position of the stars, more or less accurately; but calculations according to them can not be done easily without the help of the board (*paṭṭa*). His object was to write a book which would help to calculate the position of the stars easily without using a *paṭṭa*. What Keśava undoubtedly meant was that the calculations required for his handbook were to be so simple that they could be done mentally. (Cf. *Ganaka Tarangini*, Sudhakara Dvivedi, Benares, 1892, p. 58). My attention was drawn to this passage by Mr. Gurugovinda Chakrabarti.

literature we find separate mention of three classes, viz., *mudrā*, *gaṇanā*, and *saṅkhyāna*. Following a Simhalese commentator, Rhys Davids takes *mudrā* as meaning "counting on the fingers."[1] From the Tibetan source, Schiefner gives it the German title "Handrechnen," while Franke translated it as "Finger Rechnen." *Gaṇanā* is rendered by Rhys Davids as "counting without using the fingers," meaning "mental arithmetic, pure and simple." All these meanings are quite in agreement with the interpretations of the eminent Buddhist commentator Buddhaghosa, although Rhys Davids is not sufficiently clear and precise, perhaps due to his failure to grasp fully the true significane of *saṅkhyāna*, taking it to mean "summing up large totals." It really means arithmetic in general—literally "the science of numbers." What was undoubtedly intended by the term *saṅkhyāna* is the upper class of arithmetic in contradistinction to the other two classes mentioned. One of the earliest enumerations of these three classes occurs in the *Digha Nikāya*,[2] and it is also found in the *Vinaya Piṭaka*,[3] *Divyāvadāna*[4] and *Milindapanho*.[5] Thus as early as the fifth or the sixth century before the Christian era the Indians were accustomed to distinguish between three classes of arithmetic; (1) mental arithmetic, pure and simple; (2) arithmetic with the use of the fingers; and (3) higher arithmetic in general. From the way in which this classification was made it is evident that some kind of writing material was essential for this third class of arithmetic. The term *pāṭi-gaṇita* must have been originated, in a later age, as a specific and distinctive name for it.

It is also noteworthy that the above classification of arithmetic is met with in Buddhist writings, not in other Indian literatures. Even in the former the different classes of arithmetic are not always distinguished, arithmetic in general being referred to as *gaṇanā*, *gaṇita*, or *saṅkhyāna*.[6] For example, in the *Vinaya Piṭaka*,[7] as also in the Hathigumpa inscriptions,[8] there is only general mention of the *gaṇanā* (arithmetic) together with *lekhā* (writing) and *rupa* (drawing)[9] as a subject of elementary study for a beginner. Of course in

[1] T. W. Rhys Davids, *Dialogues of the Buddha*, vol. 1, London, 1899, p. 21; *Milinda-pañho*, English translation by Rhys Davids, Oxford, 1890, p. 91 footnote. The Simhalese commentator is very explicit; according to him *mudrā* denotes "the finger-ring art, so-called from seizing on the joints of the fingers, and using them as signs."

[2] I. 51.

[3] IV. 7.

[4] *Divyāvadāna*, edited by E. B. Cowell and R. A. Neil, Cambridge, 1886, pp. 3, 26, 58.

[5] *Milinda pañho*, loc. cit., p. 91.

[6] For instance see *Jātaka*, I. 29; *Visuddhi mägga*, p. 278; *Sutta-Nipāta*, verse 677, *Milindapañho*, p. 79.

[7] I, 77; IV, 129.

[8] Hathigumpa and three other inscriptions, ed. Bhagawanlal Indraji, p. 22.

[9] *Rupa* includes painting as well as drawing. For amongst the illustrations of *rupa* enumerated in the Buddhist *Dhammasangini*, 617 (ed. by Edward Müller, London, 1885, p. 139, and English transla-

some of those instances, *ganaṇā* may be taken to mean mental arithmetic—the class (1) mentioned before; but in others it certainly denotes arithmetic in general. Similarly, in the Kauṭilaya's *Arthaśāstra*,[1] the word *saṅkhyāna* has been used instead of *ganaṇā*. The Jaina literatures speak of *Saṅkhyāna* (arithmetic) and *Jyotiṣa* (astronomy) as two of the *Brahmanical sciences*, as early as 300 B.C.[2] In the *Amarakoṣa* (c. 450 A.D.), the *gaṇita* and the *saṅkhyāta* are used synonymously. In several Sanskrit works an expert arithmetician is spoken of as a *saṅkhyā-gaṇita-tattvajna*.[3]

From what has been stated above, it follows that the name *paṭi-gaṇita* originated probably in a non-Brahmanical literature of India, a vernacular of northern India. This accounts to a certain extent for the presence in it of a word having the stamp of non-Sanskrit influence. The term *pāṭi-gaṇita* as well as the word *pāṭi*, however, entered into the Sanskrit literature in a later age, probably about the seventh century A.D.

Turning now to Arabia, we find strikingly close similarity in the title of several treatises on arithmetic belonging to the two countries. Two instances of a literal translation have been noticed before. There are others in which the titles of the Arabic treatises are directly adopted from the Hindu source. The title of Al-Kālwādani's arithmetical treatise *Kitāb al-takht fi'l hisāb al-Hindi* is obviously the Arabic version for *Hindu pāṭi-gaṇita* and Al-Antāki's *Kitāb al-takht al-kabir fi'l hisāb ab-Hindi* is nothing but the Arabic for "The great Hindu pāṭi-ganita." Suter has noticed eight Arabic treatises in the title of which occurs the word *takht* and he has always connected it with Hindu arithmetic.[4] This is very significant, and it all goes to show that the source of the Arabic term *Hisāb al-takht* for arithmetic must be traced to India.[5] The Arabs seem to have even followed the old Hindu practice of dividing arithmetic into the *pāṭi-gaṇita* and the *mudrā* (*-gaṇita*). This is shown by the titles of the works of Al-Antāki; for besides the one on *Hisāb al-takht* we find a title of another work by the same writer, *Kitāb al-hisāb bila takht bal bi'l yad* ("The book on arithmetic with the hand without the board)."

tion by Caroline A. F. Rhys Davids, *Buddhist Psychology*, London, 1900, p. 123) there are colors as well as geometric figures. The date of this book is 350 B.C. or earlier.

[1] Kauṭilya's *Arthaśāstra*, ed. and English Translation by R. Shamashastri, i. 5, 2.

[2] The *Kalpasūtra of Bhadrabāhu*, ed. by H. Jacobi, Leipzig, 1897, *Bhagabati-sūtrā*, Bombay, 1918, p. 112.

[3] For instance *Mahābhārata*, i. 293; *Mrcchaṭika*, i. 15; *Kathāsaria-sāgara*, vi. 32; *Brihat Sanilutā* xi. 2.

[4] H. Suter, *Die Mathematiker und Astronomer der Araber und ihre Werke*, Leipzig, 1900, pp. 31, 64, 66, 74, 149, 160.

[5] Cf. Dr. Solomon Gandz, *On three interesting terms relating to area*, this Monthly, vol. 34 (1927), pp. 80–86. Dr. Gandz has shown that all those three terms entered into Arabia from India. He has further remarked, "Naturally we turn to Hindu literature, the source of so many ideas of Al-Khowârîzmî . . ." (p. 83).

Nearly five months after the foregoing portion of the paper left my hands, there came through the courtesy of the author, a reprint of a paper by Dr. Solomon Gandz of New York, *Did the Arabs know the abacus?*, which had appeared in a previous issue of this Monthly.[1] As it deals primarily with the problem which forms the subject matter of the present study, it is proper to refer to it here. The learned writer, after making a retrospective study of what various other scholars have stated before him about the existence of the abacus among the Arabs, discusses anew, as an introduction to his own theory on the subject, the question of how the term *takht* or *Kitāb al-takht* came to be identified with the name of the science of arithmetic among the eastern Arabs. This question was raised and discussed previously by Professors Smith and Mourad, and an exactly similar question has been raised by Indologists about the identification of the term *pāṭi* with the name of arithmetic among the Hindus. It has been said very rightly that as all treatises of the eastern Arabs named *Kitāb al-takht* or *Hisāb al-takht* and those of the western Arabs named *Ilm al-ghobār* or *Hisāb al-ghobār* dealt with the Hindu arithmetic, any theory to be correct and reliable must have reference to this latter source. Smith and Mourad are of the opinion that the Hindu arithmetic was so named by the Arabs on account of the ancient practice of computations carried out on a wooden board sprinkled with a thin layer of dust, with the help of a stylus; that is, on the dust-board. Such was also the opinion of Smith and Karpinski[2] and of Weissenborn;[3] and a similar hypothesis, as already stated, was adumbrated by Bayley and Fleet as regards the origin and growth of the name *Pāṭīgaṇita*.

Gandz, however, does not think that the use of the dust-board as a crude writing device gave rise to the name of the science of arithmetic. "It can hardly be assumed," says he, "that a kind of a board tablet of the Arabic school is meant. There is nothing characteristic in such a board to give the name to the science of arithmetic" (p. 313). He therefore ventures to make the bold conjecture that the words *takht*, *ghobār*, and *turāb* are simply the Arabic terms for the translation of the Greek term *abakion* (abacus)—the Semitic *abaq*. He observes further that they need not necessarily refer to the dust-abacus alone, but the abacus in general, including the counter- and line-abacus. As regards the *raison d'être* of the Hindu arithmetic being called the abacus arithmetic, Gandz remarks: "Now we know the characteristic of the Hindu numerals was the place value and the zero. This was also the characteristic of the abacus. It would therefore be proper to call the Hindu arithmetic the abacus arith-

[1] Vol. 34 (1927), pp. 308–316. I am sorry that this interesting article escaped my notice when it first appeared.

[2] Smith-Karpinski, *The Hindu-Arabic Numerals*, p. 65.

[3] Quoted in extenso by Dr. Gandz.

zero. This theory is more comprehensive than that of Gandz, since it embraces also the Hindu side which he has overlooked. Indeed Gandz's theory fails to explain the origin and source of the Hindu terms. Again, from the fact that the western Arabs adopted and continued to use a term which went practically into disuse in the land of its birth, there follows the irresistible conclusion that the Hindu arithmetic reached the Arabs of the West long before it reached the Arabs of the East. Hence the present theory is in perfect accord with and is a kind of corroboration of the ingenious theory of Woepcke relating to the differences between the Hindu numerals of the eastern and the western Arabs.[1]

THE USE OF THE DISCRIMINANT IN FINDING THE SINGULAR SOLUTION OF $f(x, y, p)=0$

By G. H. LIGHT, University of Colorado

The theory of singular solutions, as given by Cayley,[2] is founded upon the fact that the singular solution must satisfy the differential equation $f(x, y, p)=0$, since x, y, and p are the same for the singular solution and the general solution of $f(x, y, p)=0$ at all points on the singular solution. In order to separate the singular solution from the general solution of $f(x, y, p)=0$ we must examine more closely the behaviour of p. To do this, suppose $f(x, y, p)=0$ to be written in the form

(1) $$y = f_1(x,p) \quad \text{or (2)} \quad x = f_2(y,p).$$

This will not change the singular solution. Hence the singular solution will still be present when $f(x, y, p)=0$ has been changed into either

$$\phi\left(x, p, \frac{dp}{dx}\right) = 0 \quad \text{or} \quad \psi\left(y, p, \frac{dp}{dy}\right) = 0$$

by differentiating (1) and (2), respectively; only its form will now be $r(x, p)=0$, $s(y, p)=0$, or $t(p)=0$—since it is a function of x and y only, say $h(x, y)=0$. Therefore dp/dx or dp/dy must represent the change that takes place in p along the general solution only, having nothing to do with the singular solution; in other words, when $f(x, y, p)=0$ is changed into

$$\phi\left(x, p, \frac{dp}{dx}\right) = 0 \quad \text{or} \quad \psi\left(y, p, \frac{dp}{dy}\right) = 0$$

by differentiating (1) and (2) with respect to x and y, respectively, then

[1] *Journ. Asiat.* t. 1 (6), 1863; also compare Florian Cajori, *History*, pp. 100–101.

[2] Messenger of Mathematics, vol. 2 (1873), pp. 6–12, vol. 6 (1877), pp. 23–37.

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$$\phi\left(x, p, \frac{dp}{dx}\right) = 0 \quad \text{or} \quad \psi\left(y, p, \frac{dp}{dy}\right) = 0$$

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$h(x, y)=0$ changes into $r(x, p)=0$, $s(y, p)=0$, or $t(p)=0$; and the left member of these equations must now appear as a factor of

$$\phi\left(x, p, \frac{dp}{dx}\right) \text{ or } \psi\left(y, p, \frac{dp}{dy}\right).$$

This means that $\phi(x, p, dp/dx)$ has at least two known factors and may be written in the form

$$\phi(x,p,dp/dx) \equiv \lambda_1 r(x,p)\phi_1(x,p,dp/dx), \tag{3}$$

where λ_1 contains the remaining factors of $\phi(x, p, dp/dx)$. Similarly,

$$\psi(y,p,dp/dy) \equiv \lambda_2\ s(y,p)\ \psi_1(y,p,dp/dy), \tag{4}$$

and λ_2 contains the remaining factors of $\psi(y, p, dp/dy)$. When $h(x, y)=0$ changes into $t(p)=0$, both (3) and (4) must have $t(p)$ as a factor.

Since the vanishing of the discriminant is the condition that the variable, in this case p, shall have at least two equal roots, the singular solution must be represented by an equation $h(x, y)=0$, where $h(x, y)$ is a factor of $B(x, y)$ and $B(x, y)=0$ is the discriminant equation of $f(x, y, p)=0$.

This is exactly what takes place in practice; for, when we change $f(x, y, p)=0$ into

$$(1) \qquad y = f_1(x,p) \quad \text{or} \quad (2) \quad x = f_2(y,p)$$

by solving $f(x, y, p)=0$ for y and x, respectively, and then eliminating y and x from (1) and (2) by differentiation, we at once get

$$\phi\left(x, p, \frac{dp}{dx}\right) = 0 \text{ or } \psi\left(y, p, \frac{dp}{dy}\right) = 0.$$

These decompose into (3) or (4), and the singular solution is in the form $r(x, p)$, $s(y, p)$, or $t(p)$, while the general solution[1] is obtained from

$$\phi_1\left(x, p, \frac{dp}{dx}\right) = 0 \text{ or } \psi_1\left(y, p, \frac{dp}{dy}\right) = 0.$$

The purpose of this paper is to show how the factor $r(x, p)$, $s(y, p)$, or $t(p)$ can be found before solving $f(x, y, p)=0$. In what follows, $f(x, y, p)=0$ is to be considered:

(a) as a polynomial in x, y, p,

(b) as not being factorable as to p,

(c) as having a singular solution.

[1] The general solution is found by solving $\phi_1(x, p, dp/dx)=0$ or $\psi_1(y, p, dp/dy)=0$, obtaining $p=f_1(x, c)$ or $p=f_2(y, c)$ and using the p thus found to eliminate p from $f(x, y, p)=0$.

Consider the type of differential equations that can be written in the form

$$f(x,y,p) = 0, \quad p = dy/dx,$$

which are of degree n in p, $n \geqq 2$.

Let the p-discriminant of $f(x, y, p) = 0$ be written in the form

$$B(x,y) = Cf(x)g(y)h(x,y).$$

As far as this paper is concerned, the factors $f(x)$ and $g(y)$ are considered as being trivial, since they lead to the obvious values ∞ and 0, respectively, for p, where p is the value of p along the singular solution.

In order to find p for the singular solution, we proceed as follows. Let $h(x, y) = 0$. Then $p = -(h_x/h_y)$. We now make use of $h(x, y) = 0$ to express $p = -(h_x/h_y)$ as a polynomial in x and p, say $r(x, p) = 0$; or as a polynomial in y and p, say $s(y, p) = 0$; or as a polynomial in p, say $t(p) = 0$.

Let us take $r(x, p)$[1] $= 0$ as the form that $h(x, y) = 0$ assumes; when the p, obtained by solving $r(x, p) = 0$ for p, along with the y we get by solving $h(x, y) = 0$ for y, satisfy $f(x, y, p) = 0$, then this p becomes the p of the singular solution. We then solve $f(x, y, p) = 0$ for y, getting (1) $y = f_1(x, p)$, and we eliminate y from (1) by differentiating it with respect to x, thus obtaining $\phi(x, p, dp/dx) = 0$, in which the differential equation of the general solution is distinguished from the differential equation of the singular solution by the new variable dp/dx which cannot be present in $r(x, p)$ since the singular solution is a function of x and y only and since differentiating $h(x, y) = 0$ with respect to x can only introduce p into the new form, $r(x, p)$, for $h(x, y) = 0$.

Therefore, when $f(x, y, p) = 0$ is changed into $\phi(x, p, dp/dx) = 0$ by changing $f(x, y, p) = 0$ into (1) $y = f_1(x, p)$ and by eliminating y from (1) by differentiating it with respect to x, then $r(x, p)$ must appear as a factor $\phi(x, p, dp/dx)$; and another factor becomes $\phi_1(x, p, dp/dx)$, this factor leading to the general solution.

The same argument applies when $s(y, p) = 0$ is taken instead of $r(x, p) = 0$. We have only to interchange y and x in what has been said, obtaining $\psi(y, p, dp/dy) = 0$; and $s(y, p)$ must appear as a factor of $\psi(y, p, dp/dy)$ leaving $\psi_1(y, p, dp/dy) = 0$ to furnish the general solution. It sometimes happens that the polynomial in p is a function of p only, say $t(p)$; this occurs only when $h(x, y) = 0$ has a particular form, and does not affect the argument just given. When this occurs, $r(x, p) = s(y, p) = t(p)$, and the factor $t(p)$ will be present in both

$$\phi\left(x, p, \frac{dp}{dx}\right) \text{ and } \psi\left(y, p, \frac{dp}{dy}\right),$$

[1] This does not mean that we cannot take $s(y, p) = 0$ and arrive at the same general solution.

and we can use either

$$\phi_1\left(x, p, \frac{dp}{dx}\right) = 0 \text{ or } \psi_1\left(y, p, \frac{dp}{dy}\right) = 0$$

to obtain the general solution.

It remains to find what relation must hold between the coefficients and exponents of $h(x, y) = 0$ in order to determine whether $r(x, p)$, $s(y, p)$, or $t(p)$ is to be the factor that occurs in $\phi(x, p, dp/dx)$ or $\psi(y, p, dp/dy)$. The three most common forms for $h(x, y) = 0$ are considered but the method is general.

Case 1.

$$h(x, y) = ax^m + by^n = 0.$$

Here

(1) $$p = -ab^{-1} \cdot mn^{-1} \cdot x^{m-1}y^{-(m-1)},$$

(a) $$y^{n-1} = (-ab^{-1})^{(n-1)/n} \cdot x^{m(n-1)/n},$$

(b) $$x^{m-1} = (-ba^{-1})^{(m-1)/m} \cdot y^{n(m-1)/m}.$$

From (1) and (a) we get

(c) $$r(x, p) \equiv bn^n p^n + am^n x^{m-n},$$

while (1) and (b) give

(d) $$s(y, p) \equiv bn^m p^m + am^m y^{m-n}.$$

Evidently (c) and (d) are the same when $m = n$, and we have $t(p)$.

Example 1.

(1) $$x^2p^2 - 3xyp + 2y^2 + x^3 = 0.$$

Here $h(x, y) \equiv 4x^3 - y^2$. Since $m \neq n$, it is not possible to have $t(p)$ as a factor and we must choose between $r(x, p) \equiv p^2 - 9x$ and $s(y, p) \equiv 2p^3 - 27y$. Since we would naturally solve (1) for y, $r(x, p)$ must appear as a factor in $\phi(x, p, dp/dx) = 0$. Thus,

$$\phi(x, p, dp/dx) = 8x^3[p^2 - 9x][2 - x(dp/dx)].$$

Here

$$\lambda_1 = 8x^3, \quad \phi_1(x, p, dp/dx) = 2 - x(dp/dx).$$

Example 2.

(1) $$p^3 - 4xyp + 8y^2 = 0.$$

Here $h(x, y) \equiv 4x^3 - 27y$. From (c) and (d), we have

$$r(x, p) \equiv 9p - 4x^2, \quad s(y, p) \equiv p^3 - 4y^2.$$

Solving (1) for x, we have

$$\phi(y, p, dp/dy) \equiv [p^3 - 4y^2][p - 2y(dp/dy)],$$

and $s(y, p)$ appears as a factor. Had (1) been solved for y, we should have had

$$\phi(x,p,dp/dx) \equiv p^3[9p - 4x^2][(dp/dx) - 2p],$$

in which $r(x, p)$ appears as a factor, with

$$\lambda_1 = p^3, \quad \phi_1(x,p,dp/dx) \equiv (dp/dx) - 2p.$$

Example **3.**

$$p^2 - xp + y = 0. \tag{1}$$

Here $h(x, y) \equiv x^2 - 4y$. From (c) and (d), we have

$$r(x,p) \equiv 2p - x, \quad s(y,p) \equiv p^2 - y.$$

Since (1) is linear in both x and y we can solve for either one. Solving for y, we have

$$\phi(x,p,dp/dx) \equiv (2p - x)(dp/dx),$$

and $r(x, p)$ appears as a factor. Solving for x gives

$$\psi(y,p,dp/dy) \equiv (p^2 - y)(dp/dy),$$

and $s(y, p)$ appears as a factor.

Example **4.**

$$y^2p^3 + 2xp - y = 0. \tag{1}$$

Here $h(x, y) \equiv 32x^3 + 27y^4$, and (c) and (d) give

$$r(x,p) \equiv 8xp^4 + 3, \quad s(y,p) \equiv 2yp^3 + 1.$$

Solving (1) for x, we shall have

$$\psi(y,p,dp/dy) \equiv 2[2yp^3 + 1][p + y(dp/dy)],$$

and $s(y, p)$ appears as a factor, with $\lambda_2 = 2$. Solving (1) for y leads to

$$\phi(x, p, dp/dx) \equiv 16p^8[8xp^4 + 3][p^6 - 2x(dp/dx)].$$

Here $\lambda_1 = 16p^8$, and $r(x, p)$ appears as a factor.

Example **5.**

$$xp^2 - 2yp + ax = 0. \tag{1}$$

Here $h(x, y) \equiv ax^2 - y^2$. From (c) and (d), we have

$$r(x,p) = s(y,p) = t(p) \equiv p^2 - a.$$

Then

$$\phi(x,p,dp/dx) \equiv 2[p^2 - a][p - x(dp/dx)],$$

Using (*a*) with (*b*) and (*a*) with (*c*), we get

(*d*) $$r(x,p) \equiv an^n p^n + (-1)^n\, bm^n x^{-(m+n)},$$

(*e*) $$s(y,p) \equiv bn^m p^m + (-1)^m\, am^m y^{m+n}.$$

Clearly (*d*) and (*e*) are not the same unless $m=n=0$: If this should occur $h(x, y)$ would become a constant and $f(x, y, p)$ would be factorable as to p, which was ruled out.

From a study of these three cases we see that $t(p)$ can only occur when $m=n$, and in Case I only are m and n unrestricted in their values.

Example **1.**

$$x^3p^2 + x^2yp + a^3 = 0.$$

Here $h(x, y) \equiv xy^2 - 4a^3$. From (*d*) and (*e*) we have

$$r(x,p) \equiv x^3p^2 - a^3, \quad s(y,p) \equiv 8a^3p + y^3.$$

Then

$$\phi(x,p,dp/dx) \equiv x[x^3p^2 - a^3][2p + x(dp/dx)],$$

and $r(x, p)$ is a factor, with $\lambda_1 = x$; and $\phi_1(x, p, dp/dx) \equiv 2p + x(dp/dx)$.

Example **2.**

$$x^4p^2 - xp - y = 0.$$

Here $h(x, y) \equiv 4x^2y + 1$. From (*d*) and (*e*), we have

$$r(x,p) \equiv 2x^3p - 1, \quad s(y,p) \equiv p^2 - 16y^3.$$

Then

$$\phi(x,p,dp/dx) \equiv [2x^3p - 1][2p + x(dp/dx)],$$

and $r(x, p)$ is a factor, with $\lambda_1 = 1$; and $\phi_1(x, p, dp/dx) \equiv 2p + x(dp/dx)$.

COORDINATES USED IN HINDU ASTRONOMY

By SUKUMAR RANJAN DAS, Calcutta, India

Coordinates in astronomy are the quantities by means of which the position in space of a heavenly body is defined. The systems of coordinates used in Hindu astronomy are nearly the same as those used in modern astronomy. Strictly speaking the Indian Astronomical Works do not very clearly point out in a systematic manner how the position of a heavenly body is to be defined with regard to the fundamental planes. The position is indicated whenever occasion arises, but there is no formal statement how the indication is to be made and how the position is to be pointed out. Information on this subject is scattered throughout almost every book, but the major portion is contained

Using (*a*) with (*b*) and (*a*) with (*c*), we get

(*d*) $$r(x,p) \equiv an^n p^n + (-1)^n\, bm^n x^{-(m+n)},$$

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Clearly (*d*) and (*e*) are not the same unless $m=n=0$: If this should occur $h(x, y)$ would become a constant and $f(x, y, p)$ would be factorable as to p, which was ruled out.

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Here $h(x, y)\equiv xy^2-4a^3$. From (*d*) and (*e*) we have

$$r(x,p) \equiv x^3p^2 - a^3, \quad s(y,p) \equiv 8a^3p + y^3.$$

Then

$$\phi(x,p,dp/dx) \equiv x[x^3p^2 - a^3][2p + x(dp/dx)],$$

and $r(x, p)$ is a factor, with $\lambda_1=x$; and $\phi_1(x, p, dp/dx)\equiv 2p+x(dp/dx)$.

Example **2.**

$$x^4p^2 - xp - y = 0.$$

Here $h(x, y)\equiv 4x^2y+1$. From (*d*) and (*e*), we have

$$r(x,p) \equiv 2x^3p - 1, \quad s(y,p) \equiv p^2 - 16y^3.$$

Then

$$\phi(x,p,dp/dx) \equiv [2x^3p - 1][2p + x(dp/dx)],$$

and $r(x, p)$ is a factor, with $\lambda_1=1$; and $\phi_1(x, p, dp/dx)\equiv 2p+x(dp/dx)$.

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in the chapter on the construction of the armillary sphere, i.e. *Golabandhādhikāra*.

As in modern astronomy, the first system determines the position of a body by its azimuth and altitude or zenith distance. In measuring the azimuth the Indian mode differs from the modern one. In modern astronomy it is the angle between the meridian and the vertical circle through the center of the body under observation, being measured from the north point of the horizon all round through 360° in the direction north, east, south and west. In Hindu astronomy it is the angle which the vertical circle through the centre of the body makes with the prime vertical, being measured either way from the east and west points of the horizon separately. It never exceeds 90° and is north or south from the east or west. Altitude and zenith distance are precisely the same as in modern astronomy.

The only attempt made by Varāhamihira to define the coordinates systematically is in the chapter on the construction of astronomical instruments in *Pañcha Siddhāntikā*. In chapter XIV, he defines the horizon as the circle in which the sky is joined as it were to the earth, in it the east-west and north-south lines are drawn (verse 17). "The interval between the pole and the horizon is the terrestrial latitude; and the difference of the latitude and ninety degrees is called co-latitude, which declines from the zenith towards the pole. The day circle is what intervenes between the rising and setting of the sun etc." (verse 18).

Brahmagupta more explicitly defines the first system of coordinates. He says: "One circle called the *Samamaṇḍala* or the prime vertical has its plane stretching east and west; another stretching north and south is termed the *Yāmyottaravṛtta* or meridian; another called the *Kṣitija* or horizon encircles the other two like a girdle. At the common centre of these circles is the observer located on the earth." (*Brāhma Sphuta Siddhānta*, *Golādhyāya*, verse 48.)

The azimuth circle is thus described by the same writer: "In as much as an observer on the earth finds a planet in the circumference of the upper half of the azimuth circle (*Dṛgmaṇḍala*) it turns with the planet." (Verse 55, *Golādhyāya*.)

Thus we get the three fixed planes—the meridian, the prime vertical, and the horizon; and these give the first system of coordinates.

One noticeable point is the importance attached to the particular azimuth circles which stand midway between the meridian and the prime vertical bisecting the right angles which they make with each other. Brahmagupta, Lalla and Bhāskara have mentioned this fact. Lalla writes

> Pürvāparamürdhvamadhaḥ prathamaṃ samamaṇḍalaṃ jagur vṛttam/
> Yāmyottaraṃcha tadvattathaiva vṛttadvayaṃ cha vidis'oḥ

Āveṣṭyamānametānyardhachchhedena yadbhavedṛttam/
Taṭ kṣitija maṇḍalaṃ syādudayāstamayāviha dyusadām//2/
(Golabandhādhikāra)

Here the azimuth circles are named as the vertical circles of intermediate directions. The word *vidiś'a* literally means "of no direction."

Regarding the use of the first system of coordinates we get the following information in Bhāskara's *Siddhānta S'iromaṇi:* "The arc of the horizon measured in degrees, intercepted between the east point and the point at which the azimuth circle under consideration meets the horizon is called the *digaṃs'a* or azimuth. The *jyā* or linear sine of *digaṃs'a* is termed the *drigjyā*. For the western part of the horizon the measurement is similar." (Gaṇitādhyāya, *Tripras'na*, verse 45.)

About the position of the heavenly body in the azimuth circle or *dṛgmaṇḍala*, i.e. about its zenith distance and altitude, Bhāskara says: "the position of a planet is to be given not by the zenith distance or *nata* and altitude or *unnata* but by *drigjyā* and *s'aṃku*." (*Golabandha, Vāsana bhāṣya*).

The terms *drigjyā* and *s'aṃku* are explained by Brahmagupta who writes, "the linear sine of the zenith distance is the *drigjyā*, that of the altitude is the *s'aṃku*, the altitude and the zenith distance are reckoned in the azimuth circle." (*Brāhma Sphuta Siddhānta, Golādhyāya*, verse 63.)

The relation between altitude and zenith distance is thus given by Bhāskara; "90 − zenith distance = altitude" (Ganitādhyāya, *Tripras'na*, 32.)

The four cardinal points are called the *Uttara-svastika* (north point), *prāk-svastika* (east point), *dakṣiṇa-svastika* (south point) and *apara-svastika* (west point). The zenith and the nadir are the *kha-svastika* and *adhaḥ-svastika*.

The second system of coordinates consists of the hour angle and declination. These are known as the *nataghati* and *sphuta-krānti*, briefly mentioned as *nata* and *krānti*. There is besides, an *unnata-ghati* or time-altitude. It is the angle which the hour circles of a heavenly body at its rise and at a given time make with each other.

Of a star the time-altitude or *unnata-ghati* is the sidereal time elapsed since its appearance on the horizon; for the sun it is measured by clock-time; for the moon or a planet it is the sidereal time elapsed diminished by the change of right ascension during it, neglecting in all cases the change of declination. The *nata-ghati* or hour angle is half day of the body(that is the sidereal time taken by it to pass from the horizon to the meridian) diminished by the time-altitude.

For the western sky the *nata-ghati* is the sidereal time elapsed since the star under observation crossed the meridian and the *unnata-ghati* is the sidereal measure of the time from the moment under consideration to the setting of the star or in other words its half day diminished by the nata-ghati. For other

heavenly bodies change of right ascension is to be taken into account as in the case of eastern sky. The word *ghati* means twenty four minutes. It is either sidereal (*ārkṣa*) or mean solar (*sāvana*). Like the word "hour angle" *nata-ghati* signifies the time equivalent of an angle; *ghati* in this as "hour" in that means time. The following definition of *nata* and *unnata* is given by Bhāskara: "considering the part of the day that is over and the part that yet remains, the less is the *unnata* or time-altitude; half the time diminished by it is called the *nata* or hour angle." (Gaṇitādhyāya, *Tripras'na*, verse 53).

The above verse is intended to indicate the hour angle of the sun, but its application may be extended to include all heavenly bodies. We may for that purpose take day in the sense of the time from the rising to the setting of a heavenly body and reduce it to the standard of a complete revolution round the earth.

The *Mahā-siddhānta* of Āryabhaṭa II gives a similar definition in the chapter on *Tripras'na* or "Triple Enquiry." He says: "The part of the day that is over is the time-altitude or *unnata* in the eastern sky, while the part that yet remains is the *unnata* in the western sky. Half the length of day diminished by the *unnata* is the *nata* or hour angle."

About declination Bhāskara says that it is the distance of a heavenly body from the equator along its secondary through the body. He says: "The perpendicular distance from the equator of a planet lying in its orbit is its declination."

The equator from which declination is measured lies in relation to the observer as described in Brahmagupta's *Brāhma Sphuta Siddhānta:* "The equator meets the horizon at the cardinal points east and west, the top lying to the south of the prime vertical by the latitude of the observer and the bottom as much to the north of it." (*Golabandha*, verse 51.)

The equator and the meridian are evidently the two fixed planes of reference of this second system of coordinates, though importance is attached to the six hour circle or *unmaṇḍala*. The *unnata* or time-latitude is related to it just as the hour angle or *nata* is related to the meridian, though not reckoned in the same way.

Brahmagupta describes the six-hour circle as below: "The *unmaṇḍala* or six-hour circle meets the horizon at the east and west points and is above and below it from the north and south cardinal points by the latitude of the observer. The increase and decrease in the duration of day and night are due to it." (*Golādhyāya*, 50).

The fixed planes of the two systems are together in Brahmagupta's words: "The prime vertical, the meridian, the horizon, the equator and the six-hour circle which are fixed planes." (*Golādhyāya*, 67).

Of the two modern systems of coordinates which are independent of diurnal

these are called *kālāṃs'as* or degrees of time. But the right ascension of any heavenly body whatsoever or right ascension in the amplitude of its modern application does not appear in Hindu astronomy.

This absence of the recognition of right ascension may be conjectured as an internal evidence of the remote date of the existence of the sidereal system. If the initial point is different from the vernal equinox, simple ascertainment of right ascension is not possible. This might have led to the neglect of such an useful coordinate.

However, it may be said in conclusion that the coordinates used in Hindu astronomy, though not explicitly stated, were more or less the same as are used in modern astronomy.

THE EQUATION OF TIME IN HINDU ASTRONOMY

By SUKUMAR RANJAN DAS, Calcutta, India

The *Sūrya Siddhānta* recognises that the apparent solar day (the interval of time from one sunrise to the next) is variable, as is evident from the following division of time which it gives:

"Time that is measurable is that which is in common use, beginning with the *prāṇa*. The *pala* or *bināḍi* contains six *prāṇas*. The *ghalikā* or *nāḍi* is 60 *palas*, and the *nakṣatra ahórātra*, or sidereal day and night, contains 60 *ghalikās*. A *nakṣatra māsa* or sidereal month, consists of 30 sidereal days."[1]

The same work gives the following rule for finding the length of an apparent solar day in sidereal units: "Multiply the diurnal motion (in minutes) of a planet by the number of *prāṇas* which the sign in which the planet is takes in its rising (at a given place); divide the product by 1800′ (the number of minutes which each sign or *rās'i* of 30° of the ecliptic contains); add the quotient, in *prāṇas*, to the number of *prāṇas* contained in one sidereal day (*i.e.*, $6\times60\times60$, or 21,600 *prāṇas*); the sum will be the apparent day of that planet in *prāṇas*."[2]

Here 1800 is the number of minutes of one *rās'i* or sign of the ecliptic. Let the time taken by a particular sign to rise above the horizon be p *prāṇas;* d, the daily motion of a planet in minutes; and x, the time in *prāṇas* by which a planet's day exceeds the sidereal day, this excess being due to its eastward motion among the stars; therefore, $1800:d::p:x$ or $x=dp/1800$. This is the rule.

This rule is based on the knowledge of two facts: (1) that the daily motion in longitude of any planet is a variable quantity, (2) that the different signs of

[1] *Sūrya Siddhānta*, chapter V, verses 11 and 12.

[2] *Sūrya Siddhānta*, chapter II, verse 59.

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the zodiac take different intervals of time to rise above the horizon at any station, and at any place on the equator.

The *Sūrya Siddhānta* then proceeds to find out the corrections to longitude of planets due to one part of the equation of time: "Multiply the diurnal motion of a planet by the number of minutes contained in the first equation of the sun, and divide the product by the number of minutes contained in a circle, i.e., by 21,600′; add or subtract the quotient, in minutes, according as the sun's equation is additive to or subtractive from the place of the planet, which is found from the *ahargaṇa* at the mean midnight at Lankā; the result will be the place of the planet at the true midnight at Lankā."[1] This is called the *bhujāntara* correction in minutes.

The *bhujāntara* correction is to be applied to the place of a planet found from the *ahargaṇa* for finding the place of the planet at the true midnight at Lankā, arising from that portion of the equation of time which is due to the sun in the ecliptic.

The *Siddhānta S'iromaṇi*[2] of Bhāskarāchārya has, however, laid down more accurate rules:

"Multiply the sun's equation of center by the time the sign of the zodiac, in which the sun is, takes to rise at any place on the equator, and divide the product by 1800; multiply the quotient thus obtained by the daily motion of a planet and divide by the number of *prāṇas* in a whole day; then apply this last quotient positively or negatively to the longitude of a planet, according as the sun's equation of center is applied positively or negatively. This correction is called *bhujāntara*."

These rules will be evident from the way in which the longitude of a planet is calculated. The equation of time is the difference of the right ascensions of the true sun moving along the ecliptic and the mean sun moving along the equator. It has two parts; the difference in the right ascensions of the true sun and a fictitious star moving along the ecliptic with the mean motion of the sun is one part of the equation of time; while the difference of the right ascensions of the mean sun moving along the equator and this fictitious star is the other part. The former is called *bhujāntara* and the latter *udayāntara* in Hindu astronomy. The *Sūrya Siddhānta* is content with only having applied the *bhujāntara* correction. It was Bhāskara who first of all Hindu astronomers detected this *udayāntara* correction.

Bhāskara gives the following proof of *bhujāntara:*

"The longitude that has been found for the mean sunrise is converted to that for the apparent sunrise. First convert the sun's equation of center into

[1] *Sūrya Siddhānta*, chapter II, verse 46.

[2] *Siddhānta S'iromaṇi*, Grahagaṇita, chapter VIII, verse 61.

prāṇas by proportion; if the number of minutes in a sign rises on the equator, in how many *prāṇas* will the arc of the equation of center rise? Then take another proportion; if the number of *prāṇas* of a day changes the longitude of a planet by the amount of daily motion, what is the change for these *prāṇas*? The result in minutes should then be applied negatively or positively as the apparent sun rises before or after the mean sun. Hence the rule is proved."[1]

Then Bhāskara gives the rule for the *udayāntara* correction:

"To the mean sun add the total amount of the precession of the equinoxes; find the sign in which the sun now falls and the degrees of it which are passed over. Multiply the degrees of the sign by that sign when on the equator and divide by 30°. The result is the number of *bhukta-asus* or *prāṇas* taken by that sign. Now add up the *asus* (*prāṇas*) while on the equator of the whole signs passed over by the sun and add to this sum the *bhukta-asus* (or *prāṇas* taken) found above; the result is the number of *prāṇas* of the right ascension of the mean sun. Next reduce the mean longitude of the sun to which the total amount of precession has been added to minutes of arc. Take the difference of these minutes and those *prāṇas*, multiply the daily motion of a planet by it and divide by the total number of *asus* of a whole day; add the result, which is in minutes of arc, to the longitude of the planet if the number of *asus* is greater than the number of minutes and subtract if the minutes are greater in number. This correction is called *udayāntara*."

Bhāskara has given the following proof of his rule:

"The *ahargana* (the number of days elapsed since creation) that has been found elsewhere (for the calculation of longitudes) is of mean solar units (days), as the apparent solar day is variable. The sixty sidereal *ghatis* increased by the number of *asus* (which are equal to the minutes of arc) in the mean daily motion of the sun, is equal to one mean solar day i.e., 60 *ghatis* 59 *prāṇas* and 8 sixtieths (is equal to 24 hrs. 3 min. 56 secs. Twenty-two sixtieths in sidereal time). But the apparent solar day is 60 ghatis and the time in which the arc of the daily motion rises is a variable quantity, because every day the daily motion is different and also because the times of rising of the different signs of different months are different. As the apparent *ahargaṇa* is not obtained in the way described before, we find the mean *ahargaṇa* and we do not get the longitudes at the time of sunrise at Lankā (the City of Rāvana); we get them sometimes for a moment a little before sunrise and sometimes for a moment a little after sunrise. Thus we find the mean *ahargaṇa* when the mean sun is near the horizon at Lankā."

We next proceed to find the difference between the mean and the apparent

[1] See Vasāna Bhāṣya under verses 62, 63, 64 of *Grahagaṇita*, chapter VIII. See also chapter V, verse 43 and the footnote, *Golādhyāya* (B. D. Sastri's edition).

ahargaṇa. Add up the *asus* at the time of the rise on the equator of the signs passed over by the sun from the beginning of the sign *Meṣa*. The resulting *asus* is the excess of the *ahargaṇa* over the sidereal day. Hence the difference between the *asus* and the minutes of arc is the difference between the two *ahargaṇas*. Now apply the proportion; if the *asus* of a whole day changes the longitude by the daily motion, what will be the change for this difference of *asus*? The result is additive if the *asus* are greater than the minutes, otherwise it is to be subtracted. (See also chapter IV, verses 19, 20, 21, 22, 23 of Golādhyāya and the footnote by Bapu Dev Sastri.)

Bhāskara has given further explanation of his *udayāntara* correction:

"If the ecliptic be divided into four quadrants beginning from the vernal equinox, each quadrant will rise on equator in 15 *ghatis* (or 6 sidereal hours); but each sign will not rise in 5 *ghalis* (2 sidereal hours) and the *udayāntara* correction goes on increasing till the middle point of the first quadrant, and then continues to decrease. Hence at the end of each quadrant it vanishes (i.e., four times a year) and attains the greatest value at the middle point of each quadrant. The above is a rough method and meant for a beginner. The method by which the *udayāntara* correction is more accurately obtained is as follows: to the mean longitude of the sun add the total amount of the precession of the equinoxes, now find the *jyā* of the longitude ($R \times$ sine longitude) and the *kotijyā* of the corresponding declination ($R \times$ cosine declination); multiply this *jyā* by the *kotijyā* of the declination of the last point of the third sign and divide by the *kotijyā* of the declination. Find the arc in *asus* of which this is the *jyā;* and diminish, by these *asus*, the number of minutes in the mean longitude of the sun as increased by the total amount of precession; the result is the correct number of asus of *udayāntara*. In this way at the middle of the quadrant the *udayāntara* becomes a little greater than 26 *palas*" (i.e., 10 minutes 24 seconds).

Now if this *udayāntara* correction is so necessary, being that part of the equation of time due to the obliquity of the ecliptic, why was it ignored by the former astronomers? Bhāskara replies by saying that, "after all, this correction is a variable and a small quantity and vanishes at the four ends of the four quarter years." Kamalākarabhatta in his Siddhāntatattvaviveka tried to ignore this *udayāntara* correction, but his arguments are not very convincing.[1] Bhāskara is the only Indian astronomer who found out the two parts of the equation of time.

[1] See page 251 of *Bhāratīya Jyotih S'āstra* by S. B. Dikshit.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. STROPHOIDAL CURVES AND CUBICS

By R. M. MATHEWS, West Virginia University and OTTO DUNKEL, Washington University

1. Problem 2928 [1921, 466] states the following theorem: *A line drawn through the end, F, of the loop of a right strophoid or of a folium of Descartes cuts the curve again in Q and R such that the segment QR subtends a right angle at the node. This article will furnish answers to the questions raised in that problem, some references to the history of the curves possessing this property, and the development of certain other properties.*

This property of the right strophoid can be traced as far back as Barrow's *Lectiones Geometricae* (1669), p. 69; and for the oblique strophoid to Quetelet and Dandelin.[1] For these curves the locus of the centers of the circles on QR as diameter is a straight line.

Barbarin[2] has generalized the notion by defining strophoidal curves as follows: we are given two fixed points O and F and a curve Γ; for each point K on Γ we take Q and R on KF such that KQ and KR are each equal to KO; the locus of Q and R is a strophoidal curve of Γ.

2. The Equation of Strophoidal Curves. If F be the pole, FO the polar axis with $FO=a$, and Γ be defined by $\rho_1=f(\theta)$, then the equation of the locus is

$$\rho^2 - 2\rho f(\theta) - a^2 + 2a\cos\theta f(\theta) = 0.$$

Barbarin found some general properties of such curves and later De Longchamps[3] discussed the construction of their tangents.

For cartesian coördinates take O as origin with F on the x axis at $(a, 0)$. Let the locus of K be $f(\alpha, \beta)=0$. Then the equations of the line KF and of the required circle are respectively

$$\alpha y + \beta(a - x) - ay = 0,$$
$$2\alpha x + 2\beta y - (x^2 + y^2) = 0.$$

From these we find

$$\alpha:\beta:1 = [(x - a)(x^2 + y^2) + 2ay^2]:y(x^2 + y^2 - 2ax):2(x^2 + y^2 - ax),$$

[1] *Mémoire sur quelques propriétés de la focale parabolique*, Nouveaux Mémoires de l'Académie Royale des Science et Belles-lettres, Bruxelles vol. 2, (1882), pp. 169–202.

[2] *Courbes strophoïdales*, Revue de Mathématiques Spéciales, vol. 2, (1892–94), pp. 298–299.

[3] *Sur les strophoïdales*, Mathesis (2), vol. 4, (1894), pp. 138–141.

and on inserting these values of α and β in the equation $f(\alpha, \beta) = 0$ of Γ, we have the equation of the locus desired.

When Γ is a straight line the locus is a circular cubic; that is, it passes through the circular points at infinity. It is obvious from the equations that it passes through O and F. Moreover, it follows that F is a focus of the cubic. The foci of a curve are defined as the intersections of the tangents to the curve which pass through the circular points at infinity. If I and J denote the circular points at infinity, then on the line FI we have R at I. The perpendicular to OI is OI itself, and hence Q coincides with R. Hence FI and FJ are tangents to the cubic at I and J, and we may for this reason call F a singular focus. Thus strophoidal circular cubics must contain their singular foci. These curves have been studied synthetically by Lagrange.[1]

If the line Γ passes through a point common to the circles of the pencil, the curve is an oblique, or general strophoid. The singular focus F is at the end of the loop only for the right strophoid or Booth's logocyclica. Take Γ as the y-axis, $\alpha = 0$, and a as negative, and the usual equation for this curve results.

If the locus of centers Γ be other than a straight line, the degree of the strophoidal curve will be higher than three, in general. It may be three for special cases as is indicated by the fact that the property holds for the folium which is not a circular cubic. We shall now find the form of the equation for the general cubic.

3. Strophoidal Cubics. Let a fixed straight line cut a cubic in the points O, S, F, and let any other line through F cut the curve in the variable points Q and R We shall determine the form of the equation so that $\angle QOR$ shall be a right angle.

With axes as before, S at $(b, 0)$, finite, and F at $(a, 0)$, the equation of the cubic may be written

$$Ax(x - a)(x - b) + Bxy^2 + Cyx^2 + Dy^3 + A'xy + B'y^2 + C'y = 0,$$

where neither A nor a is zero. Writing the equation of FQR in the form $\lambda y = x - a$, we eliminate a factor $(x - a)$ between the two equations and obtain

$$\lambda Ax(x - b) + Bxy + Cx^2 + Dy^2 + A'x + B'y + C' = 0.$$

On making this quadratic equation homogeneous by use of the factor $1 = (x - \lambda y)/a$, we have as the equations of the lines OQ and OR

$$(a^2D - \lambda aB' + \lambda^2C')y^2 + (aA' - \lambda abA + \lambda a^2A + a^2C + C')x^2 + (\lambda^2 abA + a^2B - aA'\lambda + aB' - 2C'\lambda)xy = 0.$$

Since the product of the slopes of these two lines is to be -1, the sum of the coefficients of x^2 and y^2 must be zero; thus

[1] *Sur les cubiques strophoïdales*, Nouvelles Annales de Mathématiques, (3), vol. 1 (1900), pp. 66–74.

In the second case we find

$$(4)\quad A(\alpha - a)^2 + [2C\alpha - aC - aD](\alpha - a)\beta + (2B\alpha + A - aB)\beta^2 + 2D\beta^3 = 0.$$

These equations show that a cubic arises as a strophoidal curve only from a directrix Γ curve which is a cubic with a singular point at the proposed focus F. Indeed these equations include the circular cubics, for with $B=A$ and $C=D$ we get for Γ

$$[A(b - 2\alpha) - 2C\beta][(\alpha - a)^2 + \beta^2] = 0,$$

or a straight line and the null circle at F.

5. In closing we note certain conditions for the degeneration of the strophoidal curve. If n be the order of the curve Γ of centers K, the order of the strophoidal curve S will be $3n$ in general. For a straight line from F will cut Γ in n points K each of which will give two points Q and R of S on this straight line. The multiplicity of F will be n since the perpendicular bisector of FO will cut Γ in n points K such that $KO=KF$. Hence a straight line through F different from these KF lines cuts S in $2n+n=3n$ points.

Suppose now that a pair of these points on the perpendicular bisector happens to be conjugate points K', K'' on the minimal lines through F. Then the circle with K' as center passing through O is satisfied by every point on FK'. The same reasoning applies to FK''. Now the two lines FK' and FK'' are defined by the equation of the null circle at F. Hence a factor must separate from the equation for S which set equal to zero gives this null circle. If the multiplicity of K' and K'' is i, then the separation of the corresponding factors reduces the degree of S by $2i$.

Again, when Γ passes through F, then for the circle of center F and radius FO, every pair of diametrically opposite points Q and R are collinear with F and subtend a right angle at O. Therefore, if F be of multiplicity r on Γ, a factor must separate which equated to zero gives r of these circles, and the degree of S is reduced by $2r$. Thus by the separation of these two kinds of factors, the degree of S is reduced to $3n-2(i+r)$.

It is easily seen from the equations (3) and (4) that $i=1$, and, as we have already observed, $r=2$. Hence the degree of S in this case is $9-2(3)=3$. Moreover, these conditions imposed upon a cubic equation enable us to arrive at the equations (3) and (4).

II. Logarithms of Large Numbers[1]

By C. C. Camp, University of Nebraska

1. Briggs' tables to fourteen places are extremely scarce and the Tables du Cadastre and those of Sang are still unpublished. The scarcity of Vlacq's and

[1] Read before the Illinois Section of the Mathematical Association of America, May 13, 1927.

In the second case we find

$$(4)\quad A(\alpha - a)^2 + [2C\alpha - aC - aD](\alpha - a)\beta + (2B\alpha + A - aB)\beta^2 + 2D\beta^3 = 0.$$

These equations show that a cubic arises as a strophoidal curve only from a directrix Γ curve which is a cubic with a singular point at the proposed focus F. Indeed these equations include the circular cubics, for with $B=A$ and $C=D$ we get for Γ

$$[A(b - 2\alpha) - 2C\beta][(\alpha - a)^2 + \beta^2] = 0,$$

or a straight line and the null circle at F.

5. In closing we note certain conditions for the degeneration of the strophoidal curve. If n be the order of the curve Γ of centers K, the order of the strophoidal curve S will be $3n$ in general. For a straight line from F will cut Γ in n points K each of which will give two points Q and R of S on this straight line. The multiplicity of F will be n since the perpendicular bisector of FO will cut Γ in n points K such that $KO=KF$. Hence a straight line through F different from these KF lines cuts S in $2n+n=3n$ points.

Suppose now that a pair of these points on the perpendicular bisector happens to be conjugate points K', K'' on the minimal lines through F. Then the circle with K' as center passing through O is satisfied by every point on FK'. The same reasoning applies to FK''. Now the two lines FK' and FK'' are defined by the equation of the null circle at F. Hence a factor must separate from the equation for S which set equal to zero gives this null circle. If the multiplicity of K' and K'' is i, then the separation of the corresponding factors reduces the degree of S by $2i$.

Again, when Γ passes through F, then for the circle of center F and radius FO, every pair of diametrically opposite points Q and R are collinear with F and subtend a right angle at O. Therefore, if F be of multiplicity r on Γ, a factor must separate which equated to zero gives r of these circles, and the degree of S is reduced by $2r$. Thus by the separation of these two kinds of factors, the degree of S is reduced to $3n-2(i+r)$.

It is easily seen from the equations (3) and (4) that $i=1$, and, as we have already observed, $r=2$. Hence the degree of S in this case is $9-2(3)=3$. Moreover, these conditions imposed upon a cubic equation enable us to arrive at the equations (3) and (4).

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Vega's tables has been compensated for by the new ten place tables of Peters, but the demand for larger tables still exists. So great is this demand that the French and Germans are preparing tables to 14 and 15 decimals, respectively, while the British, not to be outdone, are preparing a standard table to 20 figures.[1] When a table is available there remain two possible difficulties. One is the obvious inconvenience due to the size of the book and the other is the excessive amount of numerical work still necessary in computing the logarithm of a number with ten or more significant digits. It is with a view to overcoming these that the author became interested in comparing the various methods which have been employed in finding the logarithms of large numbers.

2. A simple method used by Briggs involves a very small so-called brief table which may be illustrated as follows. Separate π into the factors[2]

$$3,\ 1.04,\ 1.0^{2}6,\ 1.0^{3}9,\ 1.0^{4}1 \text{ and } 1.0^{5}52171,$$

where $1.0^{s}m$ implies s zeros between the decimal point and the number m. If one desires a ten place logarithm, the logarithms of the factors may be taken from the table to twelve decimals. For the last factor, if the natural logarithm is sought one simply takes the decimal part. For the common logarithm one multiplies the decimal part of this factor by the modulus M using a multiplication table, usually given in an auxiliary table. Such a brief table is found in Schroen's *Logarithmen* on the last page. It extends to 16 decimals. A similar table is given by Hedrick in the *Macmillan Tables*, to 15 places.[3]

3. Fédor Thoman modifies this process so as to obtain a logarithm correct to 26 figures "by an easy and simple process without division, without interpolation and without a formula"! He points out that it is no fault of the ordinary seven place table that the result of dividing 321473×819255 by 452604×595118 gives 0.9777831 instead of 0.9777822. In such calculations it is safer to use an extra digit or two in the logarithms in order to be sure of the answer. His method will be illustrated by a 15-place logarithm altho his tables run to 27 decimals. To get $\log \pi$, he uses the multipliers[4]

[1] *Logarithmetica Britannica*, by A. J. Thompson, Parts IX and VIII have now appeared, containing numbers from 80,000 to 100,000 and some auxilliary tables.

[2] To separate a number into such factors, divide the number by the first significant digit unless it is unity, in which case divide by the number represented by the first two significant digits regardless of the number of zeros between; continue this process, separating successive quotients until there are half as many zeros after unity (indicated by s) as the number of figures in the mantissa of the desired logarithm. The divisions may be done by a machine, a multiplication table, or by simple short division.

[3] See a table by D. H. Lehmer, this Monthly, vol. 32 (1925), where such factors are used to compute cube and fifth roots.

[4] The purpose of multiplying by 32 is to obtain a number beginning with 10. The table gives reciprocals of all two-digit numbers, also their logarithms. The other factors are used to create more zeros and so to increase s. Logarithms of reciprocals of such factors are included to 27 decimals.

$$.32,\ 1-.0^{2}5,\ 1-0^{3}2,\ 1-.0^{4}8,\ 1-.0^{5}3,\ 1-.0^{7}3,$$

obtaining $1.0^{8}76,30152,82$. The logarithm of the last number is found from this table, two digits of the decimal part being taken at a time. He then adds the logarithms of the reciprocals of the multipliers taken out to 17 decimals. The whole calculation appears on one page with the result, 0.49714, 98726, 94134.

4. An alternate method is due to Flower who used multipliers of the form $1.0^{\bullet}m$. For example to find log π, divide by 4 and multiply successively by

$$1.2,\ 1.05,\ 1.01,\ 1.0^{3}5,\ 1.0^{5}2,\ 1.0^{6}5,\ 1.0^{7}4,\ 1.0^{8}8,\ 1.0^{9}4,\ 1.0^{10}1 \text{ and } 1.0^{11}4.$$

The multipliers[1] are so chosen that the successive products approach unity from below. The final product in this case is $1-x$, where x is $.0^{12}4105140237$. If S denotes the sum of the logarithms of the eleven multipliers, then $\log \pi = \log 4 + \log (1-x) - S$.

5. The method of Briggs has been extended by Gray to 24 decimal places and to factors which contain three significant figures besides unity. His book is scarce.

6. Steinhauser carries the same method to factors containing blocks of four digits. To illustrate,

$$\pi = 3.141 \times 1.0^{3}1886 \times 1.0^{7}8307 \times 1.0^{11}5638,\ 2819,\ 6-.$$

The last number may be separated into the factors $1.0^{11}5638$, $1.0^{15}2819$ and $1.0^{19}6-$, which are sufficiently accurate for a twenty place logarithm. The first three factors are found in tables A, B, C, respectively, the others in table D, which is really a multiplication table for the modulus. Peters' four by two multiplication table or a calculating machine would expedite the above divisions. An extension of Flower's method would be a distinct advantage here. To determine the first multiplier one may use the reciprocal of the first four digits from Barlow's table, taken to four digits and diminished if necessary.

One may guard against errors by factoring the given number in two independent ways. Some such check is valuable no matter what tables are used.

If one desires a logarithm to seventeen decimals, two divisions or multiplications are sufficient. In such a case the first three factors are found in tables A, B, C and the fourth in D.

7. The method of factoring is combined with simple interpolation for the second factor in Scott's table. Incidentally an error was found in log 1.00038

[1] The rule is to divide by the first digit plus one, then to multiply by a factor beginning with unity and having another significant digit obtained by subtracting the multiplicand, digit by digit, from 9 until a remainder is not zero. One might have used 1.06 instead of the two factors 1.05 and 1.01. Then the next multipliers would have been $1.0^{3}9$, $1.0^{4}7$, etc. If natural logarithms are sought this method requires the briefest table.

in his illustrative problem: to find log 8.463,298,746. The factors are 8.46 and 1.0003899227. The result given is 0.9275396714. Andoyer in his thirteen-place table carries the same method to second differences. Crelle's multiplication table is used and the second factor is taken to 14 places. The first and second differences in this case are 43412734 and 433, respectively.

8. One may use a corrected first difference instead of a second difference with Peter's table but by either method it may be surprising that it requires 4 or 5 minutes to take out the logarithm of a ten-digit number. The last decimal is always in doubt. A seven place logarithm table or a calculating machine is a valuable aid in the interpolation.

9. Some of the methods mentioned above may be applied in Thompson's table. Eventually it will have the advantage for five-figure numbers when the logarithm is required to more than ten places and up to 20. Extra work is now necessary to bring numbers within the range of the table. If one divides by the first five digits it is possible to interpolate in an auxiliary table for the second factor. First differences are sufficient for a 14-decimal logarithm, while second differences are needed to obtain a logarithm to 20 places. A preferable method for the latter is first to calculate by interpolation, using central differences, the logarithm of the first eight digits, then to divide the so called transformed number by the latter. Either method requires extensive interpolations with Everett's "coefficients" and withal a tremendous amount of arithmetic.

10. The most complicated method found in any brief table was that of Börgen, involving antilogarithms to 10 or 11 places corresponding to four-decimal logarithms. His table is based on something analogous to addition logarithms. Let us illustrate by taking the number $A = 8463298746$, previously treated. One finds in the first table the antilogarithm $B = 8462525688.1$ such that $A > B$. Here $\log B = .9275$, (one disregards characteristics). Find a second number $B_1 < A - B = 773{,}057.9$ such that its logarithm has the same fourth digit 5, namely $B_1 = 771{,}791.5$. Log $B_1 = .8875$. Perform the process once more finding $B_2 = 1{,}266.2$ such that $\log B_2 = .1025$. The residue $A - (B + B_1)$ is 1,266.4. Hence there is a remainder of .2. If we consider

$$A = B + B_1 + B_2 + B_3 = B(1 + B_1B^{-1}) \cdot (1 + B_2B^{-1}) \cdot (1 + B_3B^{-1}) \cdot (1 - B_1B_2B^{-2}),$$

approximately, then we have A factored. Write $L_1 = \log B - \log B_1 = .040$ and $\Delta L_1 = \log (1 + B_1B^{-1})$, and similarly for L_2, L_3. Then $\log A = \log B + \Delta L_1 + \Delta L_2 + \Delta L_3 - \Delta(L_1 + L_2)$, approximately. The last two items are found by the approximation formulas $B_3 \Delta L_B$, and $-\Delta L_1 \Delta L_2 / M$ with corrected decimal point. There are two cases for determining the position of the decimal point in the items taken from table B: Case I, $\log B < .637$; Case II, $\log B > .637$; and the items are taken as follows:

$\log B =$	.927 5000 000		from Table A
$\Delta L_1 =$	396 063.2		from Table B, column 1,
$\Delta L_2 =$	649.8		from Table B, column 2,
$B_3 \Delta L_B =$	0.1	$= B_3(.514)$	from Table B, column 2,
$-\Delta(L_1+L_2) =$	−0.1		
	.927 5396 713.2		

The result is quite accurate, but the method seems very awkward and difficult to master.

A comprehensive bibliography of logarithmic tables has recently appeared as No. XIII of *Tracts for Computers*, published by the Cambridge University Press. By a comparison of all the methods one is led to believe that the simplest method for computing a ten place logarithm is that of Steinhauser[1] in connection with Schroen's *Tafel* I. A more extensive logarithm can easily be calculated with a slight increase in the arithmetic. Below is the total work necessary to find log 8.463298746 when one writes down even the work of interpolation in the seven place table:

8463,298746	1.000,0353,003
.4752934	
87	.9275 2434 0875
.4753021	
.9275243	1533 0325
9.5477778	
47	0130
31	.9275 3967 13

III. On Definite Algebraic Quadratic Forms

By Leonard M. Blumenthal, Johns Hopkins University

1. Introduction. A fundamental theorem in the theory of algebraic quadratic forms is that in which necessary and sufficient conditions that the form be definite are obtained.[2] It is the purpose of this paper to obtain conditions for the case of properly definite, non-singular forms by a method involving only elementary considerations. The usual proofs (which this paper avoids), though simplified somewhat for the special case treated here, are believed to be not so

[1] Steinhauser, Logarithmen-Hilfstafeln, Wien, Druck und Verlag von Carl Gerold's Sohn, is still listed for sale by G. E. Stechert and Co., New York, N. Y.

[2] See L. E. Dickson's *Modern Algebraic Theories*, Chapter 4, and Maxime Bôcher's *Introduction to Higher Algebra*, p. 150; see also Ex. 4, 5, p. 147.

It may be observed that since the order of the variables is, of course, immaterial, a necessary and sufficient condition that a form be properly definite is that (1) every principal minor determinant be positive for positively definite forms, (2) that every principal minor determinant be plus or minus according as its order is even or odd, for negatively definite forms.

IV. A Reading Course in Mathematics

By Tomlinson Fort, Lehigh University

The Mathematics Department at Lehigh University is making an experiment[1] which we regard with much interest. We have established a course which is described in the catalogue as follows:

"Math. 15. Reading Course in Mathematics. Credit not to exceed one hour per semester, total credit not to exceed three hours, approval of program and written report required."

Permission to give such a course was obtained from the faculty in 1927 and it was advertised on the Mathematics bulletin board. At the beginning of the second semester of 1927–28 it was elected by one senior, three sophomores, and eight freshmen. The distribution is interesting. Even this amount of interest in a course that was not yet in the catalogue was gratifying, particularly as it was in each instance in addition to an otherwise full program, and was at Lehigh where the majority of the students are studying engineering. No student who had made less than a "B" in mathematics the previous semester was eligible. The senior was assigned to Professor Lamson, the freshmen to Professor Smail, and the sophomores to me. We undertake to advise each student separately as to mathematical reading suitable in amount and difficulty to his advancement and ability. The student undertakes to report to his adviser once in two weeks, more frequently if he desires, and to submit a brief written report each time he completes an assignment. These assignments are made after the freest possible discussion with him. The work is going very well. The enthusiasm of the students varies, but on the whole there is more enthusiasm than we expected.

Each adviser is provided with a list of books which are in our library and which seem, in part at least, readable by an undergraduate and likely to interest him. He has this before him when talking the matter over with the student. And inasmuch as teachers at other institutions may be interested in establishing such a course themselves, this list is given here. Of course, it is not comprehensive. We shall add to it and probably strike off some that are on it. It is interesting to know, however, that in a library of moderate size so much could

[1] This experiment at Lehigh University, which involves collateral reading on the part of the student, was begun and the list of readings was prepared before the "Committee on Collateral Reading" of the Mathematical Association of America published its report in the May, 1928 issue of this Monthly.

Vol. 32); *Mathematical Preparation for Physical Chemistry*, by F. Daniels; *Differential Equations in Applied Chemistry*, by F. L. Hitchcock and C. S. Robinson; *The Mathematics of Biology*, by O. Richards (This Monthly, Vol. 32); *Vector Analysis*, by J. B. Coffin; *Vector Analysis*, by Willard Gibbs, E. B. Wilson; *Elementary Vector Analysis*, by C. E. Weatherburn; *Introduction to Quaternions*, by P. Kelland and P. G. Tait; *Elements of Quaternions*, by G. H. Hardy; *Empirical Formulas*, by T. R. Running; *Synthetic Projective Geometry*, by D. N. Lehmer; *Mathematical Theory of Limits*, by J. B. Leatham; *Introductory Mathematical Analysis*, by T. Walmsley; *Life Assurance Primer*, by H. Moir; *Curve Tracing*, by W. W. Johnson; *Theory of Numbers*, by R. D. Carmichael; *Graphical Methods*, by T. R. Running; *HigherMathematics*, by Mansfield Merriman and R. S. Woodward.

V. Two Mutual Relations Between Pairs of Tetrahedra

By Paul Wernicke, Washington, D. C.

There may exist between two tetrahedra a certain reciprocal relation, which we call *orthogrammacy* since the shorter words orthography and orthographic have been preëmpted to designate a type of projection.

The four vectors a, b, c, d, representing by their lengths the areas of the sides of a tetrahedron $ABCD$, to which they are outward normals, will, if placed end to end, form a closed figure, a tetragram in space. For, taking D as origin and denoting the vectors DA, DB, DC by α, β, γ, the sides will be represented by the vector products (in Gibb's notation):

$$2a = \beta \times \gamma, \quad 2b = \gamma \times \alpha, \quad 2c = \alpha \times \beta,$$
$$2d = (\gamma - \alpha) \times (\beta - \alpha) = -(\beta \times \gamma + \gamma \times \alpha + \alpha \times \beta),$$

so that $a+b+c+d=0$. Their tetragram $abcd$ defines by its vertices and by the planes laid through its four pairs of consecutive sides, a second tetrahedron the sides of which are represented by the vectors: $4a \times b = \gamma v$, $-4b \times c = \alpha v$, $4c \times d = (\alpha - \beta)v$, $-4d \times a = (\beta - \gamma)v$, where v is the (pseudo-) scalar product $[\alpha\beta\gamma]$ which is the six-fold volume of tetrahedron $ABCD$. The alternation in sign is necessary to cause all these vectors to point outward.

Apart from the irrelevant similarity factor v, which may be made unity, the relation between the tetrahedra is thus mutual. The six-fold volume of the second tetrahedron is v^2.

Further, there is this to be observed: The three arrangements $acbd$, $abcd$, $abdc$, distinct by their pairs of opposite sides, are neither cyclically permutable into one another nor cyclic permutations reversed in order. They furnish, in general, three distinct tetrahedra orthogrammatic to the given one and corresponding to the three tetragrams formed with the latter's edges.

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Further, there is this to be observed: The three arrangements *acbd*, *abcd*, *abdc*, distinct by their pairs of opposite sides, are neither cyclically permutable into one another nor cyclic permutations reversed in order. They furnish, in general, three distinct tetrahedra orthogrammatic to the given one and corresponding to the three tetragrams formed with the latter's edges.

The sides of $ABCD$ are ordered into a reversible cycle by the selection of such a tetragram, and since the latter defines its tetrahedron it seems more accurate to speak of orthogrammacy between *tetragrams*, in which case the relation is strictly one-to-one reciprocal.

A second mutual relation between certain pairs of tetrahedra is *orthology*. Steiner[1] introduced the term "orthologiques" for a pair of triangles ABC and $A'B'C'$ so situated that the perpendiculars from A, B, C, on the sides $B'C'$, $C'A'$, $A'B'$ concur (in a point P). It is then proved, commonly by the aid of trilinear coördinates, that this relation is mutual, i.e. that the perpendiculars from A', B', C' to BC, CA, AB are likewise concurrent (in a point O).

It is found that a demonstration of this theorem by vector analysis is directly applicable to tetrahedra in S_3 and further to simplicia [i.e. $(n+1)$-points] in n-dimensional space S_n. We here indicate it for two tetrahedra $ABCD$ and $A'B'C'D'$.

Let the perpendiculars from A', B', C', D' to the sides $a=BCD$, $b=CAD$, $c=ABD$, $d=ABC$ concur in O, which we take as origin. Then, denoting the vectors from O to A, B, C, D, by α, β, γ, $\delta=d_0\alpha+d_1\beta+d_2\gamma$ and those to A', B', C', D', by α', β', γ', δ', we have:

$$\alpha' = a(\delta\times\gamma+\gamma\times\beta+\beta\times\delta) = [(d_1+d_2-1)\beta\times\gamma - d_0\gamma\times\alpha - d_0\alpha\times\beta]a,$$

$$\beta' = b(\delta\times\alpha+\alpha\times\gamma+\gamma\times\delta) = [-d_1\beta\times\gamma + (d_2+d_0-1)\gamma\times\alpha - d_1\alpha\times\beta]b,$$

$$\gamma' = c(\delta\times\beta+\beta\times\alpha+\alpha\times\delta) = [-d_2\beta\times\gamma - d_2\gamma\times\alpha + (d_0+d_1-1)\alpha\times\beta]c,$$

$$\delta' = d(\alpha\times\beta+\beta\times\gamma+\gamma\times\alpha) = [\beta\times\gamma+\gamma\times\alpha+\alpha\times\beta]d.$$

The normals from the vertices A, B, C, D to the sides of A', B', C', D', are given by

$$\alpha + x'(\delta'\times\gamma'+\gamma'\times\beta'+\beta'\times\delta') = \alpha + x[-(cd+bc+bd)\alpha + cd\beta + bd\gamma + bc\delta],$$

$$\beta + y'(\delta'\times\alpha'+\alpha'\times\gamma'+\gamma'\times\delta') = \beta + y[cd\alpha - (da+ac+cd)\beta + da\gamma + ca\delta],$$

$$\gamma + z'(\delta'\times\beta'+\beta'\times\alpha'+\alpha'\times\delta') = \gamma + z[bd\alpha + da\beta - (bd+ab+da)\gamma + ab\delta],$$

$$\delta + t'(\beta'\times\gamma'+\gamma'\times\alpha'+\alpha'\times\beta') = \delta + t[bc\alpha + ca\beta + ab\gamma - (bc+ca+ab)\delta],$$

where x', y', z', t' by their (real) values determine the points on each normal, and $x=x'(1-d_0-d_4-d_2)\,v$, $y=y'(1-d_0-d_4-d_2)\,v$ etc., v again being $[\alpha\beta\gamma]$, the six-fold volume of $ABCD$.

It is now clear that the point

$$(bcd\alpha + cad\beta + abd\gamma + abc\delta)/(bcd + cad + abd + abc)$$

[1] Gergonne's Annales de Mathématiques Pures et Appliquées, vol. 19.

lies on all four normals and corresponds to the particular values

$$x = a/n, \quad y = b/n, \quad z = c/n, \quad t = d/n, \quad \text{where} \quad n = bcd + cad + abd + abc.$$

This proves the theorem that *if the perpendiculars from the vertices of a first tetrahedon to the sides of a second concur in a point, those from the vertices of the second to the sides of the first will likewise concur.*

Using Steiner's expression, two such tetrahedra are *orthologic.*

VI. A Solution of the Quartic Equation

By Raymond Garver, University of California at Los Angeles

It is the purpose of this note to present a solution of the general quartic equation which follows as a direct corollary from two theorems on Tschirnhaus transformations. It does not, however, at all resemble the usual solution by the Tschirnhaus transformation, which requires, first, reduction to the trinomial form $y^4+c_2y^2+c_4=0$ and, secondly, expression of the roots of the original quartic in terms of those of the transformed equation. Nor does it seem to be similar to any of the known solutions.

The two theorems required are included in an article of mine which was recently published in the Messenger of Mathematics.[1] I shall re-state them here, together with an outline of the methods used in their proof. I gave an earlier proof of the first theorem in a paper entitled: *A new normal form for quartic equations,*[2] but it is not well suited to the present purpose.

Theorem I. *Any quartic equation in the form*

$$x^4 + a_2x^2 + a_3x + a_4 = 0 \tag{1}$$

can be transformed, by a quadratic Tschirnhaus transformation

$$y = x^2 + k_1x + k_2, \tag{2}$$

into a quartic whose second term is missing, and which has a double root.

The required transformation has $k_2=a_2/2$, while k_1 may be either square root of any one of the roots of the cubic

$$k^3 + 2a_2k^2 + (a_2^2 - 4a_4)k - a_3^2 = 0. \tag{3}$$

This result is obtained easily. Call the roots of (1) x_1, x_2, x_3, x_4 and those of the transformed equation y_1, y_2, y_3, y_4. Then if we wish, say, y_1 to equal y_2, we may use (2) to obtain a necessary and sufficient condition on k_1. We find $k_1=-(x_1+x_2)=(x_3+x_4)$. Similarly, if $k_1=(x_i+x_j)$, $i, j=1, 2, 3, 4, i\neq j$, the

[1] Vol. 57 (Nov. 1927), pp. 99–101.

[2] Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 310–314.

Now if the transformation $z=P(x)$ leads to a transformed equation with a triple root, so will the transformation $z=aP(x)$, where a is any constant. That is,

$$z = x^3 + (\tfrac{1}{2}k_1 + \tfrac{1}{2}c_1k_1^{-1})x^2 + \cdots \tag{8}$$

leads to an equation with a triple root; and by Theorem II the coefficient of x^2 is the root x_4 of (1).

By simplifying $\frac{1}{2}c_1k_1^{-1}$, we may write $x_4=(k_1+k_1'+k_1'')/2$, where $k_1=x_k+x_3$, $k_1'=x_4+x_2$, $k_1''=x_4+x_1$, and where these three expressions are square roots of roots of (3). Similarly we see that we have a root x_i provided $k_1=x_i+x_j$; $k_1'=x_i+x_k$; $k_1''=x_i+x_l$; where i, j, k, l are the numbers 1, 2, 3, 4 in some order. To obtain a root, we choose for k_1 either square root of any root of (3); this is of the form x_i+x_j. Choose for k_1' either square root of either of the other roots of (3); this has one of the x's in common with k_1, and is thus of the form x_i+x_k. Now k_1'' is determined uniquely by the condition $(x_i+x_j)(x_i+x_k)(x_i+x_l)=-a_3$. This final process leads to four roots, and is of course essentially the same as that arrived at in most of the other solutions of the quartic.

In our work we assumed that (1) did not have a double root; but since the expression for x_i in the preceding paragraph is simply an identity, it is true for this case as well. In fact, we might now disregard our framework, and present a solution based simply on the preceding paragraph and a knowledge of the form of the roots of (3). This would, it seems, be as short a solution as could possibly be obtained.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

College Algebra, by N. R. WILSON and L. A. H. WARREN. Oxford, University Press. viii+451 pages. $2.08.

A text by two members of the faculty of the University of Manitoba.

Precis d'analyse mathematique, par E. LAINÉ, Paris, Librairie Vuibert. Vol. I, 232 pages, 30 fr. Vol. II, 336 pages, 40 fr.

"Preliminaires, fonctions de variables réelles, fonctions analytiques; équations differentielles, géométrie infinitséimale, équations aux dérivées partielles."

Elementary Calculus, by F. S. WOODS and F. H. BAILEY. Revised Edition. Boston, Ginn and Company. x+385 pages. $3.00.

A new edition of a standard and popular text.

* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

New Analytic Geometry, by P. F. SMITH, A. S. GALE, and J. H. NEELLEY. Revised Edition. Boston, Ginn and Company. $2.00.

A new edition, of a standard and popular text.

The Logic of Modern Physics, by P. W. BRIDGMAN. New York, The Macmillan Company. xiv+228 pages.

"This essay will be largely concerned with the fundamental concepts; it will appear that almost all the concepts can profit from re-examination."

Les espaces abstraits, par M. FRECHET. Paris, Gauthier-Villars. xi+296 pages.

A systematic exposition of the new concepts which constitute the general analysis.

Lecons sur les nombres transfinis, par WACLAW SIERPINSKI. Paris, Gauthier-Villars. vi+249 pages. 40 fr.

"The book offers the reader a clear and easy exposition of the theory of assemblages and to that of the theory of transfinite numbers. To read this book, one has no need of preliminary special studies. It ought to interest not only the professional mathematician, but also the philosopher."

A History of Mathematical Notations, by FLORIAN CAJORI. Volume I, Elementary Mathematics. Chicago, Open Court Publishing Company. xvi +451 pages. $6.00.

The first volume of a highly significant historical work.

Plane and Spherical Trigonometry, by J. SHIBLI. Boston, Ginn and Company xii+220+94 pages.

A thorough and complete text.

Algebra for Secondary Schools, by S. EMORY and E. E. JEFFS. New York, D. Van Nostrand Company. x+626 pages. $1.85.

This text is distinguished by the very large number of problems serially numbered, and totalling over 11000.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Dynamical Systems. By GEORGE D. BIRKHOFF. American Mathematical Society Colloquium Publications, Volume IX. New York, 1927. viii+295 pages.

According to the author a large part of the material presented in the book is from his colloquium lectures, delivered at the University of Chicago before the American Mathematical Society, September 5–8, 1920. The book is not a treatment of dynamical systems in general but rather it concerns itself with those fields of dynamics that have been of particular interest to the author in the last few years. Particular emphasis is given to the author's own researches

sition: Of all polygons with sides all given but one (and in a definite order), the maximum can be inscribed in a semicircle which has the undetermined side for its diameter.

Prove that there is one and only one maximum. Given the lengths of the sides and their order, compute the radius of the semicircle.

3356. Proposed by L. S. Johnston, Pennsylvania State College.

Consider a triangle ABC, with angles at A and B acute and the angle at A greater than the angle at B. Let A', B', and C' be the feet of the respective perpendiculars from A, B, and C to the opposite sides. Let the perpendiculars from A to the lines $A'B'$ and $A'C'$ meet those lines at R_1 and R_2, respectively; let the perpendiculars from B to the lines $B'C'$ and $B'A'$ meet those lines at S_1 and S_2, respectively; let the perpendiculars from C to the lines $C'A'$ and $C'B'$ meet those lines at T_1 and T_2, respectively. Show that the conic with A' and B' as foci and S_2 and R_1 as vertices is tangent to AB at C'; that the conic with B' and C' as foci and T_2 and S_1 as vertices is tangent to BC at A'; that the conic with C' and A' as foci and R_2 and T_1 as vertices is tangent to CA at B'.

3357. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through a given point to draw a line so that the segment intercepted on it by two given intersecting lines shall be divided in a given ratio by the foot of the perpendicular dropped upon it from the point of intersection of the two given lines.

3358. Proposed by J. V. Uspensky.

Let p be a prime such that $p \equiv 5 \pmod 6$ and $p = 4l+1$ or $p = 4l+3$. Show that in this case

$$f(l) = 1 + \frac{1}{4}\left(\frac{l}{1}\right)^2 + \frac{1}{4^2}\left(\frac{l(l-1)}{1\cdot 2}\right)^2 + \cdots \equiv 0 \pmod 6.$$

Show also that if $p = 4l+1 = 12n+1$,

$$f(l) \equiv (-1)^n \cdot C_{6n}^{3n} \cdot C_{6n}^{2n} / C_{6n}^{n} \pmod p;$$

and that if $p = 4l+3 = 12n+7$,

$$f(l) \equiv (-1)^{n+1} \cdot 2 \cdot C_{6n+3}^{2n+1} \cdot C_{6n+3}^{3n+1} / C_{6n+3}^{n} \pmod p.$$

3359. Proposed by Clifford N. Mills, Normal, Illinois.

A B, C, D are four points in a plane. Let the centroids of the triangles BCD, CDA, DBA, ABC be respectively a, b, c, d. Then Aa, Bb, Cc, Dd meet in a point, and are divided in the same ratio at this point. Also the quadrilateral $abcd$ is similar to $ABCD$.

3360. Proposed by Bernardo Ig. Baidaff, Buenos Aires.

Solve the system $x_i = a^i + bx_{i-1} + b^2 x_{i-2}$ $(i = 1, 2, \cdots, n)$, with $x_{-1} = x_0 = 0$. Generalize.

SOLUTIONS

446 [1914, 191]. Proposed by S. G. Barton, University of Pennsylvania.

Prove any one or more of the sixteen theorems, stated without proof, in the article in this issue of the MONTHLY (June, 1914, pages 182–184) on "Properties of the Normals to a Conic."

SOLUTION BY OTTO DUNKEL, Washington University

Theorem (1) gives the equations of the coincident and single normals to the parabola $y^2 = 4px$ from a point (a, b), the coördinates of the intersections of these normals with the curve, and some facts obvious by inspection of these results. These results will be derived.

If x_1, y_1 is a point on the parabola, we easily find the slope of the normal to be $m = -y_1/2p$, $x_1 = pm^2$, $y_1 = -2pm$. The equation of the normal is then $pm^3 + (2p-x)m + y = 0$. Suppose that two normals through (a, b) coincide and have the slope m_1, while the third (single) normal through the same point has the slope m_2. Then from the above equation we find $m_2 = -2m_1$, $m_1^2 + 2m_1 m_2 = (2p-a)/p$, $m_1^2 m_2 = -b/p$. From the first and third of these we get $m_1 = (b/2p)^{1/3}$; from the first and second, $3m_1^2 = (a-2p)/p$; and

finally $(b/2p)^2=[(a-2p)/3p]^3$. Hence $m_1=[(a-2p)/3p]^{1/2}$, $m_2=-2m_1$, where b is taken as positive. The point on the curve corresponding to (a, b) for the slope m_1 is $(pm_1^2, -2pm_1)$, and similarly for m_2. Hence the equation of the double normal and the coördinates of the corresponding point on the curve are

$$y-b=\left(\frac{a-2p}{3p}\right)^{1/2}(x-a), \quad \left[\frac{a-2p}{3}, \quad -2\left(\frac{(a-2p)p}{3}\right)^{1/2}\right],$$

while for the single normal they are

$$y-b=-2\left(\frac{a-2p}{3p}\right)^{1/2}(x-a), \quad \left[\frac{4}{3}(a-2p), \quad 4\left(\frac{(a-2p)p}{3}\right)^{1/2}\right].$$

478 [1915, 338]. Proposed by J. A. Campbell, St. Johnsbury, Vt.

Construct a triangle having given the difference of the segments of the base formed by the foot of the perpendicular from the vertical angle, the difference of the base angles, and the sum of the three sides.

523 [1917, 426]. Proposed by H. Campbell, St. Johnsbury, Vt.

Given the difference of the segments of the base made by the perpendicular let fall from the vertical angle, the difference of the base angles, and the sum of the sides of a triangle, to construct the triangle.

3324 [1928, 261]. Proposed by H. Campbell, St. Johnsbury, Vt.

Given the difference of the segments of the base made by the perpendicular to the base from the vertical angle, the difference of the base angles, and the difference of the sides including the vertical angle, to construct the triangle.

Solution by Nathan Altshiller-Court, University of Oklahoma

Let ABC be the required triangle and O its circumcenter. The parallel through A to BC meets the circumcircle (O) in A'. It is readily seen that the segment AA' is equal to the difference, d, of the segments into which the foot of the altitude from A divides BC; hence $AA'=d$ is known. Since $A'C=AB$, we have angle $ABA'=$angle $ABC-$angle $A'BC=$angle $ABC-$angle ACB. But the difference of the last two angles is given; hence the angle ABA' is known. Thus in the circle (O) we know the chord AA', and the angle ABA' inscribed, which this chord subtends; hence the circle (O) is known. Suppose then that this circle is drawn and that the chord AA' is placed in it.

Let $EF=2R$ be the diameter of (O) perpendicular to BC, the points A and E being assumed to lie on opposite sides of BC. This diameter is also perpendicular to AA'; hence the points E, F are known. The line AE is therefore known both in magnitude and position.

The point E is the center of a circle (E) passing through B, C and cutting AE in the incenter I and in the excenter I' relative to the side BC. The perpendicular $I'X$ from I' to AB determines on AB a segment AX equal to half the sum of the sides of the triangle ABC. The inscribed angles BAE, BFE being equal as subtending the same chord, the right triangles $AI'X$, BEF are similar; hence

$$AX/BF = AI'/EF. \tag{1}$$

If we denote by x the radius of the circle (E), by t the known segment AE, and by $2p$ the given perimeter of the triangle ABC, we have $AX=p$, $EF=2R$, $AI'=t+x$, $BF^2=4R^2-x^2$. The equation (1), when rationalized, becomes

$$(t+x)^2(4R^2-x^2)=4R^2p^2,$$

where p, R, and t are known segments. The radius x of (E) thus being determined the circle (E) is known. The common chord of the two circles (O), (E) is the base BC of the required triangle ABC.

Since the radius, x, of (E) depends upon an irreducible biquadratic equation, the proposed problem cannot be solved by ruler and compasses.

Note by Otto Dunkel, Washington University

Setting $\angle ABA'=\delta$, we have $t=2R\cos(\delta/2)$, $x=2R\sin(A/2)$, and the above equation becomes

$$(p/2R)=u=[\cos(\delta/2)+\sin(A/2)]\cos(A/2), \quad 0\leqq A\leqq 180^\circ-\delta.$$

finally $(b/2p)^2=[(a-2p)/3p]^3$. Hence $m_1=[(a-2p)/3p]^{1/2}$, $m_2=-2m_1$, where b is taken as positive. The point on the curve corresponding to (a, b) for the slope m_1 is $(pm_1^2, -2pm_1)$, and similarly for m_2. Hence the equation of the double normal and the coördinates of the corresponding point on the curve are

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The point E is the center of a circle (E) passing through B, C and cutting AE in the incenter I and in the excenter I' relative to the side BC. The perpendicular $I'X$ from I' to AB determines on AB a segment AX equal to half the sum of the sides of the triangle ABC. The inscribed angles BAE, BFE being equal as subtending the same chord, the right triangles $AI'X$, BEF are similar; hence

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where p, R, and t are known segments. The radius x of (E) thus being determined the circle (E) is known. The common chord of the two circles (O), (E) is the base BC of the required triangle ABC.

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$$(p/2R)=u=[\cos(\delta/2)+\sin(A/2)]\cos(A/2), \quad 0\leqq A\leqq 180^\circ-\delta.$$

The triangle AFE is the ambiguous case of triangles. If $p-q<(a+c)\sin\frac{1}{2}(C-A)$ it cannot be constructed. If $p-q$ is greater than this there will be two possibilities for AFE. Call the second triangle AFE'. Call the resulting solution $AB'C'$. Locate N, the mid-point of EE'. The perpendicular bisector of FN will be equidistant from the perpendicular bisectors of FE and FE' which determined B and B'. It will, therefore, bisect BB' in a point M. But M is the midpoint of AF and therefore AB' equals BF which was equal to BC. By considering right triangles AFN and ANE we can show that angle AEN is $\frac{1}{2}(A+C)$. Then CAC' is $A+C$ and $B'AC'$ is, therefore, equal to C. Then C' is equal to A. The two solutions are, therefore, congruent.

2695 [1918, 170]. Proposed by Frank Irwin, University of California.

A positive number, which for convenience we will write as a fraction, a/b, is developed into a continued fraction,

$$a_1 \frac{1}{-a_2} \frac{1}{-a_3} \cdots,$$

by the following process:

$$a = a_1 b - r_1, \qquad b = a_2 r_1 - r_2, \qquad r_1 = a_3 r_2 - r_3, \text{ etc.}$$

Here $a_1, a_2, \cdots$ are positive integers, and, if we suppose a, b to have been taken positive, $b>r_1>r_2>\cdots>0$. Show that if a/b be but slightly larger than a positive integer or zero, the numerators (as likewise the denominators) of the successive convergents will for a time be in arithmetical progression, and determine how long this phenomenon will continue.

Solution by Charles H. Smiley.

By the conditions of the problem, $a/b=a_1-1+\epsilon_1$, where a_1 is a positive integer equal to or greater than one, and ϵ_1 is a positive quantity less than unity. We may write

$$\begin{aligned} a/b &= a_1 - (1-\epsilon_1) = a_1 - 1/(1+\epsilon_2), \quad \epsilon_2 > \epsilon_1, \\ &= a_1 - \frac{1}{2-(1-\epsilon_2)} = a_1 - \frac{1}{2-\dfrac{1}{1+\epsilon_3}}, \quad \epsilon_3 > \epsilon_2, \\ &= a_1 - \frac{1}{2-\dfrac{1}{2-(1-\epsilon_3)}}, \end{aligned}$$

and so on until we reach $\epsilon_i \geqq 1$.

In this case $0<1-\epsilon_{i-1}\leqq 1/2$, $1>\epsilon_{i-1}\geqq 1/2$, and we find in turn

$$1 > \epsilon_{i-1} \geqq 1/2 > \epsilon_{i-2} \geqq 1/3 > \epsilon_{i-3} \geqq 1/4 \cdots 1/(i-1) > \epsilon_1 \geqq 1/i.$$

Conversely, if $1/(i-1)>\epsilon_1\geqq 1/i$, we find that $\epsilon_i\geqq 1$ and that

$$a_2 = a_3 = a_4 = \cdots = a_{i-1} = 2, \quad a_i > 2,$$

and for the successive convergents, we have

$$\begin{aligned} k &= 0 \quad 1 \quad\quad 2 \quad\quad \cdots (i-2), \\ P &= a_1 \quad (2a_1-1) \quad (3a_1-2)\cdots(i-1)a_1-(i-2), \\ Q &= 1 \quad 2 \quad\quad 3 \quad\quad \cdots (i-1). \end{aligned}$$

We find that under the conditions named above, the numerators of the first $(i-1)$ convergents are in arithmetical progression, common difference, (a_1-1). Also the denominators are the first $(i-1)$ integers and, therefore, are in arithmetical progression. In case $\epsilon_1=1/i$, we find that, in addition, $a_i=2$ and hence the numerators (as well as the denominators) of the first (i) convergents are in arithmetical progression.

The triangle AFE is the ambiguous case of triangles. If $p-q<(a+c)\sin\frac{1}{2}(C-A)$ it cannot be constructed. If $p-q$ is greater than this there will be two possibilities for AFE. Call the second triangle AFE'. Call the resulting solution $AB'C'$. Locate N, the mid-point of EE'. The perpendicular bisector of FN will be equidistant from the perpendicular bisectors of FE and FE' which determined B and B'. It will, therefore, bisect BB' in a point M. But M is the midpoint of AF and therefore AB' equals BF which was equal to BC. By considering right triangles AFN and ANE we can show that angle AEN is $\frac{1}{2}(A+C)$. Then CAC' is $A+C$ and $B'AC'$ is, therefore, equal to C. Then C' is equal to A. The two solutions are, therefore, congruent.

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by the following process:

$$a = a_1 b - r_1, \qquad b = a_2 r_1 - r_2, \qquad r_1 = a_3 r_2 - r_3, \text{ etc.}$$

Here $a_1, a_2, \cdots$ are positive integers, and, if we suppose a, b to have been taken positive, $b>r_1>r_2>\cdots>0$. Show that if a/b be but slightly larger than a positive integer or zero, the numerators (as likewise the denominators) of the successive convergents will for a time be in arithmetical progression, and determine how long this phenomenon will continue.

Solution by Charles H. Smiley.

By the conditions of the problem, $a/b=a_1-1+\epsilon_1$, where a_1 is a positive integer equal to or greater than one, and ϵ_1 is a positive quantity less than unity. We may write

$$\begin{aligned} a/b &= a_1 - (1-\epsilon_1) = a_1 - 1/(1+\epsilon_2), \quad \epsilon_2 > \epsilon_1, \\ &= a_1 - \frac{1}{2-(1-\epsilon_2)} = a_1 - \frac{1}{2-\dfrac{1}{1+\epsilon_3}}, \quad \epsilon_3 > \epsilon_2, \\ &= a_1 - \frac{1}{2-\dfrac{1}{2-(1-\epsilon_3)}}, \end{aligned}$$

and so on until we reach $\epsilon_i \geqq 1$.

In this case $0<1-\epsilon_{i-1}\leqq 1/2$, $1>\epsilon_{i-1}\geqq 1/2$, and we find in turn

$$1 > \epsilon_{i-1} \geqq 1/2 > \epsilon_{i-2} \geqq 1/3 > \epsilon_{i-3} \geqq 1/4 \cdots 1/(i-1) > \epsilon_1 \geqq 1/i.$$

Conversely, if $1/(i-1)>\epsilon_1\geqq 1/i$, we find that $\epsilon_i\geqq 1$ and that

$$a_2 = a_3 = a_4 = \cdots = a_{i-1} = 2, \quad a_i > 2,$$

and for the successive convergents, we have

$$\begin{aligned} k &= 0 \quad 1 \quad\quad 2 \quad\quad \cdots (i-2), \\ P &= a_1 \quad (2a_1-1) \quad (3a_1-2)\cdots(i-1)a_1-(i-2), \\ Q &= 1 \quad 2 \quad\quad 3 \quad\quad \cdots (i-1). \end{aligned}$$

We find that under the conditions named above, the numerators of the first $(i-1)$ convergents are in arithmetical progression, common difference, (a_1-1). Also the denominators are the first $(i-1)$ integers and, therefore, are in arithmetical progression. In case $\epsilon_1=1/i$, we find that, in addition, $a_i=2$ and hence the numerators (as well as the denominators) of the first (i) convergents are in arithmetical progression.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor G. D. BIRKHOFF of Harvard University delivered a lecture on "The Mathematical Basis of Art" at the Ohio State University on December 3, 1928.

Professor E. R. HEDRICK, of the University of California at Los Angeles, filled the following schedule of lectures during November and December, 1928: University of Missouri, November 26; Ohio State University, November 27–28; University of Cincinnati, November 30–December 3; University of Kentucky, December 4; Oberlin College, December 5–6; University of Michigan, December 7; University of Wisconsin, December 10; Northwestern University, December 11; University of Indiana, December 12; University of Pittsburgh, December 13–17; University of West Virginia, December 18; University of Pennsylvania, December 19.

At the thirteenth annual meeting of the Optical Society of America, held in Washington, October 31 to November 3, 1928, under the joint auspices of that society and the Bureau of Standards, was celebrated the semicentennial of the publication of Professor A. A. Michelson's first communication on the velocity of light. On this occasion, Professor Michelson and his collaborators, Messrs. Pease and Pearson, delivered an address on *Results of repetition of the Michelson-Morley experiment.*

Professor RAYMOND PEARL, of Johns Hopkins University, has been elected president of the International Union for the Study of Demographic Problems.

The following have been appointed to National Research Fellowships in mathematics for 1928–29; A. A. Albert, W. L. Ayres, F. R. Bamforth, Theodore Bennett, Alonzo Church, T. F. Cope, C. M. Cramlet, Jesse Douglas, D. A. Flanders, J. J. Gergen, M. S. Knebelman, Lincoln La Paz, Morris Marden, T. W. Moore, Hillel Poritsky, N. E. Rutt, I. M. Sheffer, Marie J. Weiss. Messrs. Ayres, Church, and Douglass hold fellowships supported by the International Education Board.

Dr. H. A. BENDER of the University of Illinois, has been appointed assistant professor of mathematics at the Municipal University of Akron, Ohio.

Mr. W. M. BOND has been appointed professor of mathematics at Waynesburg College, Waynesburg, Pa.

Mr. P. T. COPP has been appointed head of the department of mathematics at Valparaiso University.

Professor TOMLINSON FORT, of Lehigh University, paid a visit to the mathematics clubs of Rutgers University on November 9 and delivered a lecture on "The place of Sturm's theorem in algebra."

Dr. MARY G. HASEMAN has been appointed professor of mathematics at Hartwick College, Oneonta, N. Y.

Associate Professor B. V. HILL, of Phillips University, Enid, Oklahoma, has been promoted to a professorship of mathematics.

Associate Professor L. S. JOHNSTON, of Pennsylvania State College, has been appointed head of the department of engineering mathematics at the University of Detroit.

Dr. O. J. PETERSON has been appointed head of the department of mathematics at the State Teachers' College at Emporia, Kansas.

Assistant Professor J. B. SCARBOROUGH, of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

Dr. S. A. SCHELKUNOFF, of the State College of Washington, has been promoted to an assistant professorship of mathematics.

Professor A. A. TITSWORTH has retired after forty-two years of service at Rutgers University.

Associate Professor MARY E. WELLS, of Vassar College, has been promoted to a professorship of mathematics.

Associate Professor C. H. Yeaton, of Oberlin College, has been promoted to a professorship of mathematics.

The following appointments to instructorships in mathematics are announced:

George Washington University, Mr. Edgar W. Woolard of the United States Weather Bureau;

University of Michigan, Mr. P. M. SWINGLE;

Northwestern University, Mr. R. H. BARDELL;

Pennsylvania State College, Messrs. L. T. DUNLAP, W. O. GORDON, J. A. HAMILTON, P. V. KUNKEL, and R. E. PETERSON;

St. John's College, Mr. G. C. VEDOVA.

Professor T. A. MARTIN, Berea College, Berea, Kentucky, died on September 11, 1928 after a lingering illness. He had been head of the department of mathematics for seven years and had been a member of the Mathematical Association since December, 1925.

Miss MARION REILLY, of Bryn Mawr College, died in January, 1928. Miss Reilly has been a member of the American Mathematical Society since 1913.

Professor C. D. RICE, of the University of Texas, died on May 28, 1928, at the age of sixty-five. Professor Rice has been a member of the American Mathematical Society since 1920.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December 28-29, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.
INDIANA, Butler University, May 11-12.
IOWA, Grinnell College, May 4-5.
KANSAS, Topeka, Kan., February 4.
KENTUCKY, Louisville, Ky., April 21.
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.
MICHIGAN, Ann Arbor, Mich., March 31.
MINNESOTA, St. Joseph, Minn., May 19.
MISSOURI, Kansas City, Mo., November.
NEBRASKA, Midland College, April 28.
OHIO, Columbus, Ohio, April 5.
PHILADELPHIA, Philadelphia, Pa., December 1.
ROCKY MOUNTAIN, Golden, Colo., April 20-21.
SOUTHEASTERN, Dunham, N. C., March.
SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.
TEXAS, Texas A. and M. College, Jan. 28.

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By

RAYMOND W. BRINK, PH.D.

ASSOCIATE PROFESSOR OF MATHEMATICS AT THE
UNIVERSITY OF MINNESOTA

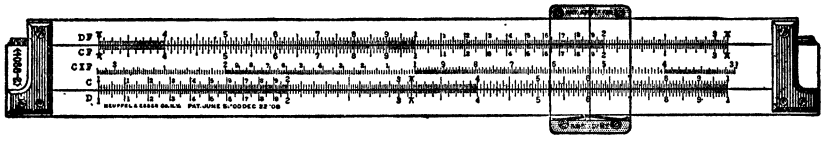
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